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APPLICATION OF CONDITIONAL GRADIENT METHOD TO A CORPORATE FINANCING PROBLEM

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Abstract: The determination of optimal financing sources is crucial in efficient and profitable corporate affairs. In this paper, we consider a model, first proposed by Davis and Elzinga [3], which is one of the well known and pioneering optimal control models in deterministic financial modeling. We show that the problem is nonconvex optimal control problem and did attempt to improve a result obtained in [11] by applying the conditional gradient algorithm starting from different initial controls.

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1. Introduction

Financial modeling has experienced dramatic improvements since the beginning of the 20^{th} century. Especially stochastic financial modeling was developed significantly. In 1970 Davis [4] stated a deterministic model for optimal corporate finance in form of a next non-linear optimal control problem:

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$$\max\left\{P(T)e^{-\rho T}+\int_0^T e^{-\rho t}[1-u_r(t)]rE(t)dt\right\} \tag{1}$$

$$\dot{P}(t)=c[\{1-u_r(t)\}\,rE(t)-\rho P(t)] \tag{2}$$

$$\dot{E}(t)=rE(t)\left[u_r(t)+u_s(t)\left\{1-\frac{E(t)}{(1-\delta)P(t)}\right\}\right] \tag{3}$$

$$u(t)\in U=\left\{u\in\mathbb{R}^2|u_r+u_s\le k/r<1,u_r\ge 0,u_s\ge 0\right\} \tag{4}$$

where $P(0)=P_0$, $E(0)=E_0$ and the terminal condition is given by planning horizon T[1].

Then in joint paper with Elzinga [3] Davis provided a possible analytical solution to his problem (1)-(4). In 2005 Chen and Sardar [11] suggested a computational method for solving Davis and Elzinga model and provided computer software SCOM[2]. However, regardless of the nonconvex structure[3] of the problem, they applied Pontryagin's maximum principle to the problem finding a local solution.

In this paper we suggest the conditional gradient method [1] with appropriate algorithm for finding local optimums of the original problem.

2. Conditional Gradient Method

Let us rewrite state variables as $x_1(t)=P(t),\ x_2(t)=E(t)$ and restate the model as follows:

$$\max\left\{x_1(T)e^{-\rho T}+\int_0^T e^{-\rho T}[1-u_r(t)]rx_2(t)dt\right\} \tag{5}$$

[1] $P(t)$ - market price of a share of stock, $E(t)$ - equity per share of outstanding common stock (net worth of utility divided by the number of shares outstanding), u_{r} - retention rate which describes the fraction of earnings retained for increasing the capital assets, u_{s} - stock financing rate concerning new money invested in the company, ρ - market capitalization rate (or investor discount rate), k - maximum investment rate, r - rate of return to equity (maximum return allowed by government), δ - discount on share price resulting from flotation cost, c - a positive constant denoting the responsiveness of the price to changes in earnings and dividends, T - planning horizon of the optimal financing program.

[2] In 2001 Craven and Islam first developed this package and used for solving optimal control problems.

[3] In (3) right side of the differential equation has a state variable in quadratic form ($E^2(t)$) which results in nonconvex form of the reachable set.

$$\dot{x_1}(t) = c[\{1 - u_r(t)\}\, rx_2(t) - \rho x_1(t)] \tag{6}$$

$$\dot{x_2}(t) = rx_2(t)\left[u_r(t) + u_s(t)\left\{1 - \frac{x_2(t)}{(1-\delta)x_1(t)}\right\}\right] \tag{7}$$

$$x_1(0) = x_1^0,\ x_2(0) = x_2^0 \tag{8}$$

$$u(t) \in U = \left\{u \in \mathbb{R}^2 | u_r + u_s \le k/r < 1, u_r \ge 0, u_s \ge 0\right\} \tag{9}$$

We introduce a new variable x_3 as follows:

$$\begin{aligned} &x_3(\tau) = \int_0^\tau e^{-\rho t}[1-u_r]rx_2(t)dt,\\ &\dot{x_3}(t) = e^{-\rho t}[1-u_r]rx_2(t),\\ &x_3(0) = 0. \end{aligned}$$

Then we can restate (5)-(9) in a terminal functional form:

$$\min\left\{\varphi(x(T)) = -x_1(T)e^{-\rho T} - x_3(T)\right\} \tag{10}$$

$$\dot{x_1}(t) = c[\{1 - u_r(t)\}\, rx_2(t) - \rho x_1(t)] \tag{11}$$

$$\dot{x_2}(t) = rx_2(t)\left[u_r(t) + u_s(t)\left\{1 - \frac{x_2(t)}{(1-\delta)x_1(t)}\right\}\right] \tag{12}$$

$$\dot{x_3}(t) = e^{-\rho t}[1-u_r]rx_2(t) \tag{13}$$

$$x_1(0) = x_1^0,\ x_2(0) = x_2^0 \tag{14}$$

$$u(t) \in U = \left\{u \in \mathbb{R}^2 | u_r + u_s \le k/r < 1, u_r \ge 0, u_s \ge 0\right\} \tag{15}$$

Problem (10)-(15) is nonconvex optimal control problem, and therefore applying Pontryagin's maximum principle cannot always guarantee a global solution. As control constraint set U is compact, we can use conditional gradient method in infinite dimensional space.

Let us construct Pontryagin's function as follows:

$$H(\Psi, x, u, t) = \Psi_1(t)c[(1-u_r)rx_2 - \rho x_1] + \Psi_2(t)rx_2\left[u_r + u_s\left\{1 - \frac{x_2}{(1-\delta)x_1}\right\}\right] +$$

$$+\Psi_3 e^{-\rho t}[1-u_r]rx_2 \tag{16}$$

where $\Psi_1,\ \Psi_2,\ \Psi_3$ are the solutions of the next system of adjoint differential equations:

$$\begin{cases} \dot{\Psi}_1(t) = -\frac{\partial H}{\partial x_1} = c\Psi_1\rho + \frac{x_2^2 r\Psi_2 u_s}{x_1^2(1-\delta)} \\ \dot{\Psi}_2(t) = -\frac{\partial H}{\partial x_2} = r(c\Psi_1 + \Psi_3 e^{-\rho t}) - r[\Psi_2 - c\Psi_1 - \Psi_3 e^{-\rho t}]u_r - \\ -r\Psi_2\left\{1 - \frac{x_2}{(1-\delta)x_1}\right\}u_s + \left\{\frac{\Psi_2 r x_2}{(1-\delta)x_1}\right\}u_s \\ \dot{\Psi}_3(t) = -\frac{\partial H}{\partial x_3} = 0 \\ \Psi_1(T) = -\frac{\partial\varphi}{\partial x_1} = e^{-\rho T} \\ \Psi_2(T) = 0 \\ \Psi_3(T) = 1 \end{cases} \tag{17}$$

The objective functional of (5)-(9) is $J(u) = \varphi(x(T))$. It's gradient can be calculated as follows:

$$\dot{J}(u) = -\frac{\partial H}{\partial u} = -\left\{\frac{\partial H}{\partial u_r},\ \frac{\partial H}{\partial u_s}\right\},$$

where,

$$\begin{cases} -\frac{\partial H}{\partial u_r} = \Psi_1 c r x_2 - \Psi_2 r x_2 + \Psi_3 e^{-\rho T} r x_2 \\ -\frac{\partial H}{\partial u_s} = -\left[\Psi_2 r x_2\left(1 - \frac{x_2}{(1-\delta)x_1}\right)\right] \end{cases} \tag{18}$$

Then (5)-(9) can be restated as follows:

$$\begin{aligned} &\min_{u\in U} J(u),\ U \subset H, \\ &J(u) \in C^1(H) \end{aligned} \tag{19}$$

where, U is weak compact, H - Hilbert space, $C^1(H)$ - space of functionals continuously differentiable on H.

3. Computational Algorithm

Algorithm 1 (Conditional gradient)

Step 1. Choose arbitrary initial control $u^0 = (u_r^0, u_s^0) \in U$, set $u^k,\ k := 0$.
Step 2. Solve (6)-(7) for $u = u^k$ and compute values of $x^k(t) = (x_1^k(t), x_2^k(t))$.
Step 3. Solve system (17) and compute values of $\Psi^k(t) = (\Psi_1^k(t), \Psi_2^k(t), \Psi_3^k(t))$.
Step 4. Compute functional's gradient $\dot{J}(u^k)$ using (18):

$$\dot{J}(u^k) = \left(-\frac{\partial H(\Psi^k, x^k, u^k, t)}{\partial u_r}, -\frac{\partial H(\Psi^k, x^k, u^k, t)}{\partial u_s}\right),$$

where $H(\Psi^k, x^k, u^k, t)$ is computed by (16).
Step 5. Solve next linear programming problem:

$$\min_{u \in U} \left[\langle \dot{J}(u^k), u \rangle = -\frac{\partial H}{\partial u_r} u_r - \frac{\partial H}{\partial u_s} u_s \right].$$

Let $\bar{u}^k = (\bar{u}_r^k, \bar{u}_s^k)$ be a solution of the above problem, i.e.:

$$\langle \dot{J}(u^k), \bar{u^k} \rangle = \min_{u \in U} \langle \dot{J}(u^k), u \rangle, \ t \in [0, T].$$

Step 6. If $\langle \dot{J}(u^k), \bar{u^k} - u^k \rangle = 0$ holds then u^k is optimal. If $\langle \dot{J}(u^k), \bar{u^k} - u^k \rangle \neq 0$ holds proceed to **Step** 7.
Step 7. Construct a segment $u^k(\alpha)$:

$$u^k(\alpha) = u^k + \alpha(\bar{u}^k - u^k), \ 0 \leq \alpha \leq 1.$$

Step 8. Choose α_k to hold next conditions:

$$u^k(\alpha_k): \ \min_{0 \leq \alpha \leq 1} J(u^k(\alpha)) = J(u^k(\alpha_k)).$$

Step 9. Construct a control u^{k+1}:

$$u^{k+1} := u^k(\alpha_k) = u^k + \alpha_k(\bar{u}^k - u^k).$$

Step 10. Set $k := k + 1$ and proceed to **Step** 2.

Next theorem provides the convergence of the above algorithm:

Theorem 1. *[20] Sequence $\{u^k\}$ generated by **Algorithm 1** according to differential maximum principle of Pontryagin converges to stationary point of problem (5)-(9) as follows:*

$$\lim_{k \to \infty} \langle \dot{J}(u^k), \bar{u}^k - u^k \rangle = 0$$

4. Numerical Experiments

As mentioned above Ping Cheng and Islam Sardar [11] provided their numerical solution. They used next values of system parameters $\rho = 0.1, \ k = 0.15, \ r = 0.2, \ \delta = 0.1, \ c = 1, \ P(0) = 0.5, \ E(0) = 1$ and achieved maximum value of objective functional $f^* = 2.04$ with terminal values of states $P(T) = x_1(T) =$

Table 1: $u_r(0) = 0,\ u_s(0) = 0$

t	$x_1(T)$	$x_2(T)$	f^*
10	1.363990024	0.005175294	**2.12754**
20	1.360725818	0.002854255	2.12284
30	1.359562271	0.002308791	2.12111
50	1.358626698	0.001938102	2.12111
70	1.358227107	0.001795339	2.12111
100	1.357928621	0.001694181	2.11866

Table 2: $u_r(0) = 0.375,\ u_s(0) = 0.375$

t	$x_1(T)$	$x_2(T)$	f^*
10	1.353424465	0.00353208	2.11288
20	1.355974687	0.00233097	2.11705
30	1.356695453	0.002011626	2.11796
50	1.357103472	0.001781852	**2.11826**
70	1.357210419	0.001689978	2.11823
100	1.357257493	0.001623573	2.11813

Table 3: $u_r(0) = 0.75,\ u_s(0) = 0$

t	$x_1(T)$	$x_2(T)$	f^*
10	1.353424465	0.00353208	2.11288
20	1.355974687	0.00233097	2.11705
30	1.356695453	0.002011626	2.11796
50	1.357103472	0.001781852	**2.11826**
70	1.357210419	0.001689978	2.11823
100	1.357257493	0.001623573	2.11813

1.99, $E(T) = x_2(T) = 1.96$. To find the best solution of the problem, as U is a triangular set, we chose 4 initial controls as follows: $u_a^0 = \{0, 0\}$, $u_b^0 = \{0.375, 0.375\}$, $u_c^0 = \{0.75, 0\}$, $u_d^0 = \{0, 0.75\}$.

From this we can see that the solution we found is $f^* = 2.12754 > 2.04$, which is better than the previous solutions proposed by Chen and Sardar.

Table 4: $u_r(0) = 0, \ u_s(0) = 0.75$

t	$x_1(T)$	$x_2(T)$	f^*
10	1.354151332	0.002330051	**2.11096**
20	1.352959975	0.001681294	2.10937
30	1.352536452	0.001508424	2.10878
50	1.352196923	0.001383136	2.1083
70	1.35205227	0.001332709	2.1081
100	1.35194438	0.001296109	2.10794

5. Conclusions

In this paper we considered Davis-Elzinga optimal financing problem. We showed that previous numerical solution [11] is not a global solution, and proposed the conditional gradient algorithm in order to improve the existing solution.

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