

**MATHEMATICAL DEFINITION OF DURABILITY OF  
A MECHANICAL SYSTEM IN CASE OF  
EXTREME LOADING CONDITIONS**

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**Abstract:** The article is dedicated to the mathematical definition of durability of a mechanical system taking into account the appearance of extreme loads acting on it for definite time. Materials of research represent a continuation of several works of authors on the development of methods and techniques for determining reliability and durability to control safety by a mechanical system – load-bearing structure of the bridge of metallurgical crane based on interdisciplinary approach.

The emergence of “surges” when loading mechanical systems for the period of time is taken by independent rare events that are subject to the law of K. Pearson  $\chi^2$ -distribution.

The resulting relationship takes into account the probability of a single load excess of the permissible level (the emergence of “surges”), arising from system operating conditions and mode of loading, which can cause the emergence of such “surges”. Mathematical modeling and determination of durability of a mechanical system in case of extreme loading conditions allows making the right engineering decisions to control technogenic safety of certain productions or their major components.

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## 1. Introduction

Assessment of technogenic safety is associated with reliability and durability of various mechanical systems. The relevance of such assessment is obvious in view of providing efficiency, reliability and safety of such systems at all stages of the life cycle [1, 2, 3], [6, 7, 8] and [11, 12, 13, 14, 15, 16, 17, 18, 19, 20].

We assume the emergence of “surges” when loading mechanical systems for the period of time as independent rare events [4] and [5] that are subject to the law of K. Pearson  $\chi^2$ -distribution. This distribution (when  $\chi^2 \geq 0$ ) has probability density of:

$$p(\chi^2) = \frac{1}{\psi(n)} (\chi^2)^{n-1} e^{-\frac{(\chi^2)^2}{2}}, \quad (1)$$

where  $n$  is the number of degrees of freedom;  $\psi(n)$  is the normalization constant, expressed through a complete gamma function:

$$\psi(n) = \int_0^\infty (\chi^2)^{n-1} e^{-\frac{(\chi^2)^2}{2}} d(\chi^2) = 2^{\frac{n-2}{2}} \Gamma\left(\frac{n}{2}\right); \quad (2)$$

$\Gamma\left(\frac{n}{2}\right)$  – complete gamma function at  $\frac{n}{2} < 1$  and  $\frac{n}{2} > 2$ ,

$$\Gamma\left(\frac{n}{2}\right) = \left(\frac{n}{2} - 1\right) \Gamma\left(\frac{n}{2} - 1\right). \quad (3)$$

Distribution law is the sum of normally distributed values. The function of K. Pearson  $\chi^2$ -distribution  $F(\chi, n)$  is as follows:

$$F(\chi, n) = \frac{1}{\psi(n)} \int_0^\chi u^{n-1} e^{-\frac{u^2}{2}} d(u). \quad (4)$$

## 2. Methods

Mathematical modeling and determination of durability of a mechanical system – load-bearing structure of the main beam of metallurgical traveling crane is conducted based on the current state of the theory of probability and mathematical analysis.

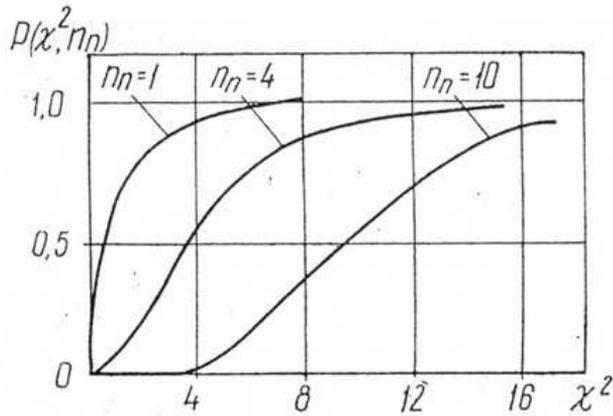


Figure 1: Integral function of  $K$ . Pearson  $\chi^2$ -distribution.

### 3. Results

Instead of  $F(\chi, n_n)$ , the expression of function  $P(\chi^2, n_n)$  is often used (Figure 1):

$$P(\chi^2, n) = 1 - F(\chi, n) = \frac{1}{\psi(n)} \int_{\chi} u^{n-1} e^{-\frac{u^2}{2}} du. \tag{5}$$

Function  $P(\chi^2, n_n)$  is reduced to the tables.

The average of values of  $\chi_n^2$  distribution and its dispersion are expressed in terms of parameter  $n$ :

$$M[\chi_n^2] = n \text{ and } D_{\chi^2} = 2n.$$

The use of this distribution for applied issues of the theory of predictive calculations of reliability of mechanical systems is based on the fact that it is the limiting distribution of the peak values of narrowband Gaussian random process [4].

At  $n = 2$ , value  $x = \sqrt{\chi_1^2 + \chi_2^2}$  is distributed under the following law:

$$p(x) = \frac{x}{\sigma_x^2} \exp\left(-\frac{x^2}{2\sigma_x^2}\right), \tag{6}$$

where  $x$  is distribution parameter.

This distribution is known in the theory of probabilities (along with  $\chi^2$ -distribution) as Rayleigh law, to which correspond the “surges” of a random

process exceeding the permissible level (distribution of maximums), and refers to the description of rare events.

Rayleigh cumulative distribution function at  $x > 0$  is as follows:

$$P(x) = 1 - \exp\left(-\frac{x^2}{2\sigma_x^2}\right), \tag{7}$$

and the expectation and dispersion are defined as follows [4]:

$$M_x = \left(\frac{\pi\sigma_x^2}{2}\right)^{\frac{1}{2}} \text{ and } D_x = 0,429\sigma_x^2.$$

When  $n = 3$ , K. Pearson  $\chi^2$ -distribution is converted into Maxwell distribution, and when  $n > 30$ , value  $\rho = (2\chi_n^2)^{\frac{1}{2}}$  is distributed approximately under normal distribution with average  $M(\rho) = (2n - 1)^{\frac{1}{2}}$  and dispersion  $D_\rho = 1$ .

Then, expression (4) from [9]:

$$P\left(\ln(\Delta\varepsilon) > \frac{\ln(\Delta\varepsilon)}{T}\right) \approx \frac{T\omega_0}{2\pi} P_0\left(\ln(\Delta\varepsilon) > \frac{\ln(\Delta\varepsilon)}{T_0}\right), \tag{8}$$

where  $\omega_0$ [omega 0] is the average frequency of the process;

$P\left(\ln(\Delta\varepsilon) > \frac{\ln(\Delta\varepsilon)^*}{T_0}\right)$  is the probability of a single excess of the level within one period  $T_0$  determined by the time of one cycle of system operation:

$$P\left(\ln(\Delta\varepsilon) > \frac{\ln(\Delta\varepsilon)}{T_0}\right) = \exp\left\{-\frac{[\ln(\Delta\varepsilon) - n \ln(\frac{\bar{\sigma}}{E})]^2}{2\sigma_{\ln(\Delta\varepsilon)}^2}\right\}. \tag{9}$$

Equation (8) is the probability of failure of a component of technological equipment by constitutive parameter – the limit value of residual deformation.

Let's express function  $P(\ln(\Delta\varepsilon) > \frac{\ln(\Delta\varepsilon)^*}{T})$  in equation (8) through error integral  $I_p$  of Pearson distribution. In this case:

$$T \approx \frac{2\pi I_p}{\omega_0 P_0\left(\ln(\Delta\varepsilon) > \frac{\ln(\Delta\varepsilon)^*}{T}\right)}, \tag{10}$$

where

$$I_p = \int_{\Delta \ln(\Delta\varepsilon)^0} \Delta \ln(\Delta\varepsilon)^{n-1} \exp\left[-\frac{\Delta \ln(\Delta\varepsilon)^2}{2}\right] d(\Delta \ln(\Delta\varepsilon)) = \\ = \psi(n)P[(\Delta \ln(\Delta\varepsilon)^0)^2, n]; \tag{11}$$

$P[(\Delta \ln(\Delta \varepsilon)^0)^2, n]$  is a function of Pearson  $\chi^2$ -distribution reduced to the tables;  $\psi(n)$  is the normalization distribution constant:  $\psi(n) = 2^{\frac{n-2}{2}} \Gamma(\frac{n}{2})$ ;  $\Delta \ln(\Delta \varepsilon)$  is the natural logarithm of residual (plastic) deformation. In dimensionless form from [9]:

$$Ln(\Delta \varepsilon) = nLn\left(\frac{\sigma}{\varepsilon}\right). \tag{12}$$

$\Delta \ln(\Delta \varepsilon)^0$  is residual deformation having no significant effect on load-bearing capacity of a mechanical system.  $n$  is the parameter of K. Pearson  $\chi^2$ -distribution; for a particular case, it is determined based on the relationship between the magnitude of the load at the “output” of the system perceived by a mechanical system and residual deformation.

Then, taking into account the expression for  $I_p$  and  $\psi(n)$ , equation (10) takes the following form:

$$T \approx \frac{2^{\frac{n}{2}} \pi \Gamma(\frac{n}{2}) P[(\Delta \ln(\Delta \varepsilon)^0)^2, n]}{\omega_0 P_0 \left( \ln(\Delta \varepsilon) > \frac{\ln(\Delta \varepsilon)^*}{T} \right)}, \tag{13}$$

where  $\omega_0$  is associated with the effective period  $T_{eff}$  of change in a random function  $\Delta \ln(\Delta \varepsilon) : \omega_0 = \frac{2\pi}{T_{eff}}$ .

The resulting relationship takes into account the probability of a single load excess of the permissible level (the emergence of “surges”), arising from system operating conditions and mode of loading, which can cause the emergence of such “surges”.

#### 4. Discussion

Knowing the value of a dangerous, catastrophic state of the load-bearing structure of metallurgical traveling crane, it is possible to determine the probability of failure of such system and control technogenic safety of part or the entire production as a whole.

#### 5. Conclusions

Thus, mathematical modeling and determination of durability of a mechanical system in case of extreme loading conditions allows making the right engineering decisions to control technogenic safety of certain productions or their major components.

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