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NUMERICAL INTEGRATION USING HYBRID OF BLOCK-PULSE FUNCTIONS AND LAGRANGE POLYNOMIAL

G. Uma[1,2], V. Prabhakar[1], S. Hariharan[1]
[1]School of Advanced Sciences
VIT University Chennai, INDIA
[2]S.A. Engineering College
Chennai, INDIA

Abstract: This paper attempts to solve the definite integrals numerically using Gaussian quadrature rule with variable spacing. In this paper, the orthogonal polynomial is chosen to be the hybrid function formed from the block-pulse function of order N and Lagrange basis polynomial of order M. Here, the domain of the integral is partitioned into N sub-intervals, and the M roots of the Legendre polynomial of order M, are chosen as the nodes in each sub-interval. Having identified M nodes in each sub-interval, the Lagrange polynomial basis of order M is constructed. The total number of abscissas (or quadrature points) in the domain of the definite integral is NM. The weights can be easily obtained from the Lagrange polynomial. A significant advantage of this method is that the Hybrid matrix turns out to be an Identity matrix of order NM and the hybrid coefficients are simply the value of the integrand at the nodal points. The method is explained for single definite integrals, and this rule is extended to cover definite double and triple integrals with constant or variable limits. A comparative study of this method for both definite single and double integrals with Haar Wavelet and Hybrid function [2] reveals that better accuracy can be achieved with this quadrature rule with less number of points. It can be considered to cover a broad class of integrands.

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1. Introduction

Numerical integration plays an important role in the field of Engineering and Sciences. A lot of work in the literature [1-4] done in the numerical integration using quadratures, reveal that quadrature rules based on wavelets and hybrid functions are becoming popular and have a considerable advantage over classical quadratures in several applications [5-9] in terms of better accuracy and faster convergence.

Siraj-ul-islam et al., [1] considered a comparative study of quadrature rule based on Haar wavelet, Hybrid function of block-pulse and Legendre polynomial(say *HBL1*) for finding the approximate value of the definite integrals with constant limits. Imran Aziz et al.,[2] extended the procedure for the numerical solution of double and triple integrals with variable limits using the quadrature rule based on Haar wavelet, Hybrid function of block-pulse and Legendre polynomial (say *HBL2*) to approximate the value of the integration. In their work they proved the supremacy of Hybrid functions with equally spaced nodes over Haar wavelets in terms of accuracy. Maleknejad and Kajani [6] considered Galerkin method to convert the integral equations to a system of linear equations and achieved good accuracy by combining Legendre polynomial with block-pulse function. Further Shojaeizadeh et al., [7] applied Hybrid of Block pulse and Legendre polynomial for solving Fredholm and Volterra integral equations of second kind. They discussed some properties of Block pulse functions and Legendre Polynomials. Also, they used operational matrices of integration and product to reduce the computation of integral equation into some algebraic equations. Chun-Hui-Hsiao [8] also solved Fredholm and Volterra integral equations by means of Hybrid of block-pulse and Legendre polynomials scheme. Marzban et al. [9] applied the Hybrid functions of block-pulse and Lagrange basis polynomial approach for Lane-Emden type nonlinear initial-value problems and proved that the method provides better accuracy. Using the properties of hybrid of block-pulse functions and Lagrange interpolating polynomials they reduced the nonlinear initial-value problems to a system of non-algebraic equations. Motivated by these results, the current work focuses on the Gaussian Quadrature rule for the numerical evaluation of definite integrals using the Hybrid of block-pulse functions and Lagrange basis polynomial. The procedure is extended for definite triple integrals.

The structure of this paper is as follows: Preliminaries are given in Section 2 followed by Section 3 which discuss about the Function approximation using Hybrid of block-pulse and Lagrange polynomial basis functions. In Section 4 the numerical integration of single, double and triple integrals are discussed using hybrid functions and Haar wavelets. Finally, the comparison results are provided in Section 5 followed by the conclusion in Section 6.

2. Preliminaries

2.1. Block-Pulse Functions

A set of block-pulse functions ϕ_n,
$n = 1 \dots N$ on the interval [0,1) are defined as

$$\phi_n(t) = \begin{cases} 1, & t_{n-1} \le t < t_n, \\ 0, & \text{otherwise}, \end{cases}$$

Here $[t_{n-1}, t_n)$ denotes the n^{th} partition (order) and N represents the number of partitions of [0,1).

2.2. Hybrid Functions of Block-Pulse and Lagrange Basis Polynomials

A set of hybrid functions $s_{nm}(t)$, $n = 1 \dots N$,
$m = 0 \dots M - 1$, on the interval [0,1) are defined as

$$s_{nm}(t) = \begin{cases} L_m(2Nt - 2n + 1), & t \in [\frac{n-1}{N}, \frac{n}{N}], \\ 0, & \text{otherwise}, \end{cases}$$

Now the Lagrange basis polynomial $L_m(t)$, is defined by

$$L_m(t) = \prod_{i=0, i \neq m}^{M-1} \frac{t - \tau_i}{\tau_m - \tau_i},$$

where τ_i, $i = 0, 1, \dots M - 1$, are the roots of the Legendre polynomial of order M, with the Kronecker delta property

$$L_m(\tau_i) = \delta_{mi} = \begin{cases} 1, & i = m, \\ 0, & i \neq m. \end{cases}$$

As the block-pulse functions and Lagrange interpolating polynomials are complete and orthogonal, $\{s_{nm}\}$ is a complete orthogonal set in the Hilbert space $L^2[0, 1)$, see [11].

3. Function Approximation

3.1. Function Approximation for f(t)

A function $f(t) \in L^2[0, 1)$ can be approximated as :

$$f(t) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} c_{nm} s_{nm}(t). \tag{1}$$

In practice

$$f(t) = \sum_{n=1}^{N} \sum_{m=0}^{M-1} c_{nm} s_{nm}(t). \tag{2}$$

For a fixed N and M, the number of nodes considered for the function approximation is NM. To find the co-efficients c_{nm}, consider the nodal points r_{nm} which are the corresponding Gaussian nodes at the n^{th} sub interval $\left[\frac{n-1}{N}, \frac{n}{N}\right]$ given by

$$r_{nm} = \frac{1}{2N}(\tau_m + 2n - 1), \tag{3}$$

where τ_m, $m = 0$, 1... M - 1 are Gaussian nodes defined in $[-1, 1]$.
It can be shown that

$$s_{nm}(r_{ni}) = \delta_{mi}. \tag{4}$$

As the Hybrid matrix is an identity matrix of order NM, the co-efficients are simply the value of the function at nodal points.

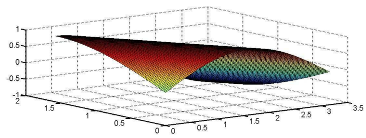
$$c_{nm} = f(r_{mn}). \tag{5}$$

3.2. Function Approximation for Two Variables

Using Hybrid functions, a function $f(t, s)$ can be approximated as :

$$f(t,\ s) \approx \sum_{n=1}^{N'} \sum_{m=0}^{M'-1} \sum_{o=1}^{N} \sum_{p=0}^{M-1} c_{nmop} s_{nm}(t) s_{op}(s), t, s \in [0, 1). \tag{6}$$

Figure 1: Graph of the function $sin(x+y)$



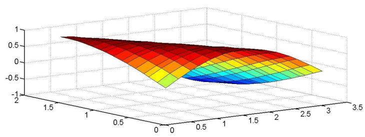


Figure 2: Approximation of $sin(x+y)$ using Hybrid function with N'=N=3 and M'=M=5.

The total number of nodes considered for the function approximation is $NMN'M'$ Using the property as in Eq.(4), function at the nodal points t,s give the coefficients.

$$c_{nmop} = f(r_{nm}, r_{op}). \tag{7}$$

Here r_{nm} and r_{op} are the Gaussian nodes along the t and s directions respectively. Figure 1 and Figure 2 shows the graph of the surface $\sin(x+y)$ and its approximation using the present Hybrid method. The number of nodes considered in x and y directions for the surface approximation is $15X15$. It is clear that both the surface and its approximation agree at the nodal points.

3.3. Function Approximation for Three Variables

Extending to three variables, the Hybrid function can be shown to approximate the function $f(t,\ s,\ u)$ defined in $0 \leq t,\ s,\ u$¡1 as given below.

$$f(t,\ s,\ u) \approx \sum_{n=1}^{N''} \sum_{m=0}^{M''-1} \sum_{o=1}^{N'} \sum_{p=0}^{M'-1} \sum_{q=1}^{N} \sum_{r=0}^{M-1} f(r_{nm}, r_{op}, r_{qr}) \times s_{nm}(t) s_{op}(s) s_{qr}(u). \tag{8}$$

The total number of nodes considered for the function approximation is $NMN'M'N''M''$. where r_{nm}, r_{op} and r_{qr} are the Gaussian nodes along the integration variables.

4. Numerical Integration using Hybrid Functions

4.1. Numerical Integration for Definite Single Integrals

Consider the definite integral

$$\int_0^1 f(t)dt. \tag{9}$$

Using Eq. (2) the integral can be approximated as follows

$$\int_0^1 f(t)\ dt = \sum_{n=1}^{N}\sum_{m=0}^{M-1} c_{nm}w_{nm}, \tag{10}$$

where $w_{nm}=\int_0^1 s_{nm}(t)\,dt$ are the weights. As $s_{nm}(t)$ is a polynomial, the weights can be easily calculated.
Substituting Eq. (5), the above equation becomes

$$\int_0^1 f(t)\ dt = \sum_{n=1}^{N}\sum_{m=0}^{M-1} f(r_{nm})w_{nm}, \tag{11}$$

It can be verified that

$$\sum_{n=1}^{N}\sum_{m=0}^{M-1} w_{nm} = 1. \tag{12}$$

Letting $t = a + (b - a)s$ in Eq. (9) leads to

$$\int_a^b f(t)\ dt \approx (b-a)\int_0^1 f(a+(b-a)s)\ ds, \tag{13}$$

Using Eq. (11),

$$\int_a^b f(t)\ dt \approx (b-a)\sum_{n=1}^{N}\sum_{m=0}^{M-1} f(a+\{(b-a)r_{nm}\})w_{nm}. \tag{14}$$

4.2. Numerical Integration for Single Integrals using Haar Wavelets

In terms of Haar wavelets,

$$\int_a^b f(t)\ dt = \frac{b-a}{2M}\sum_{k=1}^{2M} f(t_{k})$$

where $t_k = a + \frac{(b-a)(k-0.5)}{2M}$ and $M{=}2^J$. Here J is the maximum level of resolution. For fixed J, the number of nodes will be $2M$ (refer [2] for more details).

4.3. Numerical Integration for Double Integrals

Consider the following double integral

$$\int_a^b \int_{c(s)}^{d(s)} f(t,s)\, dt\, ds. \tag{15}$$

Applying Eq. (14), to the inner integral and treating the variable s as constant, the above equation can be written as,

$$\int_a^b \int_{c(s)}^{d(s)} f(t,s)\, dt\, ds \approx \int_a^b \{d(s)-c(s)\} \sum_{n=1}^{N'} \sum_{m=0}^{M'-1} f(c(s)+\{d(s)-c(s)\}\, r_{nm}, s) w_{nm}\, ds \tag{16}$$

Let

$$g(s) = \{d(s)-c(s)\} \sum_{n=1}^{N'} \sum_{m=0}^{M'-1} f(c(s)+\{d(s)-c(s)\}\, r_{nm}, s) w_{nm}. \tag{17}$$

Finally, applying Eq. (14), to the outer integral leads to

$$\int_a^b \int_{c(s)}^{d(s)} f(t,s)\, dt\, ds \approx \int_a^b g(s)\, ds \approx (b-a) \sum_{n=1}^{N} \sum_{m=0}^{M-1} g(a+\{(b-a)r_{nm}\}) w_{nm}. \tag{18}$$

In the above equation N' represents the number of partitions of the outer interval and M' denotes the order of Legendre polynomial in each partition. Thus, the present method has the flexibility of choosing the values of N', N, M' and M independently.

4.4. Numerical Integration for Double Integrals using Haar Wavelets

In terms of Haar wavelets,

$$\int_a^b \int_{c(s)}^{d(s)} f(t,s)\,dt\,ds \approx \frac{b-a}{2N}\sum_{i=1}^{2N} g(a+\frac{(b-a)(i-0.5)}{2N})$$

where $g(s)=\frac{\{d(s)-c(s)\}}{2M}\sum_{i=1}^{2M} f(t_i,s)$ with $t_i = c(s)+\frac{(d(s)-c(s))(i-0.5)}{2M}$. Here, $M=2^{J_1}$,$N=2^{J_2}$ and J_1 and J_2 are the maximum levels of resolution along t and s directions respectively. For detailed study refer to [2].

4.5. Numerical Integration for Triple Integrals

Consider the Triple integral with varying limits

$$\int_a^b \int_{c(u)}^{d(u)} \int_{e(s,u)}^{f(s,u)} g(t,s,u)\,dt\,ds\,du. \tag{19}$$

Applying Eq. (14), to the inner integral and treating the variables t, s as constant, the above equation can be written as,

$$\int_a^b \int_{c(u)}^{d(u)} \int_{e(s,u)}^{f(s,u)} g(t,s,u)\,dt\,ds\,du \approx$$
$$\int_a^b \int_{c(u)}^{d(u)} \{f(s,u)-e(s,u)\} \sum_{n=1}^{N''}\sum_{m=0}^{M''-1} g(t_{nm},s,u)w_{nm}\,ds\,du \tag{20}$$

where t_{nm}=$e(s,u)+\{f(s,u)-e(s,u)\}\,r_{nm}$.
Let

$$h(s,u) = \{f(s,u)-e(s,u)\}\sum_{n=1}^{N''}\sum_{m=0}^{M''-1} g(t_{nm},s,u)w_{nm}. \tag{21}$$

Secondly, applying Eq. (14), to the middle integral of Eq. (19) leads to the following equation,

$$\int_a^b \int_{c(u)}^{d(u)} h(s,u)\,ds\,du \approx \int_a^b du(d(u)-c(u))\sum_{o=1}^{N'}\sum_{p=0}^{M'-1}$$

$$h(s_{op}, u)w_{op}. \tag{22}$$

where $s_{op} = c(u) + (d(u) - c(u))r_{op}$.
Choose

$$L(u) = (d(u) - c(u)) \sum_{o=1}^{N'} \sum_{p=0}^{M'-1} h(s_{op}, u)w_{op} \tag{23}$$

Finally, applying Eq. (14) to the outer integral in Eq. (19) yields,

$$\int_a^b L(u)\, du \approx (b-a) \sum_{q=1}^{N} \sum_{r=0}^{M-1} L(a + \{(b-a)r_{qr}\})w_{qr}. \tag{24}$$

In the above equation N'' represents the number of partitions of the outer interval and M'' denotes the order of Legendre polynomial in each partition. Thus, the present method has the flexibility of choosing the values of N, N', N'', M, M' and M'' independently.

5. Numerical Examples

In this section the accuracy of Hybrid functions with unequal nodes are shown by means of a comparative study of the present method with earlier methods. Absolute errors of few single integrals are shown in Table 1. The integral values in the first example of Table 1 confirm that the present method and Hybrid [1] attain the accuracy of 9 and 6 decimal places when 15 nodes with Legendre polynomial of order 3 are used. By increasing the order of Legendre polynomial to 5, the present method achieves very high accuracy of 17 decimal places. The last example in this Table is considered to test the applicability of the present method to the singular integral. It can be noted that the number of nodes considered for the Hybrid methods are almost half that of Haar. When the nodes are increased the present Hybrid method gives better accuracy compared to Haar and Hybrid [1].

Absolute errors of double integrals with constant limits, variable limits and polar integrals are presented in Table 2. In the first and third integrals, the proposed method and HBL1 uses the same number of nodes say 15, 32 and 60 along the x and y axes. Here also, the proposed method uses half the number of nodes compared to Haar [1]. It is clear that the present method gives better accuracy over Haar and HBL1. A MATLAB code is written for the double integrals using HBL2 and also extended to the case when the nodes in

Table 1: Absolute errors of few single integrals

Haar [2]			Absolute errors		
Integral	J/nodes	Absolute errors	nodes $= M \times N$	HBL1	Present method
$\int_0^1 sin(x^2)\ dx$	4/32 5/64 6/128	4.3987e-5 1.0994e-5 2.7482e-6	3×5=15 4×8=32 5×12=60	3.1369e-6 2.4654e-7 1.0695e-11	2.6738e-9 6.9333e-14 5.5511 e-17
$\int_0^5 \sqrt{(x^2 - 5x + 31)}\ dx$	4/32 5/64 6/128	9.1357e-04 2.2838e-04 5.7095e-05	3×5=15 4×8=32 5×12=60	9.4395e-06 7.4155e-07 3.4330e-11	8.8480e-09 2.1316e-14 0
$\int_0^1 \frac{e^{\frac{-1}{x}}}{x^2}\ dx$	4/32 5/64 6/128	1.4951e-05 3.7423e-06 9.3557e-07	3×5=15 4×8=32 5×12=60	1.0807e-04 2.4429e-05 6.7026e-07	3.8449e-05 8.5339e-07 1.6958e-09

Table 2: Absolute errors of some double integrals

Haar [2]			Absolute errors		
Integral	J_1,J_2/ nodes	Absolute errors	M',N';M,N	HBL1 /HBL2	Present method
$\int_0^1 \int_0^1 \frac{1}{\sqrt{x^2 + y^2}}\ dx\ dy$	4,4/32×32 5,5/64×64 6,6/128×128	0.0126 0.0063 0.0032	3,5;3,5 4,8;4,8 5,12;5,12	0.0158 0.0082 0.0033	0.0082 0.0029 7.24e-04
$\int_0^{\frac{\pi}{4}} \int_0^{\sin(y)} \frac{1}{\sqrt{1 - x^2}}\ dx\ dy$	4,3/32×16 5,4/64×32 6,5/128×64	6.21e-06 1.56e-06 3.9e-07	3,5;2,4 4,8;3,5 5,12;4,8	3.68e-07 3.19e-08 1.38e-11	3.00e-09 5.18e-13 6.78e-17
$\int_0^{\pi/2} \int_a^{a(1+\cos(\theta))} r\ dr\ d\theta$	4,4/32×32 5,5/64×64 6,6/128×128	4.02e-04 1.00e-04 2.51e-05	3,5;3,5 4,8;4,8 5,12;5,12	5.23e-06 4.16e-07 7.42e-12	1.9e-09 7.57e-15 7.72e-16

x and y directions are different. The second integral in this Table uses different nodes in x and y direction for all the three methods. Here also the accuracy of the proposed method is higher than that of Haar and HBL2. Absolute errors of Triple integrals with constant limits and variable limits are presented in Table 3. This clearly shows that the present method works well for the triple integrals. These examples emphasis that better accuracy can be achieved with the proposed hybrid functions and it can be used to solve the integrals numerically. In the above examples, its seen that the present method achieves good accuracy with lesser number of nodes . Further, if singularity arises in the integrand then also the present Hybrid method shall be used to attain improved accuracy .

Table 3: Absolute errors of Triple integrals

Integral	Parameters $(M'',N'';M',N';M,N)$	Exact value	Hybrid method	Absolute errors
$\int_0^2\int_0^2\int_0^2 \frac{1}{x+z+y}\,dx\,dy\,dz$	5,2; 4,2; 8,6	3.1395	3.1387	0.00075
$\int_0^3\int_0^{3-x}\int_0^{3-x-y} xzy\,dz\,dy\,dx$	2,4; 2,4; 3,5	1.0125	1.0125	9.2e-18

6. Conclusion

In this paper, a new method of numerical integration using Hybrid of block-pulse function and Lagrange polynomial is proposed for the evaluation of definite integrals. The method is extended to cover the numerical evaluation of definite double and triple integrals. The method has the flexibility of choosing the appropriate number of nodes in the respective directions of the integrals. A comparative study of this method for both single and double integrals with the methods based on Haar wavelet, HBL1 and HBL2 reveals that the methods based on Hybrid functions give better accuracy with lesser number of nodal points than that of Haar wavelet method. Amongst the methods based on Hybrid functions, the present Hybrid method achieves better accuracy. An advantage of the present method is that it converges fast and accuracy can always be improved.

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