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DEVELOPMENT OF SIZE-BIASED MUKHERJEE-ISLAM DISTRIBUTION

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Abstract: In the present article, we derived a new Size-biased distribution from Mukherjee-Islam distribution. Flexibility of the model is derived by its moments, moment generating function (mgf) and characteristic function. Other important properties like Fisher's information matrix and Shannon's entropy are also derived which gives the measure of uncertainty in the system. Maximum likelihood estimator (MLE) and Baye's estimator of size-biased distribution are obtained and in the last, test of size biasedness is done.

Key Words: Mukherjee-Islam distribution, size-biased Mukherjee-Islam Distribution (SB-MID), Shannon's entropy, Fisher's information matrix, non-informative priors

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1. Introduction

This section gives the introduction of size-biasing,Mukherjee-Islam distribution and size-bised Mukherjee-Islam distribution.

1.1. Size-Biasing

Size-biasing is the method to introduce a weight on parent distribution in the form of variables. That is why size biased distributions sometimes known as weighted distributions also. Suppose X is a non-negative variable with its pdf $f(x, \theta)$, where θ is a natural parameter. $X^{\square}$ is the weighted version of X and its distribution related to that of X, is called weighted distribution. The distribution of $X^{\square}$ is weighted by the value, or size of X, that is why say that $X^{\square}$ has the X size-biased distribution. If a random x is selected for observation, which is an observation on $X^{\square}$, not on X.

Defnition 1. Suppose X is *discreate* distribution, then its pmf, after size biasing will be

$$dF^{\square} = \frac{xf(x, \theta)}{\theta}. \tag{1}$$

Defnition 2. Suppose X is *continuous* distribution, then its pdf after size biasing will be

$$f^{\square} = \frac{xf(x, \theta)}{\mu}, \quad x > 0. \tag{2}$$

Here μ is the first moment of $f(x, \theta)$:

$$\mu = \int_0^{\infty} xf(x, \theta)dx.$$

This form of distribution was first introduced by Fisher in [1]. Later used by Richard Arratia and Larry Goldstein in [3].

1.2. Mukherjee-Islam Distribution

The cdf of Mukherjee-Islam distribution (see [2]) of a random variable X is defined by

$$F(x) = \frac{x^p}{\theta^p} \tag{3}$$

$$=0\ otherwise, where\ p, \theta > 0, \theta \geq x > 0. \tag{4}$$

Here scale and shape parameters are θ and p, respectively.

The pdf in the equation above is

$$f(x) = \frac{px^{p-1}}{\theta^p}, \tag{5}$$

where $p, \theta > 0$, $\theta \geq x > 0$.

The function

$$R(x) = 1 - \frac{x^p}{\theta^p}, \tag{6}$$

where $p, \theta > 0$, $\theta \geq x > 0$ and the function

$$h(x) = \frac{px^{p-1}}{1 - \frac{x^p}{\theta^p}} \tag{7}$$

are the *Reliability and Hazzard rate functions.*

1.3. Size-Biased Mukherjee-Islam Distribution (SBMID)

SBMID can be obtained by applying weights x^c on Mukherjii-Islam distribution, where $c = 1$, since $\mu_1^{\prime} = E(x) = \int_0^{\infty} xf(x, \theta)$.

Now, for Mukherjee-Islam distribution, x ranges between 0 to θ, as it is clear from equations (5) and (6). Therefore, the first moment of Mukherjee-Islam distribution is

$$\mu_1^{\prime} = E(x) = \int_0^{\infty} xf(x, \theta),$$

$$\mu_1^{\prime} = \frac{\theta p}{p+1}. \tag{8}$$

From equation (1) and equation (6), we receive

$$f^{*}(x, \theta) = \frac{x \frac{px^{p-1}}{\theta^p}}{\frac{\theta p}{p+1}},$$

$$f^{*}(x, \theta) = \frac{(p+1)x^p}{\theta^{p+1}}, \tag{9}$$

or we can re-write

$$f^{*}(x, \theta) = f(x, \theta, p+1) = \frac{(p+1)x^p}{\theta^{p+1}}.$$

Now, it is easy to see

$$\int_0^\theta = \frac{(p+1)x^p}{\theta^{p+1}} = 1.$$

Hence $f(x, \theta, p+1)$ represents the pdf of SBMID:

$$f^{*}(x, \theta) = f(x, \theta, p+1) = \frac{(p+1)x^p}{\theta^{p+1}} \tag{10}$$

$$= 0, \tag{11}$$

otherwise $\theta > 0$, $p > 0$, $0 < x \leq \theta$.

2. Properties of Size-Biased Mukherjee-Islam Distribution (SBMID)

Here, we derived some characteristics like moments, moment generating function mgf and characteristic function of SBMID.

2.1. Mean, Variance and Moments

The r-th moments of SBMID, by equation (10), is

$$\mu'_r = E(x) = \int_0^\theta x^r f(x, \theta, p+1)dx$$

$$= \int_0^\theta \frac{(p+1)x^{p+r}}{\theta^{p+1}}dx,$$

$$\mu'_r = \frac{(p+1)\theta^r}{p+r+1}. \tag{12}$$

Remark. It is clear, see equation (12), that the sequence of moments of SBMID is the same as the sequence of moments of MI distribution, but shifted by "one".

The mean and the variance of the SBMID are given as

$$\text{mean} = \mu'_1 = \frac{\theta(p+1)}{p+2},$$

$$\text{variance} = \mu_2 = \frac{\theta^2(p+1)(1-2p)}{(p+2)^2(p+3)}.$$

2.2. Moment Generating Function (mgf)

The Moment generating function of SBMID is defined as:

$$M_x t = E(e^{tx}) = \int_0^\theta e^{tx} f(x, \theta, p+1)dx$$
$$= \int_0^\theta e^{tx} \frac{(p+1)x^p}{\theta^{p+1}} dx$$

$$M_x t = E(e^{tx}) = \frac{(p+1)\theta^p(-\theta t)^p((\lceil(p+1), -\theta t) - \lceil(p+1)}{\theta^{p+1} t}$$

$$M_x t = E(e^{tx}) = \frac{(p+1)(-\theta t)^p((\lceil(p+1), -\theta t) - \lceil p+1)}{\theta t} \tag{13}$$

where, $t < 0$, $p > -1$

2.3. Characteristic Function

The characteristic function of SBMID is given as:

$$\Phi_x t = E(e^{itx}) = \int_0^\theta e^{itx} f(x, \theta, p+1)dx$$
$$= \int_0^\theta e^{itx} \frac{(p+1)x^p}{\theta^{p+1}} dx$$

$$\Phi_x t = E(e^{itx}) = (p+1)(-i\theta t)^{-p}(\lceil(p+1) - (\lceil(p+1), -i\theta t) \tag{14}$$

where, $t < 0$, $p > -1$

3. Shannon's Entropy of Size-Biased Mukherjee-Islam Distribution (SBMID)

Shannon's entropy or the measure of uncertainty of a system has two definitions:

Definition 1: For a **discrete** random variable f , entropy is defined by,

$$H_q(f) = -\Sigma_{i=1}^n p(f = a) log(p(f = a)) \tag{15}$$

it is obvious that $H_q(f) \geq 0$.

Definition 2: Entropy of a **continuous** variable with *pdf* f can be defined as:

$$H(f) = -\int_0^{\infty} f(x) log f(x) = E(-log(x)) \tag{16}$$

above is true when integral exists: Shannon's entropy of size-biased Mukherjii-Islam distribution ($where\, 0 < x \leq \theta$) can be given as:

$$\begin{aligned} H[f(x,\theta,p+1] &= E[-log f(x,\theta,p+1)] \\ &= E[-log\frac{(p+1)x^p}{\theta^{p+1}}] \\ &= E[(p+1)\, log\theta - log(p+1) - p\, log(x) \end{aligned}$$

$$H[f(x,\theta,p+1] = (p+1)\, log\theta - log(p+1) - p\, E(log(x)) \tag{17}$$

$$\begin{aligned} E(log(x)) &= \int_0^{\theta} log(x)\; f(x,\theta,p+1)\; dx \\ &= \int_0^{\theta} log(x)\;\; \frac{(p+1)x^p}{\theta^{p+1}};\;\; 0\; < x \leq \theta,\;\; \theta > 0 \\ &= \frac{(p+1)}{\theta^{p+1}} \int_0^{\theta} log(x)\; x^p dx \\ &= \frac{(p+1)}{\theta^{p+1}}\;\; \frac{\theta^{p+1}\; [(p+1)\; log(\theta) - 1]}{(p+1)^2};\;\; p > -1 \end{aligned}$$

$$E(log(x)) = [\; log(\theta) - \frac{1}{p+1}] \tag{18}$$

Now by equ (18) and eqn (19) we have,

$$H[f(x,\theta,p+1)] = (p+1)\, log\theta - log(p+1) - p\,[\; log(\theta) - \frac{1}{p+1}\;]$$

$$H[f(x,\theta,p+1)] = log\frac{\theta}{p+1} + \frac{p}{p+1} \tag{19}$$

The above equation (19) depicts the Shannon's entropy of SBMID.

4. Fisher's Information Matrix of Size-Biased Mukherjii-Islam Distribution (SBMID)

For a random variable X, Fisher Information about the parameter θ is given by,

$$I(\theta) = E[\frac{\partial}{\partial\theta} log(f(x,\theta)]^2$$

If $logf(x,\theta)$ is twice differential wrt θ, Fisher's information is given by,

$$I(\theta) = E[\frac{\partial^2}{\partial^2\theta} log(f(x,\theta)]$$

Now pdf of SBMID is given by:

$$f(x,\theta,p+1) = \frac{(p+1)x^p}{\theta^{p+1}}$$

Here θ and p are scale and shape parameters respectively.

Taking log on both sides of above pdf, we have

$$logf(x,\theta,p+1) = p\, log(x) + log(p+1) - (p+1)\, log\theta \tag{20}$$

Diffrentiating eqn (15)partially w.r.t. θ and p, we get

$$\frac{\partial}{\partial\theta} f(x,\theta,p+1) = \frac{-(p+1)}{\theta} \tag{21}$$

$$\frac{\partial^2}{\partial\theta^2} f(x,\theta,p+1) = \frac{(p+1)}{\theta^2} \tag{22}$$

$$\frac{\partial^2}{\partial\theta\partial p} f(x,\theta,p+1) = \frac{-1}{\theta} \tag{23}$$

$$\frac{\partial}{\partial p} f(x,\theta,p+1) = log(x) + \frac{1}{p+1} - log(\theta) \tag{24}$$

$$\frac{\partial^2}{\partial p^2} f(x,\theta,p+1) = \frac{-1}{(p+1)^2} \tag{25}$$

$$\frac{\partial^2}{\partial p\partial\theta} f(x,\theta,p+1) = \frac{-1}{\theta} \tag{26}$$

Taking expectation of the equations(22), (23), (25)and(26) on both sides, we get

$$-E[\frac{\partial^2}{\partial\theta^2} f(x,\theta,p+1)] = \frac{-(p+1)}{\theta^2}; \quad I(1,1)$$

$$-E[\frac{\partial^2}{\partial\theta\partial p}f(x,\theta,p+1)] = \frac{1}{\theta}; \qquad I(2,1)$$
$$-E[\frac{\partial^2}{\partial p\partial\theta}f(x,\theta,p+1)] = \frac{1}{\theta}; \qquad I(1,2)$$
$$-E[\frac{\partial^2}{\partial p^2}f(x,\theta,p+1)] = \frac{1}{(p+1)^2}; \qquad I(2,2)$$

Now, Fisher's Information matrix of SBMID is given as;

$$I(\theta,p+1) = \begin{bmatrix} -E[\frac{\partial^2}{\partial\theta^2}f(x,\theta,p+1)] & -E[\frac{\partial^2}{\partial\theta\partial P}f(x,\theta,p+1)] \\ -E[\frac{\partial^2}{\partial p\partial\theta}f(x,\theta,p+1)] & -E[\frac{\partial^2}{\partial p^2}f(x,\theta,p+1)] \end{bmatrix}$$

$$I(\theta,p+1) = \begin{bmatrix} [\frac{-(p+1)}{\theta^2} & \frac{1}{\theta} \\ \frac{1}{\theta} & \frac{1}{(p+1)^2}] \end{bmatrix} \qquad (27)$$

Equation (27) gives the Fisher's information matrix of SBMID.

5. Estimation

Here Estimation of parameters of SBMID the maximum likelihood estimates(MLEs) and the Baye's estimator with Uniform prior.

5.1. Maximum Likelihood Estimators

Suppose $X_1, X_2, X_3, ...X_N$ is a sample of size n, then likelihood function of SBMID is given by:

$$L(x,\theta,p+1) = \prod_{i=1}^{n} f(x,\theta,p+1)$$
$$L(x,\theta,p+1) = \prod_{i=1}^{n} \frac{(p+1)x^p}{\theta^{p+1}}$$

$$L(x,\theta,p+1) = \frac{(p+1)^n}{\theta^{n(p+1)}}\prod_{i=1}^{n} x_i^p \qquad (28)$$

or,

$$logL^{\square}(x,\theta,p+1) = n\,xlog(p+1) + p\,\Sigma_{i=1}^{n} logx_i - n\,(p+1)\,log(\theta) \qquad (29)$$

Now diffrentiate $logL^{\star}(x, \theta, p+1)$ w.r.t.θ and p, we get

$$\frac{\partial L^{\star}}{\partial \theta} = -\frac{n(p+1)}{\theta} \tag{30}$$

$$\frac{\partial L^{\star}}{\partial p} = -\frac{n}{(p+1)} + \Sigma_{i=1}^{n} logx_i - n\, log\theta \tag{31}$$

Equating equation (30) and (31) to zero, we get MLEs of SBMI distribution.

$$\widehat{p} = -1 \tag{32}$$

$$\widehat{\theta} = e^{\frac{\Sigma_{i=1}^{n} x_i}{n}} \tag{33}$$

$$\widehat{\theta} = e^{\overline{x}}; where \overline{x} = \frac{\Sigma_{i=1}^{n} x_i}{n} \tag{34}$$

In equation (32) and (34), we get MLEs .

5.2. Bayesian Estimator of SBMID by Uniform Prior

Likelihood function of SBMID is:

$$L(x, \theta, p+1) = \frac{(p+1)^n}{\theta^{n(p+1)}} \prod_{i=1}^{n} x_i^p \tag{35}$$

since $0 < x \leq \theta$, we assume Uniform prior about θ, when p is known. Uniform prior $g(\theta) = 1$, and

$$\prod(\frac{\theta}{x}) = \frac{Lg(\theta)}{\int_0^{\theta} Lg(\theta) d\theta} \tag{36}$$

$$\prod(\frac{\theta}{x}) = \frac{\frac{(p+1)^n}{\theta^{n(p+1)}} \prod_{i=1}^{n} x_i^p}{\int_0^{\theta} \frac{(p+1)^n}{\theta^{n(p+1)}} \prod_{i=1}^{n} x_i^p d\theta} \tag{37}$$

is the posterior.

Now, Bayes's estimator of θ is:

$$\widehat{\theta} = \int_0^{\theta} \theta \prod(\frac{\theta}{x}) d\theta$$

$$\widehat{\theta} = \int_0^{\theta} \theta \frac{\frac{(p+1)^n}{\theta^{n(p+1)}} \prod_{i=1}^{n} x_i^p}{\int_0^{\theta} \frac{(p+1)^n}{\theta^{n(p+1)}} \prod_{i=1}^{n} x_i^p d\theta} d\theta$$

$$\widehat{\theta} = [1 - n(p+1)]\theta \tag{38}$$

6. Test for Size-Biasedness of Size-Biased Mukherjee-Islam Distribution (SBMID)

Let$X_1, X_2, X_3, ...X_n$ is random sample of size n drawn from size-bias Mukherjee-Islam distribution. We test the hypothesis, $H_0 = f(x, \theta, p)$ against $H_1 = f(x, \theta, p+1)$. To check whether sample of size n is drawn from the Mukherjee-Islam distribution or size-biased Mukherjee-Islam distribution, the following test is used.

$$\Delta = \frac{H_1}{H_0} = \prod_{i=1}^{n}[\frac{f(x, \theta, p+1}{f(x, \theta, p}]$$

$$\Delta = \frac{H_1}{H_0} = \prod_{i=1}^{n}[\frac{\frac{(p+1)x^p}{\theta^{p+1}}}{\frac{px^{p-1}}{\theta^p}}]$$

$$\Delta = \frac{H_1}{H_0} = [\frac{(p+1)}{p\theta}]^n \prod_{i=1}^{n} x_i$$

Null hypothesis will be rejected, if we have

$$[\frac{(p+1)}{p\theta}]^n \prod_{i=1}^{n} x_i > k$$

or equivalently, we reject the Null hypothesis where,

$$\Delta^{\square} = \prod_{i=1}^{n} x_i > k^{\square};$$

$$where, k^{\square} = k[\frac{(p+1)}{p\theta}]^n > 0$$

7. Conclusion

The aim of this paper is to introduce a new size-biased distribution that we called here SBMID, this distribution is based on Mukherjee-Islam original distribution.Thereafter we discussed it's structural properties, that includes mean, variance, moments of the distribution.The generating functions are also derived.Shannon's entropry and Fisher's information matrix has been calculated.we have also obtained estimates of parameters using classical and Baye's method.

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