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A NOTE ON FRACTIONAL NEUTRAL INTEGRO-DIFFERENTIAL SYSTEMS WITH STATE-DEPENDENT DELAY

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Abstract: Based on concepts for semigroups, fractional calculus and Banach contraction principle, this manuscript is principally involved with existence results for fractional neutral integro-differential systems (FNIDS) with state-dependent delay (SDD) in Banach spaces. An illustration is also offered to demonstrate the attained theorem.

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1. Introduction

The concept of semigroups of bounded linear operator is meticulously associated to solving differential and integro-differential equations in Banach spaces. Recently, this concept has been employed to a significant type of non-linear differential equations in Banach spaces. For more points of interest on this concept, we allude the reader to Pazy [1]. In the most recent decades, fractional differential equations (FDE) have assumed an imperative part because of the memory character of fractional derivative, which is a speculation of integer-

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order derivative and can depict numerous advancement that integer derivative are unable to outline. For essential assurances about fractional frameworks, one can make reference to the treatises [2, 3], and the papers [4, 5], and the references cited therein. Fractional equation with delay features happen in several areas such as medical and physical with state-dependent delay or non-constant delay. These days, existence results of mild solutions for such problems became very attractive and several researchers working on it, see for instance [6, 7, 8].

The existence, controllability and other qualitative and quantitative properties of FDE are the most advancing territory of exploration, for instance, see [9, 11, 12, 13, 14, 15, 16, 17, 10]. As of late, Santos et al. [11, 12] analyzed the existence of solutions for FIDE with unbounded or SDD delay in Banach spaces. Shu et al. [13] investigated the existence results for FDE with nonlocal conditions of order $\alpha \in (1, 2)$. In [14, 15], the writers offer adequate circumstances for the existence and approximate controllability of fractional order neutral differential and stochastic differential system with infinite delay. Kexue et al. [16] analyzed the controllability of nonlocal FDE of order $\alpha \in (1, 2]$. Sakthivel et al. [17] acknowledged the approximate controllability of fractional dynamical system by employing appropriate fixed point theorem. Lately, in [18, 19], the authors discussed the approximate controllability results for FNIDS with SDD by utilizing the suitable fixed point theorem. However, existence results for FNIDS with SDD in $\mathscr{B}_h$ phase space adages have not yet been totally inspected.

Motivated by the exertion of the aforementioned papers [18, 19], the principle motivation behind this manuscript is to research the existence of mild solutions for FNIDS with SDD of the model

$$
\begin{aligned}
D_t^{\alpha}\big[x(t) + \mathscr{G}(t, x_{\varrho(t,x_t)})\big] = \mathscr{A}x(t) + \int_0^t \mathscr{B}(t-s)x(s)ds \\
+ \mathscr{F}(t, x_{\varrho(t,x_t)}), \quad t \in \mathscr{I} = [0, T], \qquad (1.1)\\
x_0 = \varsigma(t) \in \mathscr{B}_h, \quad x'(0) = 0, \quad t \in (-\infty, 0], \qquad (1.2)
\end{aligned}
$$

where the unknown $x(\cdot)$ needs values in the Banach space $\mathbb{X}$ having norm $\|\cdot\|$, D_t^{α} is the Caputo fractional derivative of order $\alpha \in (1, 2)$, $\mathscr{A}, (\mathscr{B}(t))_{t\geq 0}$ are closed linear operators described on a regular domain which is dense in $(\mathbb{X}, \|\cdot\|)$ and $D_t^{\alpha}\sigma(t)$ symbolize the Caputo derivative of $\alpha > 0$ characterized by

$$
D_t^{\alpha}\sigma(t) := \int_0^t \widetilde{\mu}_{n-\alpha}(t-s)\frac{d^n}{ds^n}\sigma(s)ds,
$$

where $n \geq \alpha$ and $\widetilde{\mu}_{\beta}(t) := \frac{t^{\beta-1}}{\Gamma(\beta)}, t > 0, \beta \geq 0$. Further, $\mathscr{G}, \mathscr{F} : \mathscr{I} \times \mathscr{B}_h \to$

$\mathbb{X}, \varrho : \mathscr{I} \times \mathscr{B}_h \to (-\infty, T]$ are apposite functions, and $\mathscr{B}_h$ is a phase space characterized in Preliminaries.

For almost any continuous function x characterized on $(-\infty, T]$ and any $t \geq 0$, we designate by x_t the part of $\mathscr{B}_h$ characterized by $x_t(\theta) = x(t + \theta)$ for $\theta \leq 0$. Now $x_t(\cdot)$ speaks to the historical backdrop of the state from every $\theta \in (-\infty, 0]$ likely the current time t.

We move forward as follows. Section 2 is committed to analysis of some vital aspects that will be employed in this work to attain our principal outcomes. In Section 3, we declare and demonstrate the existence results by suggests of Banach fixed point theorem. In Section 4, as a final point, a proper case is furnished to reflect the efficiency of the abstract concept.

2. Preliminaries

In this part, we present some primary components which are required to confirm the principal outcomes.

Let $\mathscr{L}(\mathbb{X})$ symbolizes the Banach space of all bounded linear operators from $\mathbb{X}$ into $\mathbb{X}$ endowed with the uniform operator topology, having its norm recognized as $\|\cdot\|_{\mathscr{L}(\mathbb{X})}$.

Let $C(\mathscr{I}, \mathbb{X})$ symbolize the space of all continuous functions from $\mathscr{I}$ into $\mathbb{X}$, having its norm recognized as $\|\cdot\|_{C(\mathscr{I},\mathbb{X})}$. Moreover, $B_r(x, \mathbb{X})$ symbolizes the closed ball in $\mathbb{X}$ with the middle at x and the distance r.

It needs to be outlined that, once the delay is infinite, then we should talk about the theoretical phase space $\mathscr{B}_h$ in a beneficial way. In this manuscript, we deliberate phase spaces $\mathscr{B}_h$ which are same as described in [20]. So, we bypass the details.

We expect that the phase space $(\mathscr{B}_h, \|\cdot\|_{\mathscr{B}_h})$ is a semi-normed linear space of functions mapping $(-\infty, 0]$ into $\mathbb{X}$, and fulfilling the subsequent elementary adages as a result of Hale and Kato (see case in point in [21, 22]).

If $x : (-\infty, T] \to \mathbb{X}, T > 0$, is continuous on $\mathscr{I}$ and $x_0 \in \mathscr{B}_h$, then for every $t \in \mathscr{I}$ the accompanying conditions hold:

(P_1) x_t is in $\mathscr{B}_h$;

(P_2) $\|x(t)\|_{\mathbb{X}} \leq H\|x_t\|_{\mathscr{B}_h}$;

(P_3) $\|x_t\|_{\mathscr{B}_h} \leq \mathscr{D}_1(t) \sup\{\|x(s)\|_{\mathbb{X}} : 0 \leq s \leq t\} + \mathscr{D}_2(t)\|x_0\|_{\mathscr{B}_h}$, where $H > 0$ is a constant and $\mathscr{D}_1(\cdot) : [0, +\infty) \to [0, +\infty)$ is continuous, $\mathscr{D}_2(\cdot) : [0, +\infty) \to [0, +\infty)$ is locally bounded, and $\mathscr{D}_1, \mathscr{D}_2$ are independent of $x(\cdot)$.

(P_4) The function $t \to \varsigma_t$ is well described and continuous from the set

$$\mathcal{R}(\varrho^-) = \{\varrho(s,\varsigma) : (s,\varsigma) \in [0,T] \times \mathscr{B}_h\},$$

into $\mathscr{B}_h$ and there is a continuous and bounded function $J^\varsigma : \mathcal{R}(\varrho^-) \to (0,\infty)$ to ensure that $\|\varsigma_t\|_{\mathscr{B}_h} \le J^\varsigma(t)\|\varsigma\|_{\mathscr{B}_h}$ for every $t \in \mathcal{R}(\varrho^-)$.

Recognize the space

$$\mathscr{B}_T = \left\{x : (-\infty,T] \to \mathbb{X} : x|_{\mathscr{I}} \text{ is continuous and } x_0 \in \mathscr{B}_h\right\},$$

where $x|_{\mathscr{I}}$ is the constraint of x to the real compact interval on $\mathscr{I}$. The function $\|\cdot\|_{\mathscr{B}_T}$ to be a seminorm in $\mathscr{B}_T$, it is described by

$$\|x\|_{\mathscr{B}_T} = \|\varsigma\|_{\mathscr{B}_h} + \sup\{\|x(s)\|_{\mathbb{X}} : s \in [0,T]\},\ x \in \mathscr{B}_T.$$

Lemma 1. *[23, Lemma 2.1] Let $x : (-\infty,T] \to \mathbb{X}$ be a function in a way that $x_0 = \varsigma$, and if $(P4)$ hold, then*

$$\|x_s\|_{\mathscr{B}_h} \le (\mathscr{D}_2^* + J^\varsigma)\|\varsigma\|_{\mathscr{B}_h} + \mathscr{D}_1^* \sup\{\|x(\theta)\|_{\mathbb{X}} : \theta \in [0,\max\{0,s\}]\},$$
$$s \in \mathcal{R}(\varrho^-) \cup \mathscr{I},$$

where $J^\varsigma = \sup\limits_{t\in\mathcal{R}(\varrho^-)} J^\varsigma(t), \quad \mathscr{D}_1^* = \sup\limits_{s\in[0,b]} \mathscr{D}_1(s), \quad \mathscr{D}_2^* = \sup\limits_{s\in[0,T]} \mathscr{D}_2(s).$

Remark 2. *For extra information about this concept, we suggest the reader to refer [9, 11].*

3. Existence Results

In this segment, we present and demonstrate the existence of solutions for the structure (1.1)-(1.2) under Banach fixed point theorem. First, we present the mild solution for the model (1.1)–(1.2).

Definition 3. A function $x : (-\infty,T] \to \mathbb{X}$, is called a mild solution of (1.1)-(1.2) on $[0,T]$, if $x_0 = \varsigma$; $x|_{[0,T]} \in C([0,T] : \mathbb{X})$; the function $s \to \mathscr{A}\mathcal{S}_\alpha(t-s)\mathscr{G}(s, x_{\varrho(s,x_s)})$ and $s \to \int_0^s \mathscr{B}(s-\tau)\mathcal{S}_\alpha(t-s)\mathscr{G}(\tau, x_{\varrho(\tau,x_\tau)})d\tau$ is integrable on

$[0,t)$ for all $t \in (0,T]$ and for $t \in [0,T]$,

$$\begin{aligned} x(t) = & \mathcal{R}_\alpha(t)(\varsigma(0) + \mathscr{G}(0,\varsigma(0))) - \mathscr{G}(t, x_{\varrho(t,x_t)}) \\ & - \int_0^t \mathscr{A}\mathcal{S}_\alpha(t-s)\mathscr{G}(s, x_{\varrho(s,x_s)})ds \\ & - \int_0^t \int_0^s \mathscr{B}(s-\tau)\mathcal{S}_\alpha(t-s)\mathscr{G}(\tau, x_{\varrho(\tau,x_\tau)})d\tau ds \\ & + \int_0^t \mathcal{S}_\alpha(t-s)\mathscr{F}(s, x_{\varrho(s,x_s)})ds. \end{aligned}$$

Presently, we itemizing the subsequent suppositions:

(H1) The operator families $\mathcal{R}_\alpha(t)$ and $\mathcal{S}_\alpha(t)$ are compact for all $t > 0$, and there exists a constant $\mathcal{M}$ in a way that $\|\mathcal{R}_\alpha(t)\|_{\mathscr{L}(\mathbb{X})} \leq \mathcal{M}$ and $\|\mathcal{S}_\alpha(t)\|_{\mathscr{L}(\mathbb{X})} \leq \mathcal{M}$ for every $t \in \mathscr{I}$ and

$$\|(-\mathscr{A})^\vartheta \mathcal{S}_\alpha(t)\|_{\mathbb{X}} \leq \mathcal{M}t^{\alpha(1-\vartheta)-1}, \quad 0 < t \leq T.$$

(H2) The subsequent conditions are fulfilled.

(a) $\mathscr{B}(\cdot)x \in C(\mathscr{I}, \mathbb{X})$ for every $x \in [D((-\mathscr{A})^{1-\vartheta})]$.

(b) There is a function $\mu(\cdot) \in L^1(\mathscr{I}; \mathbb{R}^+)$, to ensure that

$$\|\mathscr{B}(s)\mathcal{S}_\alpha(t)\|_{\mathscr{L}([D((-\mathscr{A})^\vartheta)],\mathbb{X})} \leq \mathcal{M}\mu(s)t^{\alpha\vartheta-1}, \quad 0 \leq s < t \leq T.$$

(H3) The function $\mathscr{G}(\cdot)$ is $(-\mathscr{A})^\vartheta$-valued, $\mathscr{G} : \mathscr{I} \times \mathscr{B}_h \to [D((-\mathscr{A})^{-\vartheta})]$ is continuous and there exist $L_{\mathscr{G}}, L^*_{\mathscr{G}} > 0$ such that for all $(t, \varsigma_j) \in \mathscr{I} \times \mathscr{B}_h, j = 1, 2$;

$$\|(-\mathscr{A})^\vartheta \mathscr{G}(t,\varsigma_1) - (-\mathscr{A})^\vartheta \mathscr{G}(t,\varsigma_2)\|_{\mathbb{X}} \leq L_{\mathscr{G}}\|\varsigma_1 - \varsigma_2\|_{\mathscr{B}_h},$$
$$\|(-\mathscr{A})^\vartheta \mathscr{G}(t,\varsigma)\|_{\mathbb{X}} \leq L_{\mathscr{G}}\|\varsigma\|_{\mathscr{B}_h} + L^*_{\mathscr{G}},$$

where

$$L^*_{\mathscr{G}} = \max_{t\in\mathscr{I}} \|(-\mathscr{A})^\vartheta \mathscr{G}(t,0)\|_{\mathbb{X}}.$$

(H4) The function $\mathscr{F} : \mathscr{I} \times \mathscr{B}_h \to \mathbb{X}$ is continuous and we can find constants $L_{\mathscr{F}}, L^*_{\mathscr{F}} > 0$ in ways that

$$\|\mathscr{F}(t,\psi_1) - \mathscr{F}(t,\psi_2)\|_{\mathbb{X}} \leq L_{\mathscr{F}}\|\psi_1 - \psi_2\|_{\mathscr{B}_h}$$

and

$$L^*_{\mathscr{F}} = \max_{t\in\mathscr{I}} \|\mathscr{F}(t,0)\|_{\mathbb{X}}.$$

(H5) The following inequalities holds:

(i) Let

$$\begin{aligned}&\mathcal{M}\mathcal{M}_0 L_{\mathscr{G}}\|\varsigma\|_{\mathscr{B}_h} + \mathcal{M}_0 L^*_{\mathscr{G}}(1+\mathcal{M}) + (\mathscr{D}^*_1 r + c_n)\Bigg[\mathcal{M}_0 L_{\mathscr{G}}\\&+\frac{\mathcal{M}T^{\alpha\vartheta}}{\alpha\vartheta}L_{\mathscr{G}}\left(1+\int_0^T \mu(\tau)d\tau\right) + \mathcal{M}TL_{\mathscr{F}}\Bigg]\\&+\frac{\mathcal{M}T^{\alpha\vartheta}}{\alpha\vartheta}L^*_{\mathscr{G}} + \frac{\mathcal{M}T^{\alpha\vartheta}}{\alpha\vartheta}L^*_{\mathscr{G}}\int_0^T \mu(\tau)d\tau + \mathcal{M}TL^*_{\mathscr{F}} \le r,\end{aligned}$$

for some $r > 0$.

(ii) Let

$$\Lambda = \mathscr{D}^*_1\Bigg\{L_{\mathscr{G}}\left[\mathcal{M}_0 + \frac{\mathcal{M}T^{\alpha\vartheta}}{\alpha\vartheta}\left(1+\int_0^T \mu(\tau)d\tau\right)\right] + L_{\mathscr{F}}\mathcal{M}T\Bigg\} < 1$$

be such that $0 \le \Lambda < 1$.

Theorem 4. *Assume that the conditions (H1)-(H5) hold. Then the structure (1.1)-(1.2) has a unique mild solution on $\mathscr{I}$.*

Proof. We will transmute the structure (1.1)-(1.2) into a fixed-point problem. Recognize the operator $\Upsilon : \mathscr{B}_T \to \mathscr{B}_T$ specified by

$$(\Upsilon x)(t) = \begin{cases} \mathcal{R}_\alpha(t)(\varsigma(0) + \mathscr{G}(0,\varsigma(0))) - \mathscr{G}(t, x_{\varrho(t,x_t)}) \\ -\int_0^t \mathscr{A}\mathcal{S}_\alpha(t-s)\mathscr{G}(s, x_{\varrho(s,x_s)})ds \\ \quad -\displaystyle\int_0^t\int_0^s \mathscr{B}(s-\tau)\mathcal{S}_\alpha(t-s)\mathscr{G}(\tau, x_{\varrho(\tau,x_\tau)})d\tau ds \\ +\displaystyle\int_0^t \mathcal{S}_\alpha(t-s)\mathscr{F}(s, x_{\varrho(s,x_s)})ds, \quad t \in \mathscr{I}. \end{cases}$$

It is evident that the fixed points of the operator Υ are mild solutions of the model (1.1)-(1.2). We express the function $y(\cdot) : (-\infty, T] \to \mathbb{X}$ by

$$y(t) = \begin{cases} \varsigma(t), \quad t \le 0; \\ \mathcal{R}_\alpha(t)\varsigma(0), \quad t \in \mathscr{I}, \end{cases}$$

then $y_0 = \varsigma$. For every function $z \in C(\mathscr{I}, \mathbb{R})$ with $z(0) = 0$, we allocate as $\overline{z}$ is characterized by

$$\overline{z}(t) = \begin{cases} 0, & t \leq 0; \\ z(t), & t \in \mathscr{I}. \end{cases}$$

If $x(\cdot)$ fulfills (3.1), we are able to split it as $x(t) = y(t) + z(t)$, $t \in \mathscr{I}$, which suggests $x_t = y_t + z_t$, for each $t \in \mathscr{I}$ and also the function $z(\cdot)$ fulfills

$$z(t) = \begin{cases} \mathcal{R}_\alpha(t)\mathscr{G}(0,\varsigma) - \mathscr{G}(t, z_{\varrho(t,z_t+y_t)} + y_{\varrho(t,z_t+y_t)}) \\ -\int_0^t \mathscr{A}\mathcal{S}_\alpha(t-s)\mathscr{G}(s, z_{\varrho(s,z_s+y_s)} + y_{\varrho(s,z_s+y_s)})ds \\ -\int_0^t\int_0^s \mathscr{B}(s-\tau)\mathcal{S}_\alpha(t-s)\mathscr{G}(\tau, z_{\varrho(\tau,z_\tau+y_\tau)} + y_{\varrho(\tau,z_\tau+y_\tau)})d\tau ds \\ +\int_0^t \mathcal{S}_\alpha(t-s)\mathscr{F}(s, z_{\varrho(s,z_s+y_s)} + y_{\varrho(s,z_s+y_s)})ds. \end{cases}$$

Let $\mathscr{B}_T^0 = \{z \in \mathscr{B}_T^0\colon\ z_0 = 0 \in \mathscr{B}_h\}$. Let $\|\cdot\|_{\mathscr{B}_T^0}$ be the seminorm in $\mathscr{B}_T^0$ described by

$$\|z\|_{\mathscr{B}_T^0} = \sup_{t\in\mathscr{I}}\|z(t)\|_{\mathbb{X}} + \|z_0\|_{\mathscr{B}_h} = \sup_{t\in\mathscr{I}}\|z(t)\|_{\mathbb{X}}, \quad z \in \mathscr{B}_T^0,$$

as a result $(\mathscr{B}_T^0, \|\cdot\|_{\mathscr{B}_T^0})$ is a Banach space. We delimit the operator $\overline{\Upsilon} : \mathscr{B}_T^0 \to \mathscr{B}_T^0$ by

$$(\overline{\Upsilon}z)(t) = \begin{cases} \mathcal{R}_\alpha(t)\mathscr{G}(0,\varsigma) - \mathscr{G}(t, z_{\varrho(t,z_t+y_t)} + y_{\varrho(t,z_t+y_t)}) \\ -\int_0^t \mathscr{A}\mathcal{S}_\alpha(t-s)\mathscr{G}(s, z_{\varrho(s,z_s+y_s)} + y_{\varrho(s,z_s+y_s)})ds \\ -\int_0^t\int_0^s \mathscr{B}(s-\tau)\mathcal{S}_\alpha(t-s)\mathscr{G}(\tau, z_{\varrho(\tau,z_\tau+y_\tau)} + y_{\varrho(\tau,z_\tau+y_\tau)})d\tau ds \\ +\int_0^t \mathcal{S}_\alpha(t-s)\mathscr{F}(s, z_{\varrho(s,z_s+y_s)} + y_{\varrho(s,z_s+y_s)})ds. \end{cases}$$

It is vindicated that the operator Υ has a fixed point if and only if $\overline{\Upsilon}$ has a fixed point.

Initially, we demonstrate that $\overline{\Upsilon}$ maps $B_r(0, \mathscr{B}_T^0)$ into $B_r(0, \mathscr{B}_T^0)$. For any $z(\cdot) \in \mathscr{B}_T^0$, we sustain

$$\begin{aligned} &\|(\overline{\Upsilon}z)(t)\|_{\mathbb{X}} \\ &\leq \mathcal{M}\mathcal{M}_0 L_{\mathscr{G}}\|\varsigma\|_{\mathscr{B}_h} + \mathcal{M}_0 L_{\mathscr{G}}^*(1+\mathcal{M}) + (\mathscr{D}_1^* r + c_n)\Bigg[\mathcal{M}_0 L_{\mathscr{G}} \end{aligned}$$

$$+\frac{\mathcal{M}T^{\alpha\vartheta}}{\alpha\vartheta}L_{\mathscr{G}}\left(1+\int_0^T\mu(\tau)d\tau\right)+\mathcal{M}TL_{\mathscr{F}}\Bigg]$$

$$+\frac{\mathcal{M}T^{\alpha\vartheta}}{\alpha\vartheta}L^*_{\mathscr{G}}+\frac{\mathcal{M}T^{\alpha\vartheta}}{\alpha\vartheta}L^*_{\mathscr{G}}\int_0^T\mu(\tau)d\tau+\mathcal{M}TL^*_{\mathscr{F}}$$
$$\leq r.$$

Therefore, $\overline{\Upsilon}$ maps the ball $B_r(0,\mathscr{B}^0_T)$ into itself. Finally, we show that $\overline{\Upsilon}$ is a contraction on $B_r(0,\mathscr{B}^0_T)$. For this, let us consider $z,\overline{z}\in B_r(0,\mathscr{B}^0_T)$, we sustain

$$\begin{aligned}&\|(\overline{\Upsilon}z)(t)-(\overline{\Upsilon}\overline{z})(t)\|_{\mathbb{X}}\\&\leq\mathscr{D}^*_1\left\{L_{\mathscr{G}}\left[\mathcal{M}_0+\frac{\mathcal{M}T^{\alpha\vartheta}}{\alpha\vartheta}\left(1+\int_0^T\mu(\tau)d\tau\right)\right]+L_{\mathscr{F}}\mathcal{M}T\right\}\|z-\overline{z}\|_{\mathscr{B}^0_T}\\&\leq\Lambda\|z-\overline{z}\|_{\mathscr{B}^0_T}.\end{aligned}$$

From the assumption (H5) and in the perspective of the contraction mapping principle, we understand that $\overline{\Upsilon}$ includes a unique fixed point $z\in\mathscr{B}^0_T$ which is a mild solution of the model (1.1)-(1.2) on $(-\infty,T]$. The proof is now completed. □

References

[1] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, New York, 1983.

[2] D. Baleanu, J. A. T. Machado, A. C. J. Luo, *Fractional Dynamics and Control*, Springer, New York, USA, 2012.

[3] A. Kilbas, H. Srivastava and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier, Amesterdam, 2006.

[4] G. Bonanno, R. Rodriguez-Lopez, S. Tersian, Existence of solutions to boundary value problem for impulsive fractional differential equations, *Fractional Calculus and Applied Analysis*, 17(3)(2014), 717-744.

[5] R. Rodrguez-Lpez and S. Tersian, Multiple solutions to boundary value problem for impulsive fractional differential equations, *Fractional Calculus and Applied Analysis*, 17(4)(2014), 1016-1038.

[6] R. P. Agarwal, B. D. Andrade, On fractional integro-differential equations with state-dependent delay, *Comp. Math. App.,* 62(2011), 1143–1149.

[7] M. Benchohra, F. Berhoun, Impulsive fractional differential equations with state-dependent delay, *Commun. Appl. Anal.,* 14(2)(2010), 213–224.

[8] K. Aissani and M. Benchohra, Fractional integro-differential equations with state-dependent delay, *Advances in Dynamical Systems and Applications*, 9(1)(2014), 17-30.

[9] R. P. Agarwal, J. P. C. Dos Santos, and C. Cuevas, Analytic resolvent operator and existence results for fractional integro-differential equations, *J. Abstr. Differ. Equ. Appl.,* 2(2)(2012), 26-47.

[10] V. Vijayakumar, A. Selvakumar and R. Murugesu, Controllability for a class of fractional neutral integro-differential equations with unbounded delay, *Applied Mathematics and Computation*, 232(2014), 303-312.

[11] B. D. Andrade, J. P. C. Santos, Existence of solutions for a fractional neutral integro-differential equation with unbounded delay, *Electronic Journal of Differential Equations,* Vol. 2012 (2012), No. 90, pp. 1-13.

[12] J. P. C. Dos Santos, V. Vijayakumar and R. Murugesu, Existence of mild solutions for nonlocal Cauchy problem for fractional neutral integro-differential equation with unbounded delay, *Communications in Mathematical Analysis*, X(2011), 1–13.

[13] X. Shu and Q. Wang, The existence and uniqueness of mild solutions for fractional differential equations with nonlocal conditions of order $1 < \alpha < 2$, *Comput. Math. Appl.*, 64(2012), 2100–2110.

[14] Z. Yan and X. Jia, Impulsive problems for fractional partial neutral functional integro-differential inclusions with infinite delay and analytic resolvent operators, *Mediterr. Math.,* 11(2014), 393-428.

[15] Z. Yan and F. Lu, On approximate controllability of fractional stochastic neutral integro-differential inclusions with infinite delay, *Applicable Analysis*, (2014), 1235-1258.

[16] L. Kexue, P. Jigen, G. Jinghuai, Controllability of nonlocal fractional differential systems of order $\alpha \in (1, 2]$ in Banach spaces, *Rep. Math. Phys.* , 71(2013), 33–43.

[17] R. Sakthivel, R. Ganesh, Y. Ren, S. Marshal Anthoni, Approximate controllability of nonlinear fractional dynamical systems, *Commun. Nonlinear Sci. Numer. Simulat.*, 18(2013), 3498–3508.

[18] V. Vijayakumar, C. Ravichandran and R. Murugesu, Approximate controllability for a class of fractional neutral integro-differential inclusions with state-dependent delay, *Nonlinear Studies*, 20(4)(2013), 513-532.

[19] Z. Yan, Approximate controllability of fractional neutral integro-differential inclusions with state-dependent delay in Hilbert spaces, *IMA Journal of Mathematical Control and Information*, 30(2013), 443–462.

[20] J. Dabas and A. Chauhan, Existence and uniqueness of mild solution for an impulsive fractional integro-differential equation with infinite delay, *Mathematical and Computer Modelling*, 57(3-4)(2013), 754-763.

[21] J. Hale and J. Kato, Phase space for retarded equations with infinite delay, *Funkcial. Ekvac.,* 21(1978), 11-41.

[22] Y. Hino, S. Murakami, and T. Naito, *Functional Differential Equations with Unbounded Delay,* Springer-Verlag, Berlin, 1991.

[23] X. Fu and R. Huang, Existence of solutions for neutral integro-differential equations with state-dependent delay, *Appl. Math. Comp.,* 224(2013), 743-759.