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GREEN'S RELATIONS ON SEMIGROUPS OF REGRESSIVE TRANSFORMATIONS WITH RESTRICTED RANGE

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Abstract: Let X' be a subposet of a poset X. Define $P_{RE}(X, X')$ be the semigroup under composition of all regressive transformations from a subset of X into X'. Moreover,

$$T_{RE}(X, X') = \{\alpha \in P_{RE}(X, X') : \operatorname{dom} \alpha = X\}.$$

In 2012, C. Namnak and E. Laysirikul [3] investigated the Green's relations on $T_{RE}(X) = T_{RE}(X, X)$. Now, we aim to extend the result of them by study the Green's relations on the semigroups $T_{RE}(X, X')$ and $P_{RE}(X, X')$.

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1. Introduction

A partial transformation semigroup is the collection of functions from a subset of X into X with composition denoted by $P(X)$. In addition, the semigroups $T(X)$ and $I(X)$ are defined by

$$T(X) = \{\alpha \in P(X) : \operatorname{dom} \alpha = X\}$$

$$I(X) = \{\alpha \in P(X) : \alpha \text{ is injective}\}.$$

We can see that $T(X)$ and $I(X)$ are subsemigroups of $P(X)$. The semigroups

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$T(X)$ and $I(X)$ are called the full transformation semigroup and the symmetric inverse semigroup, respectively. It is well-known that $P(X)$ and $T(X)$ are regular and $I(X)$ is an inverse semigroup.

Let X be a poset. For $\alpha \in P(X)$, α is said to be regressive if

$$x\alpha \le x \text{ for all } x \in \operatorname{dom}\alpha.$$

A transformation semigroup on X is said to be regressive if all of its elements are regressive. Let

$$P_{RE}(X) = \{\alpha \in P(X) : \alpha \text{ is regressive}\},$$

$$T_{RE}(X) = \{\alpha \in T(X) : \alpha \text{ is regressive}\} \text{ and}$$
$$I_{RE}(X) = \{\alpha \in I(X) : \alpha \text{ is regressive}\}.$$

Then $P_{RE}(X)$, $T_{RE}(X)$ and $I_{RE}(X)$ are respectively subsemigroups of $P(X)$.

Some known results of regressive transformation semigroups are as follows. In [7], A. Umar gave a significant isomorphism theorem on full regressive transformation semigroups. For chains X and Y, $T_{RE}(X) \cong T_{RE}(Y)$ if and only if X and Y are order-isomorphic. In 2003, Y. Kemprasit [2] showed that in any regressive transformation semigroup on a poset, its idempotents and regular elements are identical. Moreover, she also characterized when $P_{RE}(X)$, $T_{RE}(X)$ and $I_{RE}(X)$ are regular semigroups and gave the necessary and sufficient condition for $P_{RE}(X)$, $T_{RE}(X)$ and $I_{RE}(X)$ to be eventually regular. In 2012, C. Namnak and E. Laysirikul [3] investigated the Green's relations on $T_{RE}(X)$.

Now, we consider subsemigroups of $P_{RE}(X)$, $T_{RE}(X)$ and $I_{RE}(X)$ as follows. Let X' be a subposet of X, define

$$P_{RE}(X, X') = \{\alpha \in P_{RE}(X) : X\alpha \subseteq X'\},$$

$$T_{RE}(X, X') = \{\alpha \in T_{RE}(X) : X\alpha \subseteq X'\} \text{ and}$$
$$I_{RE}(X, X') = \{\alpha \in I_{RE}(X) : X\alpha \subseteq X'\}.$$

It is clear that if $X' = X$, then $P_{RE}(X, X') = P_{RE}(X)$, $T_{RE}(X, X') = T_{RE}(X)$ and $I_{RE}(X, X') = I_{RE}(X)$ which implies that $P_{RE}(X, X')$, $T_{RE}(X, X')$ and $I_{RE}(X, X')$ are the generalization of $P_{RE}(X)$, $T_{RE}(X)$ and $I_{RE}(X)$, respectively.

In 2004, P. Udomkavanich and P. Jitjankarn [1] gave necessary and sufficient conditions for the semigroups $P_{RE}(X, X')$, $T_{RE}(X, X')$ and $I_{RE}(X, X')$ to be regular. In [5], they prove that for chains X and Y and subchains X' of X and Y' of Y, if $T_{RE}(X, X') \cong T_{RE}(Y, Y')$, then X' and Y' are order-isomorphic. In

[4], they also prove that if $P_{RE}(X, X') \cong P_{RE}(Y, Y')$ $[I_{RE}(X, X') \cong I_{RE}(Y, Y')]$, then X' and Y' are order-isomorphic. Moreover, in [6], they gave necessary and sufficient conditions for $T_{RE}(X, X')$ and $T_{RE}(Y, Y')$ to be isomorphic.

In this paper, we aim to extend the result of C. Namnak and E. Laysirikul [3] by study the Green's relations on the semigroups $T_{RE}(X, X')$ and $P_{RE}(X, X')$.

Note that in order for $T_{RE}(X, X')$ to be nonempty, each element of X must lie above some elements in X'. That is,

$$\forall x \in X \quad \exists x' \in X', \quad x' \leq x. \tag{1}$$

Let us remark here that if $\min X$ exists, then X', which satisfies (1), must contain $\min X$. From now on, we assume that the subposet X' of X satisfies the property (1) and for each $x \in X$ the element x' means the element in X' such that $x' \leq x$ which exists by the property (1).

For $\alpha \in P(X)$, we denote $\pi(\alpha)$ the composition of $\operatorname{dom} \alpha$ induced by α, namely

$$\pi(\alpha) = \{y\alpha^{-1} : y \in X\alpha\}.$$

and define $\alpha_* : \pi(\alpha) \to X\alpha$ by

$$P\alpha_* = x\alpha \text{ for each } P \in \pi(\alpha) \text{ and } x \in P.$$

Then $\pi(\alpha)$ is a partition of $\operatorname{dom} \alpha$ and α_* is a bijection from $\pi(\alpha)$ onto $X\alpha$.

More generally, we extend the above definition to the partial transformation semigroup with restricted range. Let X' be a subset of X. Define

$$PT(X, X') = \{\alpha \in P(X) : X\alpha \subseteq X'\}.$$

For each $\alpha \in PT(X, X')$, we define

$$\tilde{\pi}(\alpha) = \{M \cap X' : M \in \pi(\alpha) \text{ and } M \cap X' \neq \emptyset\}.$$

Moreover, we define $\tilde{\alpha} : \tilde{\pi}(\alpha) \to X\alpha$ by

$$P\tilde{\alpha} = x\alpha \text{ for each } P \in \tilde{\pi}(\alpha) \text{ and each } x \in P.$$

Let $\mathcal{A}$ and $\mathcal{B}$ be collections of subsets of X. We say that $\mathcal{B}$ is a *refinement* of $\mathcal{A}$ or $\mathcal{B}$ *refines* $\mathcal{A}$ if $\cup\mathcal{A} = \cup\mathcal{B}$ and for every $B \in \mathcal{B}$, there exists some $A \in \mathcal{A}$ such that $B \subseteq A$.

2. Green's Relations

In this section, we characterize Green's relations on $T_{RE}(X, X')$ and $P_{RE}(X, X')$.

Theorem 1. *Let $\alpha, \beta \in T_{RE}(X, X')$. Then the following statements are equivalent.*

1. $\beta \in \alpha T_{RE}(X, X')$.
2. *For every $P \in \pi(\alpha)$, there exists $Q \in \pi(\beta)$ such that $P \subseteq Q$ and $Q\beta_* \leq P\alpha_*$.*

Proof. Suppose that $\beta \in \alpha T_{RE}(X, X')$. Then there is $\gamma \in T_{RE}(X, X')$ such that $\beta = \alpha\gamma$. Let $P \in \pi(\alpha)$. Then $P\alpha_* = y$ for some $y \in X'$ which implies that $y\gamma \in X\alpha\gamma = X\beta$. Define $Q = y\gamma\beta^{-1} \in \pi(\beta)$. Let $p \in P$. Then $p\beta = p\alpha\gamma = y\gamma$ from which it follows that $p \in y\gamma\beta^{-1} = Q$. So $P \subseteq Q$. We can see that $p\beta = Q\beta_*$. Then $Q\beta_* = p\beta = p\alpha\gamma \leq p\alpha = P\alpha_*$

Conversely, Assume that (2) holds. For each $x \in X\alpha$, there exists unique $P_x \in \pi(\alpha)$ such that $P_x\alpha_* = x$. By (2), there is $Q_x \in \pi(\beta)$ such that $P_x \subseteq Q_x$ and $Q_x\beta_* \leq P_x\alpha_*$. Define $\gamma : X \to X'$ by

$$x\gamma = \begin{cases} Q_x\beta_* & \text{if} \quad x \in X\alpha; \\ x' & \text{otherwise.} \end{cases}$$

We can see that $x\gamma = Q_x\beta_* \leq P_x\alpha_* = x$ if $x \in X\alpha$ and $x\gamma = x' \leq x$, otherwise. Then $\gamma \in T_{RE}(X, X')$. Let $x \in X$. We obtain $x\alpha\gamma = Q_{x\alpha}\beta_*$. Since $x \in P_{x\alpha} \subseteq Q_{x\alpha}$, we get $x\beta = Q_{x\alpha}\beta_* = x\alpha\gamma$. Therefore, $\beta = \alpha\gamma$. □

Theorem 2. *Let $\alpha, \beta \in P_{RE}(X, X')$. Then the following statements are equivalent.*

1. $\beta \in \alpha P_{RE}(X, X')$.
2. *For every $P \in \pi(\alpha)$, $P \cap \operatorname{dom}\beta = \emptyset$ or there exists $Q \in \pi(\beta)$ such that $P \subseteq Q$ and $Q\beta_* \leq P\alpha_*$.*

Proof. Suppose that $\beta \in \alpha P_{RE}(X, X')$. Then there is $\gamma \in P_{RE}(X, X')$ such that $\beta = \alpha\gamma$. Let $P \in \pi(\alpha)$. Then $P\alpha_* = y$ for some $y \in X'$. If $y \notin \operatorname{dom}\gamma$, then $\emptyset = P \cap \operatorname{dom}\alpha\gamma = P \cap \operatorname{dom}\beta$. If $y \in \operatorname{dom}\gamma$, then $y\gamma \in X\alpha\gamma = X\beta$. Define $Q = y\gamma\beta^{-1} \in \pi(\beta)$. Let $p \in P$. Then $p\beta = p\alpha\gamma = y\gamma$ from which it follows that $p \in y\gamma\beta^{-1} = Q$. So $P \subseteq Q$. We can see that $p\beta = Q\beta_*$. Then $Q\beta_* = p\beta = p\alpha\gamma \leq p\alpha = P\alpha_*$

Conversely, Assume that (2) holds. For each $x \in X\alpha$, there exists unique $P_x \in \pi(\alpha)$ such that $P_x\alpha_* = x$. Let $A = \{x \in X\alpha : P_x \cap \operatorname{dom}\beta \neq \emptyset\}$. By (2), for each $x \in A$, there is $Q_x \in \pi(\beta)$ such that $P_x \subseteq Q_x$ and $Q_x\beta_* \leq P_x\alpha_*$. Define $\gamma : A \to X'$ by $x\gamma = Q_x\beta_*$. We can see that $x\gamma = Q_x\beta_* \leq P_x\alpha_* = x$. Then $\gamma \in P_{RE}(X, X')$. Let $x \in X$. We obtain $x\alpha\gamma = Q_{x\alpha}\beta_*$. Since $x \in P_{x\alpha} \subseteq Q_{x\alpha}$, we get $x\beta = Q_{x\alpha}\beta_* = x\alpha\gamma$. Therefore, $\beta = \alpha\gamma$. □

Theorem 3. *Let $\alpha, \beta \in T_{RE}(X, X')$. Then $\alpha\mathcal{R}\beta$ if and only if $\alpha = \beta$.*

Proof. Assume that $\alpha\mathcal{R}\beta$. Then $\beta \in \alpha T_{RE}(X, X')$ and $\alpha \in \beta T_{RE}(X, X')$. Let $x \in X$. Then $x \in P$ for some $P \in \pi(\alpha)$. By Theorem 1, there is $Q \in \pi(\beta)$ such that $P \subseteq Q$ and $Q\beta_* \leq P\alpha_*$. Hence $x \in Q$. We can see that $x\beta = Q\beta_* \leq P\alpha_* = x\alpha$. Similarly, we can show that $x\alpha \leq x\beta$. Therefore, $x\alpha = x\beta$. The converse is clear. □

Theorem 4. *Let $\alpha, \beta \in P_{RE}(X, X')$. Then $\alpha\mathcal{R}\beta$ if and only if $\alpha = \beta$.*

Proof. Assume that $\alpha\mathcal{R}\beta$. Then $\beta \in \alpha P_{RE}(X, X')$ and $\alpha \in \beta P_{RE}(X, X')$. It is easy to see that $\operatorname{dom}\alpha = \operatorname{dom}\beta$. Let $x \in \operatorname{dom}\alpha$. Then $x \in P$ for some $P \in \pi(\alpha)$ and $P \cap \operatorname{dom}\beta \neq \emptyset$. By Theorem 2, there is $Q \in \pi(\beta)$ such that $P \subseteq Q$ and $Q\beta_* \leq P\alpha_*$. Hence $x \in Q$. We can see that $x\beta = Q\beta_* \leq P\alpha_* = x\alpha$. Similarly, we can show that $x\alpha \leq x\beta$. Therefore, $x\alpha = x\beta$. The converse is clear. □

Theorem 5. *Let $\alpha, \beta \in T_{RE}(X, X')$. Then the following statements are equivalent.*

1. *$\alpha \in T_{RE}(X, X')\beta$.*
2. *For every $P \in \pi(\alpha)$, there exists $Q \in \pi(\beta)$ such that $Q \cap X' \neq \emptyset$ and $P\alpha_* = Q\beta_*$ and for every $x \in P$, $d_x \leq x$ for some $d_x \in Q \cap X'$.*

Proof. Suppose that $\alpha = T_{RE}(X, X')\beta$. Then $\alpha = \gamma\beta$ for some $\gamma \in T_{RE}(X, X')$. Let $P \in \pi(\alpha)$. Then $P\alpha_* = z$ for some $z \in X\alpha = X\gamma\beta \subseteq X\beta$. We obtain $z = x\gamma\beta$ for some $x \in X$. Let $Q = z\beta^{-1}$. We obtain $x\gamma \in z\beta^{-1} \cap X\gamma = Q \cap X\gamma \neq \emptyset$ which implies that $Q \cap X' \neq \emptyset$. Then $P\alpha_* = z = Q\beta_*$. Let $x \in P$. Then $x\gamma\beta = x\alpha = z$ which implies that $x\gamma \in Q \cap X'$ and $x\gamma \leq x$.

Conversely, suppose that (2) holds. For each $x \in \operatorname{dom}\alpha$, there exists unique $P_x \in \pi(\alpha)$ such that $x \in P_x$. Then there exist $Q_x \in \pi(\beta)$ and $d_x \in Q_x \cap X'$ such that $P_x\alpha_* = Q_x\beta_*$ and $d_x \leq x$. Define $\gamma : \operatorname{dom}\alpha \to X'$ by $x \mapsto d_x$. We obtain $\gamma \in T_{RE}(X, X')$ and $x\gamma\beta = d_x\beta = Q_x\beta_* = P_x\alpha_* = x\alpha$. Therefore, $\alpha \in T_{RE}(X, X')\beta$. □

By the same proof as above, we have the following theorem immediately.

Theorem 6. *Let $\alpha, \beta \in P_{RE}(X, X')$. Then the following statements are equivalent.*

1. *$\alpha \in P_{RE}(X, X')\beta$.*
2. *For every $P \in \pi(\alpha)$, there exists $Q \in \pi(\beta)$ such that $Q \cap X' \neq \emptyset$ and $P\alpha_* = Q\beta_*$ and for every $x \in P$, $d_x \leq x$ for some $d_x \in Q \cap X'$.*

Theorem 7. *Let $\alpha, \beta \in T_{RE}(X, X')$. Then the following statements are equivalent.*

1. *$\alpha \mathcal{L} \beta$.*
2. *$X\beta \subseteq X'\alpha$ and for every $P \in \pi(\alpha)$ such that $P \cap X' \neq \emptyset$, there exists $Q \in \pi(\beta)$ such that $Q \cap X' \neq \emptyset$ and $P\alpha_* = Q\beta_*$ and for every $p \in P$ and $q \in Q$, there exist $a \in P \cap X'$ and $b \in Q \cap X'$ such that $b \leq p$ and $a \leq q$.*

Proof. Assume that (1) holds. Then $\alpha = \gamma\beta$ and $\beta = \delta\alpha$ for some $\gamma, \delta \in T_{RE}(X, X')$. We obtain $X\beta = X\delta\alpha \subseteq X'\alpha$. Let $P \in \pi(\alpha)$. Since $\alpha = \gamma\beta \in T_{RE}(X, X')\beta$, by Theorem 5, there is $Q \in \pi(\beta)$ such that $Q \cap X' \neq \emptyset$ and $P\alpha_* = Q\beta_*$ and for each $p \in P$, $b \leq p$ for some $b \in Q \cap X'$. Similarly, we obtain there is $P' \in \pi(\alpha)$ such that $P' \cap X' \neq \emptyset$ and $P'\alpha_* = Q\beta_*$ and for each $q \in Q$, $a \leq q$ for some $a \in P' \cap X'$. Since $P\alpha_* = Q\beta_* = P'\alpha_*$, we get $P = P'$ which implies that (2) holds.

Conversely, suppose that (2) holds. Then we obtain $\alpha \in T_{RE}(X, X')\beta$ by Theorem 5. To show that $\beta \in T_{RE}(X, X')\alpha$, let $Q \in \pi(\beta)$. Then $Q\beta_* \in X\beta \subseteq X'\alpha$ from which it follows that $Q\beta_* = P\alpha_*$ for some $P \in \pi(\alpha)$. By the assumption, there exists $Q' \in \pi(\beta)$ such that $Q' \cap X' \neq \emptyset$ and $P\alpha_* = Q'\beta_*$ and for every $q \in Q'$, $a \leq q$ for some $a \in P \cap X'$. We can see that $Q\beta_* = P\alpha_* = Q'\beta_*$ which implies that $Q = Q'$. Therefore, $\beta \in T_{RE}(X, X')\alpha$ by Theorem 5. □

By using the same proof as Theorem 7, we obtain the result for $P_{RE}(X, X')$ as follows.

Theorem 8. *Let $\alpha, \beta \in P_{RE}(X, X')$. Then the following statements are equivalent.*

1. *$\alpha \mathcal{L} \beta$.*
2. *$X\beta \subseteq X'\alpha$ and for every $P \in \pi(\alpha)$ such that $P \cap X' \neq \emptyset$, there exists $Q \in \pi(\beta)$ such that $Q \cap X' \neq \emptyset$ and $P\alpha_* = Q\beta_*$ and for every $p \in P$ and $q \in Q$, there exist $a \in P \cap X'$ and $b \in Q \cap X'$ such that $b \leq p$ and $a \leq q$.*

Since $\mathcal{D} = \mathcal{L} \circ \mathcal{R}$ and $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$, we obtain the following two theorems.

Corollary 9. *Let $\alpha, \beta \in T_{RE}(X, X')$. Then the following statements hold.*

1. *$\alpha\mathcal{D}\beta$ if and only if $\alpha\mathcal{L}\beta$.*
2. *$\alpha\mathcal{H}\beta$ if and only if $\alpha = \beta$.*

Corollary 10. *Let $\alpha, \beta \in P_{RE}(X, X')$. Then the following statements hold.*

1. *$\alpha\mathcal{D}\beta$ if and only if $\alpha\mathcal{L}\beta$.*
2. *$\alpha\mathcal{H}\beta$ if and only if $\alpha = \beta$.*

In the last part, we will characterize $\mathcal{J}$-relation on $T_{RE}(X, X')$ and $P_{RE}(X, X')$. Firstly, we state the following lemma which appears in [3].

Lemma 11. *[3] If $\alpha, \beta, \delta, \gamma \in T(X)$ are such that $\alpha = \delta\beta\gamma$, then*

$$\mathcal{A} = \{\cup\mathcal{A}_Q : Q \in \pi(\beta) \text{ and } Q \cap X\delta \neq \emptyset\}$$

is a refinement of $\pi(\alpha)$ where $\mathcal{A}_Q = \{P \in \pi(\delta) : P\delta_* \in Q\}$.

Now, we extend the above lemma to the partial transformation semigroup $P(X)$.

Lemma 12. *If $\alpha, \beta, \delta, \gamma \in P(X)$ are such that $\alpha = \delta\beta\gamma$, then*

$$\mathcal{A} = \{\cup\mathcal{A}_Q : Q \in \pi(\beta), \cup\mathcal{A}_Q \neq \emptyset \text{ and } Q \cap X\delta \neq \emptyset\}$$

is a refinement of $\pi(\alpha)$ where $\mathcal{A}_Q = \{P \cap \operatorname{dom}\alpha : P \in \pi(\delta) \text{ and } P\delta_* \in Q\}$.

Proof. Let $\alpha, \beta, \delta, \gamma \in P(X)$ be such that $\alpha = \delta\beta\gamma$. We first show that $\cup\mathcal{A} = \cup\pi(\alpha) = \operatorname{dom}\alpha$. Let $x \in \operatorname{dom}\alpha \subseteq \operatorname{dom}\gamma$. Then $x \in P$ for some $P \in \pi(\delta)$. Since $x\delta\beta \in X\beta$, we obtain $x\delta\beta = Q\beta_*$ for some $Q \in \pi(\beta)$. Then $P\delta_* = x\delta \in Q$ which implies that $Q \cap X\delta \neq \emptyset$. Hence $P \cap \operatorname{dom}\alpha \in \mathcal{A}_Q$ and $x \in P \cap \operatorname{dom}\alpha \subseteq \cup\mathcal{A}_Q \subseteq \cup\mathcal{A}$. Thus $\operatorname{dom}\alpha \subseteq \cup\mathcal{A}$. By the definition of $\mathcal{A}$, it is easy to see that $\cup\mathcal{A} \subseteq \operatorname{dom}\alpha$. Let $Q \in \pi(\beta)$ be such that $\cup\mathcal{A}_Q \neq \emptyset$ and $Q \cap X\delta \neq \emptyset$. We will prove that there is $\tilde{P} \in \pi(\alpha)$ such that $\cup\mathcal{A}_Q \subseteq \tilde{P}$. Let $x \in Q \cap X\delta$. Then there exists $y \in X$ such that $y\delta = x$. We obtain $y \in P$ for some $P \in \pi(\delta)$ and $P\delta_* = y\delta$. Since $\cup\mathcal{A}_Q$ is nonempty, there is $z \in \cup\mathcal{A}_Q$. Then $z \in P' \cap \operatorname{dom}\alpha$ for some $P' \in \pi(\delta)$ and $P'\delta_* \in Q$. We can see that

$$z\delta\beta = P'\delta_*\beta = Q\beta_* = x\beta = y\delta\beta.$$

Then

$$z\alpha = z\delta\beta\gamma = y\delta\beta\gamma = y\alpha.$$

Hence $y \in \operatorname{dom}\alpha$ which implies that $y \in \tilde{P}$ for some $\tilde{P} \in \pi(\alpha)$. Let $a \in \cup\mathcal{A}_Q$. Then $a \in \bar{P} \cap \operatorname{dom}\alpha$ for some $\bar{P} \in \pi(\delta)$ and $\bar{P}\delta_* \in Q$. We obtain

$$a\alpha = a\delta\beta\gamma = \bar{P}\delta_*\beta\gamma = Q\beta_*\gamma = y\delta\beta\gamma = y\alpha = \tilde{P}\alpha_*.$$

Thus $a \in \tilde{P}$ from which it follows that $\cup\mathcal{A}_Q \subseteq \tilde{P}$. Therefore, $\mathcal{A}$ refines $\pi(\alpha)$. □

Theorem 13. *Let $\alpha, \beta \in T_{RE}(X, X')$. Then the following statements are equivalent.*

1. *$\alpha \in T_{RE}(X, X')\beta T_{RE}(X, X')$.*
2. *There exist a refinement $\mathcal{B}$ of $\pi(\alpha)$ and an injection $\varphi : \mathcal{B} \to \tilde{\pi}(\beta)$ such that for every $P \in \mathcal{B}$, $\tilde{P}\alpha_* \leq P\varphi\tilde{\beta}$ where $P \subseteq \tilde{P}$ and $\tilde{P} \in \pi(\alpha)$ and for every $x \in P$, $y \leq x$ for some $y \in P\varphi$.*

Proof. Suppose that (1) holds. Then $\alpha = \delta\beta\gamma$ for some $\delta, \gamma \in T_{RE}(X, X')$. We obtain there is a refinement $\mathcal{A}$ of $\pi(\alpha)$ as defined in Lemma 11. Define $\varphi : \mathcal{A} \to \tilde{\pi}(\beta)$ by $(\cup\mathcal{A}_Q)\varphi = Q \cap X'$. Since $\mathcal{A}$ is a partition of $\operatorname{dom}\alpha$, we get φ is well-defined. Suppose that $(\cup\mathcal{A}_Q)\varphi = (\cup\mathcal{A}_{Q'})\varphi$. Then $Q \cap X' = Q' \cap X'$ which implies that $Q = Q'$ and $\cup\mathcal{A}_Q = \cup\mathcal{A}_{Q'}$. Thus φ is an injection.

Let $\cup\mathcal{A}_Q \in \mathcal{A}$. Then $Q \in \pi(\beta)$ and $Q \cap X\delta \neq \emptyset$. Let $x \in \cup\mathcal{A}_Q$. Then $x \in P$ for some $P \in \pi(\delta)$ such that $P\delta_* \in Q$. Hence $x\delta \in Q \cap X' = (\cup\mathcal{A}_Q)\varphi$. Since $\mathcal{A}$ is a refinement of $\pi(\alpha)$, we get $x \in \cup\mathcal{A}_Q \subseteq \tilde{P}$ for some $\tilde{P} \in \pi(\alpha)$. We can see that

$$\tilde{P}\alpha_* = x\alpha = x\delta\beta\gamma = Q\beta_*\gamma \leq Q\beta_* = (\cup\mathcal{A}_Q)\varphi\tilde{\beta}.$$

Since δ is regressive, $x\delta \leq x$. In addition, $x\delta \in (\cup\mathcal{A}_Q)\varphi$. Therefore, (2) holds.

Conversely, suppose that (2) holds. For each $x \in \operatorname{dom}\alpha$, there is $P_x \in \mathcal{B}$ such that $x \in P_x$. Choose $y_x \in P_x\varphi$ such that $y_x \leq x$ and define $\delta : \operatorname{dom}\alpha \to X'$ by $x\delta = y_x$. Then $\delta \in T_{RE}(X, X')$. We can see that $\varphi\tilde{\beta} : \mathcal{B} \to X\beta$ is injective since $\tilde{\beta}$ and φ are injective. For each $x \in \mathcal{B}\varphi\tilde{\beta}$, there exists unique $P'_x \in \mathcal{B}$ such that $x = P'_x\varphi\tilde{\beta}$. By (2), there is $\tilde{P'}_x \in \pi(\alpha)$ such that $\tilde{P'}_x\alpha_* \leq P'_x\varphi\tilde{\beta}$ and $P'_x \subseteq \tilde{P'}_x$. Define $\gamma : X \to X'$ by

$$x\gamma = \begin{cases} \tilde{P'}_x\alpha_* & \text{if} \quad x \in \mathcal{B}\varphi\tilde{\beta}; \\ x' & \text{otherwise.} \end{cases}$$

By the assumption, we obtain if $x \in \mathcal{B}\varphi\tilde{\beta}$, then $x\gamma = \tilde{P'}_x\alpha_* \leq (P'_x\varphi)\tilde{\beta} = x$ and if $x \notin \mathcal{B}\varphi\tilde{\beta}$, then $x\gamma = x' \leq x$. Hence $\gamma \in T_{RE}(X, X')$. Now, we show that

$\alpha = \delta\beta\gamma$. Let $x \in \operatorname{dom}\alpha$. Then $x \in P_x$ for some $P_x \in \mathcal{B}$. By the definition of δ, we have $x\delta = y_x$ where $y_x \in P_x\varphi$ which implies that $y_x\beta = P_x\varphi\tilde{\beta} \in \mathcal{B}\varphi\tilde{\beta}$. Hence $x\delta\beta\gamma = y_x\beta\gamma = \tilde{P'}_{y_x\beta}\alpha_*$ where $\tilde{P'}_{y_x\beta} \in \pi(\alpha)$ and $P_x \subseteq \tilde{P'}_{y_x\beta}$ from which it follows that $x\alpha = \tilde{P'}_{y_x\beta}\alpha_*$. Therefore, $x\delta\beta\gamma = x\alpha$. □

By the same proof as the above theorem, we obtain the following result for $P_{RE}(X, X')$.

Theorem 14. *Let $\alpha, \beta \in P_{RE}(X, X')$. Then the following statements are equivalent.*

1. *$\alpha \in P_{RE}(X, X')\beta P_{RE}(X, X')$.*

2. *There exist a refinement $\mathcal{B}$ of $\pi(\alpha)$ and an injection $\varphi : \mathcal{B} \to \tilde{\pi}(\beta)$ such that for every $P \in \mathcal{B}$, $\tilde{P}\alpha_* \leq P\varphi\beta_*$ where $P \subseteq \tilde{P}$ and $\tilde{P} \in \pi(\alpha)$ and for every $x \in P$, $y \leq x$ for some $y \in P\varphi$.*

We obtain the following two corollaries immediately.

Corollary 15. *Let $\alpha, \beta \in T_{RE}(X, X')$. Then $\alpha\mathcal{J}\beta$ if and only if the following statements holds.*

1. *There exist a refinement $\mathcal{B}$ of $\pi(\alpha)$ and an injection $\varphi : \mathcal{B} \to \tilde{\pi}(\beta)$ such that for every $P \in \mathcal{B}$, $\tilde{P}\alpha_* \leq P\varphi\beta_*$ where $P \subseteq \tilde{P}$ and $\tilde{P} \in \pi(\alpha)$ and for every $x \in P$, $y \leq x$ for some $y \in P\varphi$.*

2. *There exist a refinement $\mathcal{B}'$ of $\pi(\beta)$ and an injection $\varphi' : \mathcal{B}' \to \tilde{\pi}(\alpha)$ such that for every $Q \in \mathcal{B}'$, $\tilde{Q}\beta_* \leq Q\varphi'\alpha_*$ where $Q \subseteq \tilde{Q}$ and $\tilde{Q} \in \pi(\beta)$ and for every $x' \in Q$, $y' \leq x'$ for some $y' \in Q\varphi'$.*

Corollary 16. *Let $\alpha, \beta \in P_{RE}(X, X')$. Then $\alpha\mathcal{J}\beta$ if and only if the following statements holds.*

1. *There exist a refinement $\mathcal{B}$ of $\pi(\alpha)$ and an injection $\varphi : \mathcal{B} \to \tilde{\pi}(\beta)$ such that for every $P \in \mathcal{B}$, $\tilde{P}\alpha_* \leq P\varphi\beta_*$ where $P \subseteq \tilde{P}$ and $\tilde{P} \in \pi(\alpha)$ and for every $x \in P$, $y \leq x$ for some $y \in P\varphi$.*

2. *There exist a refinement $\mathcal{B}'$ of $\pi(\beta)$ and an injection $\varphi' : \mathcal{B}' \to \tilde{\pi}(\alpha)$ such that for every $Q \in \mathcal{B}'$, $\tilde{Q}\beta_* \leq Q\varphi'\alpha_*$ where $Q \subseteq \tilde{Q}$ and $\tilde{Q} \in \pi(\beta)$ and for every $x' \in Q$, $y' \leq x'$ for some $y' \in Q\varphi'$.*

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