

**ON APPROXIMATE AND CLOSED-FORM SOLUTION
METHOD FOR INITIAL-VALUE WAVE-LIKE MODELS**

G.O. Akinlabi¹, S.O. Edeki² §

^{1,2}Department of Mathematics
Covenant University
Canaanland, Otta, NIGERIA

Abstract: This work presents a proposed Modified Differential Transform Method (MDTM) for obtaining both closed-form and approximate solutions of initial-value wave-like models with variable, and constant coefficients. Our results when compared with the exact solutions of the associated solved problems, show that the method is simple, effective and reliable. The results are very much in line with their exact forms. The method involves less computational work without neglecting accuracy. We recommend this simple proposed technique for solving both linear and nonlinear partial differential equations (PDEs) in other aspects of pure and applied sciences.

AMS Subject Classification: 35C05, 35C07, 74H10, 76D33

Key Words: closed form solution, modified DTM, wave-like equations

1. Introduction

Wave equation is a second order Partial Differential Equation (PDE) used in the description of waves. It is of immense application in applied Mathematics, Engineering and Physics. Wave equations can be linear, or nonlinear initial-boundary value problems. A variety of numerical, analytical and semi-analytical methods have been developed and proposed to obtain approximate, and accurate analytical solutions of various forms of differential equations in literature. Some of these methods include: Homotopy Perturbation Method

Received: February 3, 2016

Published: April 11, 2016

© 2016 Academic Publications, Ltd.

url: www.acadpubl.eu

§Correspondence author

(HPM), Homotopy Analysis Method (HAM), Adomian Decomposition Method (ADM), Variational Iteration Method (VIM), Differential Transform Method (DTM) and so on [1]-[8].

DTM is an iterative process that is based on the expansion of Taylor series. It was first proposed by Zhou in 1986 when he used it to solve linear and non-linear initial value problems in the analysis of electric circuit [9]. DTM in most cases, provides analytical approximation, and exact solutions in rapidly convergent sequence form. Despite these advantages, many researchers have improved and modified the DTM for better results and applications [10-18].

The MDTM is useful in obtaining exact and approximate solutions of linear and non-linear differential systems. It has been used by several authors to solve different systems easily and accurately.

The main idea of this work is to use the modified DTM to solve some wave-like PDEs by considering both cases of constant and variable coefficients.

2. Notion and Basic Theorems of the MDTM, see [15], [17], [18]

Let $m(x, t)$ be an analytic function at (x, t) in a domain D , then in considering the Taylor series of $m(x, t)$, regard is given to some variables $s^{ov} = t$ instead of all the variables as in the classical DTM. Thus, the MDTM of $m(x, t)$ with respect to t at t is defined and denoted by:

$$M(x, h) = \frac{1}{h!} \left[\frac{\partial^h m(x, t)}{\partial t^h} \right]_{t=t_*}. \quad (1a)$$

Thus, we have:

$$m(x, h) = \sum_{h=0} M(x, h) (t - t_*)^h. \quad (1b)$$

The equation (??) is called the modified differential inverse transform of $M(x, h)$ with respect to t .

2.1. Basic Theorems and Properties of the MDTM

Theorem 1. *If $m(x, t) = \alpha f(x, t) \pm \beta g(x, t)$, then*

$$M(x, h) = \alpha F(x, h) \pm \beta G(x, h).$$

Theorem 2. If $m(x, t) = \frac{\alpha \partial^n m_*(x, t)}{\partial t^n}$, then

$$M(x, h) = \frac{\alpha (h + n)!}{h!} M(x, h + n).$$

Theorem 3. If $m(x, t) = \frac{p(x) \partial^n m_*(x, t)}{\partial x^n}$, then

$$M(x, h) = \frac{p(x) \partial^n M(x, h)}{\partial x^n}.$$

Theorem 4. If $m(x, t) = p(x) m^2(x, t)$, then

$$M(x, h) = p(x) \sum_{r=0}^h M(x, r) M(x, h - r).$$

Theorem 5. If $m(x, t) = t^n$, then

$$M_k(x) = \delta(k - n) = \begin{cases} 1, & \text{if } k = n \\ 0, & \text{if } k \neq n \end{cases}$$

3. Illustrative and Numerical Examples

Here, we apply the proposed method to the following problems.

3.1. Cases 1 & 2 {Wave-Like Models with Variable, and Constant Coefficients}

Case Problem 1. Consider the wave-like model with variable coefficients:

$$\frac{\partial^2 u}{\partial t^2} = \frac{x^2}{2} \frac{\partial^2 u}{\partial x^2}, \tag{1}$$

subject to the initial conditions:

$$u(x, 0) = 1 \text{ and } u_t(x, 0) = x^2. \tag{2}$$

Solution procedure to Case Problem 1. Taking the modified differential transform (MDT) of both sides of (1), we get

$$\frac{(k + 2)!}{k!} U(x, k + 2) = \frac{x^2}{2} \frac{\partial^2 U(x, k)}{\partial x^2}, \quad k \geq 0. \tag{3}$$

Corresponding to (3) is the recurrence formula (4) with initial conditions in (5):

$$U(x, k + 2) = \frac{1}{(k + 1)(k + 2)} \frac{x^2}{2} \frac{\partial^2 U(x, k)}{\partial x^2}, \tag{4}$$

$$U(x, 0) = 1, \quad U(x, 1) = x^2, \quad k \geq 0. \tag{5}$$

Using (5) in (4) gives the following components:

$$\begin{aligned} U(x, 0) = 1, \quad U(x, 1) = x^2, \quad U(x, 2) = 0, \\ U(x, 3) = \frac{x^2}{3!}, \quad U(x, 4) = 0, \quad U(x, 5) = \frac{x^2}{5!}, \quad \dots \end{aligned} \tag{6}$$

In general, we have:

$$U(x, 2n) = 0, \quad U(x, 2n - 1) = \frac{x^2}{(2n - 1)!}, \quad n = 1, 2, 3, \dots \tag{7}$$

Substituting (6) and (7) into the solution series, we have:

$$\begin{aligned} u(x, t) &= \sum_{k=0} U(x, k) t^k \\ &= U(x, 0) + U(x, 1) t + U(x, 2) t^2 + U(x, 3) t^3 + U(x, 4) t^4 + U(x, 5) t^5 + \dots \\ &= 1 + x^2 \left\{ t + \frac{t^3}{3!} + \frac{t^5}{5!} + \frac{t^7}{7!} + \dots \right\} = 1 + x^2 \sum_{\eta=0} \frac{t^{2\eta+1}}{(2\eta + 1)!}. \end{aligned} \tag{8}$$

Equation (8) is the closed-form solution of *case problem 1*.

Case Problem 2. Consider the wave-like model with constant coefficients:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} - 3u, \tag{9}$$

subject to the initial conditions:

$$u(x, 0) = 0 \text{ and } u_t(x, 0) = 2 \sin x. \tag{10}$$

Solution Procedure to Case Problem 2. Taking the modified differential transform of both sides of (9), we get:

$$\frac{(k + 2)!}{k!} U(x, k + 2) = \frac{\partial^2 U(x, k)}{\partial x^2} - 3U(x, k), \quad k \geq 0. \tag{11}$$

Corresponding to (11) is the recurrence formula (12) with the initial conditions in (13):

$$U(x, k + 2) = \frac{1}{(k + 1)(k + 2)} \left[\frac{\partial^2 U(x, k)}{\partial x^2} - 3U(x, k) \right] \tag{12}$$

$$U(x, 0) = 0, \quad U(x, 1) = 2 \sin x, \quad k \geq 0 \tag{13}$$

Using (13) in (12) gives the following:

$$\begin{aligned} U(x, 0) &= 0, & U(x, 1) &= 2 \sin x, & U(x, 2) &= 0, \\ U(x, 3) &= \frac{-2^3 \sin x}{3!}, & U(x, 4) &= 0, & U(x, 5) &= \frac{2^5 \sin x}{5!}, \dots \end{aligned} \tag{14}$$

In general, we have:

$$U(x, 2n) = 0, \quad U(x, 2n + 1) = \frac{(-1)^n 2^{(2n+1)} \sin x}{(2n + 1)!}, \quad n = 0, 1, 2, 3, \dots \tag{15}$$

Substituting these into the solution series, we have:

$$\begin{aligned} u(x, t) &= \sum_{k=0} U_k(x) t^k \\ &= U(x, 0) + U(x, 1)t + U(x, 2)t^2 + U(x, 3)t^3 + U(x, 4)t^4 + U(x, 5)t^5 + \dots \\ &= 2t \sin x - \frac{(2t)^3 \sin x}{3!} + \frac{(2t)^5 \sin x}{5!} - \frac{(2t)^7 \sin x}{7!} + \dots \\ &= \sin x \left[2t - \frac{(2t)^3}{3!} + \frac{(2t)^5}{5!} - \frac{(2t)^7}{7!} + \dots \right] \\ &= \left(\sum_{j=0} (-1)^j \frac{(x)^{2j+1}}{(2j + 1)!} \right) \left(\sum_{\eta=0} (-1)^\eta \frac{(2t)^{2\eta+1}}{(2\eta + 1)!} \right). \end{aligned} \tag{16}$$

Equation (16) is the closed-form solution of *case problem 2*.

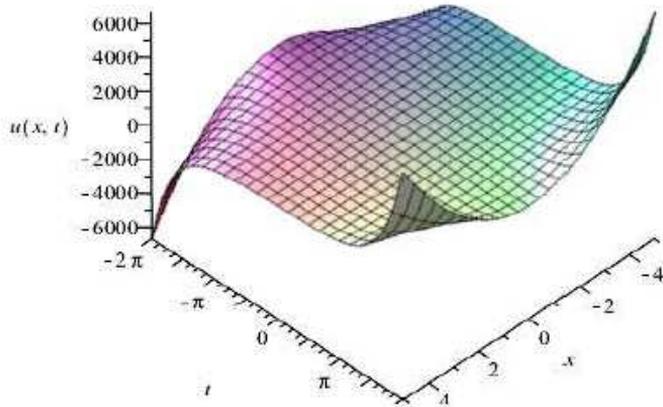


Figure 1: Exact solution of Case Problem 1.

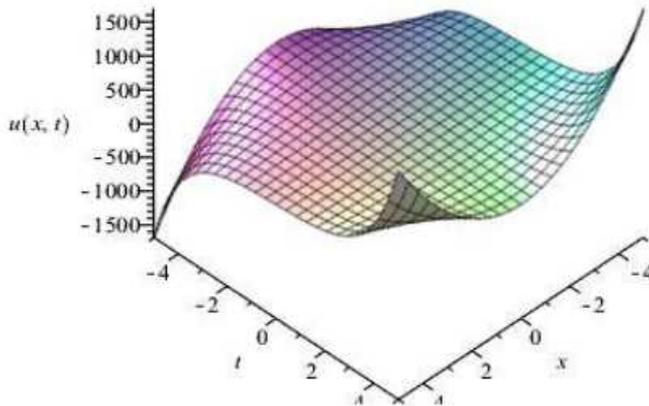


Figure 2: Approximate solution of Case Problem 1.

3.2. Discussion of Results

In this subsection, we will present graphs for the exact and the approximate solutions for discussion of results. The approximate solutions contain terms up to the power of seven (7).

Figure 1 and Figure 2 for exact and approximate solutions of *case problem 1* in that order.

4. Concluding Remarks

In this work, we solved initial-value wave-like models with variable, and constant coefficients for obtaining both closed-form and approximate solutions. For this, we used a proposed solution technique: a modified differential transform method (MDTM). Our results when compared with the exact solutions of the associated solved problems, showed that the method is simple, efficient, effective and reliable. The results are very much in line with their exact forms, even without neglecting accuracy. We therefore, recommend this solution technique for solving both linear and nonlinear partial differential equations (PDEs) in other aspects of pure and applied sciences.

Acknowledgments

The authors wish to sincerely thank Covenant University for financial support and provision of good working environment. They also wish to thank the anonymous referee(s)/reviewer(s) for their constructive and helpful remarks.

References

- [1] J. H. He, New interpretation of homotopy perturbation method, *Internat. J. Modern Phys. B* **20** (2006), 2561-2568.
- [2] S. Abbasbandy, Numerical method for non-linear wave and diffusion equations by the variational iteration method, *Int. J. Numer. Meth. Engng*, **73** (2008), 1836-1843.
- [3] Y. Keskin, A. Kurnaz, M.E. Kiris, & G. Oturanc, Approximate Solutions of Generalized Pantograph Equations by the Differential Transform Method, *International Journal of Nonlinear Sciences and Numerical Simulation*, **8**(2) (2007), 159-164.
- [4] J.H. He, Variational iteration method-a kind of non-linear analytical technique: Some examples *International Journal of Non-Linear Mechanics* **34**(4) (1999), 699-708.
- [5] S.O. Edeki1, G.O. Akinlabi and S.A. Adeosun, Analytic and Numerical Solutions of Time-Fractional Linear Schrödinger Equation, *Comm Math Appl.*, **7** (1) (2016), 1-10.
- [6] K. Tabatabaei, E. Celik and R. Tabatabaei, The differential transform method for solving heat-like and wave-like equations with variable coefficients, *Turk. J. Phys.*, **36** (2012), 87-98.
- [7] B. Jang, Two-point boundary value problems by the extended Adomian decomposition method, *J. Comput. Appl. Math.* **219** (1) (2008), 253-262.
- [8] S. O. Edeki, H.I. Okagbue, A. A. Opanuga, S.A. Adeosun A Semi-analytical Method for Solutions of a Certain Class of Ordinary Differential Equations *Applied Mathematics*, **5**, (2014), 2034-2041.

- [9] J.K. Zhou, Differential Transformation and Its Applications for Electrical Circuits. *Wuhahn Huarjung University Press*, China (in Chinese) (1986).
- [10] M. Alquran, K. Al-khaled, M. Ali, A. Ta'any, The Combined Laplace Transform-Differential Transform Method for Solving Linear Non-Homogeneous PDEs, *Journal of Mathematical and Computational Science*, **2**, 3 (2012), 690-701.
- [11] S.O. Edeki, E.A. Owoloko, A.S. Osheku, A.A. Opanuga, H.I. Okagbue, and G.O. Akinlabi, Numerical Solutions of Nonlinear Biochemical Model Using a Hybrid Numerical Analytical Technique, *International Journal Of Mathematical Analysis*, **9**, 8 (2015), 403-416.
- [12] A. Gokdogan, M. Merdan, and A. Yildirim, The Modified Algorithm for the Differential Transform Method to Solution of Genesio Systems, *Commun Nonlinear Sci Numer Simulat*, **17** (2012), 45-51.
- [13] S. Momani, V.S. Erturk, Solutions of Non-Linear Oscillators by the Modified Differential Transform Method, *Computers and Mathematics with Applications*, **55** (2008), 833-842.
- [14] M.M. Rashidi, The Modified Differential Transform Method for Solving MHD Boundary-Layer Equations, *Computer Physics Communications*, **180** (2009), 2210-2217.
- [15] B. Jang, Solving linear and nonlinear initial value problems by the projected differential transform method, *Computer Physics Communications* **181** (2010), 848-854.
- [16] S.O. Edeki, G.O. Akinlabi, S.A. Adeosun, On a modified transformation method for exact and approximate solutions of linear Schrödinger equations, *AIP Conference proceedings*, **1705**, 020048 (2016), doi: 10. 1063/1.4940296.
- [17] S.O. Edeki, O.O. Ugbebor, E.A. Owoloko, Analytical Solutions of the Black-Scholes Pricing Model for European Option Valuation via a Projected Differential Transformation Method *Entropy* (2015), **17** (11), 7510-7521.
- [18] Y. Keskin, G. Oturanc, Reduced Differential Transform Method For Solving Linear And Nonlinear Wave Equations, *Iranian Journal of Science & Technology, Transaction A*, **34** (A2) (2010), 114-122.