ρC(ℐ)-COMPACT AND ρℐ-QHC SPACES

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Abstract: In this paper we introduce and investigate two new ideal topological spaces, which are strong forms of Gupta-Noiri concepts. Interesting characterizations of this spaces are presented, as well as several useful properties of these. We compare this new spaces with C-compact and quasi-H-closed spaces.

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1. Introduction and Preliminaries

The ideal topological spaces has been introduced in Kuratowski [5] and Vaidyanathaswamy [12] books. The concept of compactness modulo an ideal was introduced by Newcomb [7], but popularized by Hamlett-Jancovic papers [3][4]. The C-compact spaces and QHC spaces were defined by Viglino [13] and Porter-Thomas [10], respectively, and are generalizations of compactness.


In this paper we introduce and study the ρC(ℐ)-compact and ρℐ-QHC spaces, which are strong forms of the Gupta-Noiri concepts. Interesting characterizations of this new spaces will also be presented, as well as their relationship with the ρℐ-compact spaces [9].

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An ideal $\mathcal{I}$ in a set $X$ is a subset of $\mathcal{P}(X)$, the power set of $X$, such that:
(i) if $A \subseteq B \subseteq X$ and $B \in \mathcal{I}$ then $A \in \mathcal{I}$, and (ii) If $A \in \mathcal{I}$ and $B \in \mathcal{I}$ then $A \cup B \in \mathcal{I}$.

Some useful ideals in $X$ are: (i) $\mathcal{P}(A)$, where $A \subseteq X$, (ii) $\mathcal{I}_f$, the ideal of all finite subsets of $X$, (iii) $\mathcal{I}_c$, the ideal of all countable subsets of $X$, (iv) $\mathcal{I}_n$, the ideal of all nowhere dense subsets in a topological space $(X, \tau)$.

If $(X, \tau)$ is a topological space and $\mathcal{I}$ is an ideal in $X$, then $(X, \tau, \mathcal{I})$ is called an ideal space.

If $(X, \tau, \mathcal{I})$ is an ideal space then the set $B = \{ U \setminus I : U \in \tau \text{ and } I \in \mathcal{I} \}$ is a base for a topology $\tau^*$, finer than $\tau$.

If $(X, \tau)$ is a topological space and $A \subseteq X$ then $\overline{A}$ (or $\text{adh}(A)$, or $\text{adh}_\tau(A)$) and $A$ (or $\text{int}(A)$, or $\text{int}_\tau(A)$) will, respectively, denote the closure and interior of $A$ in $(X, \tau)$.

If $(X, \tau)$ is a topological space and $A \subseteq X$ then $A$ is said to be regular open if $A = \overline{0_A}$, and $A$ is defined to be regular closed if $A = \underline{A}$. If $A \subseteq \overline{A}$ then $A$ is called pre-open [6]. The set of all pre-open subsets of $X$ is denoted by $PO(X)$.

If $A \subseteq \overline{A}$ then $A$ is called $\alpha$-open [8]. Clearly open $\Rightarrow$ $\alpha$-open $\Rightarrow$ pre-open.

Moreover, if $\mathcal{I}$ is an ideal in $X$ and $\mathcal{I} \cap \tau = \{ \emptyset \}$, $\mathcal{I}$ is called codense [1]. If $\mathcal{I} \cap PO(X) = \{ \emptyset \}$ then $\mathcal{I}$ is said to be completely codense [1].

2. $\rho\mathcal{I}$-QHC spaces

A topological space $(X, \tau)$ is said to be quasi-$H$-closed, or simply $QHC$ [10], if for each open cover $\{ V_\alpha \}_{\alpha \in \Lambda}$ of $X$, there exists $\Lambda_0 \subseteq \Lambda$, finite, with $X = \bigcup_{\alpha \in \Lambda_0} \overline{V_\alpha}$.

An ideal space $(X, \tau, \mathcal{I})$ is defined to be $\mathcal{I}$-compact [7] if for all open cover $\{ V_\alpha \}_{\alpha \in \Lambda}$ of $X$, there exists $\Lambda_0 \subseteq \Lambda$, finite, such that $X \setminus \bigcup_{\alpha \in \Lambda_0} \overline{V_\alpha} \in \mathcal{I}$.

The space $(X, \tau, \mathcal{I})$ is said to be $\mathcal{I}$-QHC [2] if for all open cover $\{ V_\alpha \}_{\alpha \in \Lambda}$ of $X$, there exists $\Lambda_0 \subseteq \Lambda$, finite, such that $X \setminus \bigcup_{\alpha \in \Lambda_0} \overline{V_\alpha} \in \mathcal{I}$.

It is noted that $QHC \Rightarrow \mathcal{I}$-QHC.

An ideal space $(X, \tau, \mathcal{I})$ is defined to be $\rho\mathcal{I}$-compact [9] if for each family $\{ V_\alpha \}_{\alpha \in \Lambda}$ of open subsets of $X$, if $X \setminus \bigcup_{\alpha \in \Lambda} V_\alpha \in \mathcal{I}$ there exists $\Lambda_0 \subseteq \Lambda$, finite, with $X \setminus \bigcup_{\alpha \in \Lambda_0} V_\alpha \in \mathcal{I}$.
In this section we define the $\rho I$-QHC spaces and study some of its properties and characterizations.

**Definition 2.1** If $(X, \tau, I)$ is an ideal space and $A \subseteq X$, then $A$ is said to be $\rho I$-QHC if for all family $\{V_\alpha\}_{\alpha \in \Lambda}$ of open subsets of $X$, if $A \setminus \bigcup_{\alpha \in \Lambda} V_\alpha \in I$, there exists $\Lambda_0 \subseteq \Lambda$, finite, with $A \setminus \bigcup_{\alpha \in \Lambda_0} V_\alpha \in I$. The ideal space $(X, \tau, I)$ is said to be $\rho I$-QHC if $X$ is $\rho I$-QHC.

Clearly $(X, \tau, \{\emptyset\})$ is $\rho I$-QHC $\iff$ $(X, \tau, \emptyset)$ is $I$-QHC $\iff$ $(X, \tau)$ is QHC. It is also evident that $\rho I$-QHC $\Rightarrow$ $I$-QHC and $\rho I$-compact $\Rightarrow$ $\rho I$-QHC, but the converse, in general, are not true.

**Example 2.1** We denote by $2\mathbb{Z}$ the set of even integers, and by $2\mathbb{Z} + 1$ the set of odd integers.

Let $\tau$ be the topology on $\mathbb{Z}$ given by: If $V \subseteq \mathbb{Z}$ then $V \in \tau \iff$ [if $0 \in V$ then $2\mathbb{Z} \subseteq V$, and if $1 \in V$ then $2\mathbb{Z} + 1 \subseteq V$].

Let $I = P[(2\mathbb{Z} + 1) \cup \{0\}]$. We have that:

1. $(\mathbb{Z}, \tau)$ is a QHC space, and then $(\mathbb{Z}, \tau, I)$ is $I$-QHC.
2. If $\{V_\alpha\}_{\alpha \in \Lambda}$ is a family of open subsets of $\mathbb{Z}$ and $Z = \bigcup_{\alpha \in \Lambda} V_\alpha$, then there are $\alpha_0 \in \Lambda$ and $\alpha_1 \in \Lambda$ with $0 \in V_{\alpha_0}$ and $1 \in V_{\alpha_1}$. Then $2\mathbb{Z} \subseteq V_{\alpha_0}$ and $2\mathbb{Z} + 1 \subseteq V_{\alpha_1}$, and so $Z = \overline{V_{\alpha_0}} \cup \overline{V_{\alpha_1}}$.

3. $(\mathbb{Z}, \tau, I)$ is not $\rho I$-QHC.

$Z \setminus \bigcup_{n \neq 0} \{2n\} = (2\mathbb{Z} + 1) \cup \{0\} \in I$, but if $n \neq 0$ we have that $\overline{\{2n\}} = \{0, 2n\}$, and if $\{n_1, n_2, ..., n_r\} \subseteq Z \setminus \{0\}$ then $Z \setminus \bigcup_{j=1}^r \overline{\{2n_j\}} \notin I$.

In the Examples 3.1 and 3.2 we show $\rho I$-QHC spaces.

It is easy to see that an open and closed subset of a $\rho I$-QHC space is $\rho I$-QHC.

In the next theorem we present interesting characterizations of $\rho I$-QHC spaces. The proof is similar to that of Theorems 3.2 and 3.3, so we omit it.

If $I$ is an ideal in a set $X$, a family $\mathcal{F}$ of subsets of $X$ is said to have the finite-intersection property modulo $I$, if for each $\mathcal{F}_0 \subseteq \mathcal{F}$, finite, we have that $\bigcap_{V \in \mathcal{F}_0} V \notin I$.

**Theorem 2.1** For an ideal space $(X, \tau, I)$, the following statements are equivalents:
1) \((X, \tau, \mathcal{I})\) is \(\rho\mathcal{I}\)-QHC.

2) For each family \(\{F_\alpha\}_{\alpha \in \Lambda}\) of closed subsets of \(X\), if \(\bigcap_{\alpha \in \Lambda} F_\alpha \in \mathcal{I}\), there exists \(\Lambda_0 \subseteq \Lambda\), finite, such that \(\bigcap_{\alpha \in \Lambda_0} F_\alpha \in \mathcal{I}\).

3) For each family \(\{F_\alpha\}_{\alpha \in \Lambda}\) of closed subsets, if \(\left\{\bigcap_{\alpha \in \Lambda} F_\alpha : \alpha \in \Lambda\right\}\) has the finite-intersection property modulo \(\mathcal{I}\), then \(\bigcap_{\alpha \in \Lambda} F_\alpha \notin \mathcal{I}\).

4) For each family \(\{V_\alpha\}_{\alpha \in \Lambda}\) of regular open subsets of \(X\), if \(X \setminus \bigcup_{\alpha \in \Lambda} V_\alpha \in \mathcal{I}\), there is \(\Lambda_0 \subseteq \Lambda\), finite, such that \(X \setminus \bigcup_{\alpha \in \Lambda_0} V_\alpha \in \mathcal{I}\).

5) For each family \(\{F_\alpha\}_{\alpha \in \Lambda}\) of regular closed subsets of \(X\), if \(\bigcap_{\alpha \in \Lambda} F_\alpha \in \mathcal{I}\), there is \(\Lambda_0 \subseteq \Lambda\), finite, such that \(\bigcap_{\alpha \in \Lambda_0} F_\alpha \in \mathcal{I}\).

6) For each family \(\{F_\alpha\}_{\alpha \in \Lambda}\) of regular closed subsets of \(X\), if \(\left\{\bigcap_{\alpha \in \Lambda} F_\alpha : \alpha \in \Lambda\right\}\) has the finite-intersection property modulo \(\mathcal{I}\), then \(\bigcap_{\alpha \in \Lambda} F_\alpha \notin \mathcal{I}\).

7) For each open filter base \(\Omega\) on \(X\) such that \(\Omega \subseteq \mathcal{P}(X) \setminus \mathcal{I}\), one has \(\bigcap_{V \in \Omega} V \notin \mathcal{I}\).

It follows from a result in [11] that if \((X, \tau)\) is a topological space and \(\mathcal{I}\) is a completely codense ideal in \(X\), then \((X, \tau)\) and \((X, \tau^*)\) have the same regular open subsets, and \(\text{adh}_\tau(V) = \text{adh}_{\tau^*}(V)\), for all \(V \in \tau^*\). Then the following result is clear.

**Theorem 2.2** If \(\mathcal{I}\) is a completely codense ideal in \(X\), the space \((X, \tau, \mathcal{I})\) is \(\rho\mathcal{I}\)-QHC if and only if \((X, \tau^*, \mathcal{I})\) is \(\rho\mathcal{I}\)-QHC.

Now we review the behavior of \(\rho\mathcal{I}\)-QHC spaces under continuous or open functions.

**Theorem 2.3** 1) If \((X, \tau, \mathcal{I})\) is \(\rho\mathcal{I}\)-QHC and \(f : (X, \tau) \to (Y, \beta)\) is a biyective continuous function, then \((Y, \beta, f(\mathcal{I}))\) is \(\rho f(\mathcal{I})\)-QHC, where \(f(\mathcal{I})\) is the ideal \(\{f(I) : I \in \mathcal{I}\}\).

2) If \((X, \tau, \mathcal{I})\) is \(\rho\mathcal{I}\)-QHC and \(f : (X, \tau) \to (Y, \beta)\) is a continuous function, then \((Y, \beta, \mathcal{J})\) is \(\rho\mathcal{J}\)-QHC, where \(\mathcal{J}\) is the ideal \(\{V \subseteq Y : f^{-1}(V) \in \mathcal{I}\}\).
3) If \((Y, \beta, J)\) is \(\rho J\)-QHC and \(f : (X, \tau) \to (Y, \beta)\) is a biyective and open function, then \((X, \tau, f^{-1}(J))\) is \(\rho f^{-1}(J)\)-QHC, where \(f^{-1}(J)\) is the ideal \(\{f^{-1}(V) : V \in J\}\).

**Proof.** 1) Suppose that \(\{W_\alpha\}_{\alpha \in \Lambda}\) is a family of open subsets of \(Y\) with \(Y \setminus \bigcup_{\alpha \in \Lambda} W_\alpha \in f(I)\). There exists \(I \in I\) such that \(Y \setminus \bigcup_{\alpha \in \Lambda} W_\alpha = f(I)\). Since \(X \setminus \bigcup_{\alpha \in \Lambda} f^{-1}(W_\alpha) = f^{-1}(f(I)) = I \in I\), there exists \(\Lambda_0 \subseteq \Lambda\), finite, with \(X \setminus \bigcup_{\alpha \in \Lambda_0} f^{-1}(W_\alpha) \in I\). Given that \(f\) is continuous, \(f^{-1}\left(Y \setminus \bigcup_{\alpha \in \Lambda_0} W_\alpha\right) = X \setminus \bigcup_{\alpha \in \Lambda_0} f^{-1}(W_\alpha) \subseteq X \setminus \bigcup_{\alpha \in \Lambda_0} f^{-1}(W_\alpha)\), and so \(f^{-1}\left(Y \setminus \bigcup_{\alpha \in \Lambda_0} W_\alpha\right) \in I\).

Then \(Y \setminus \bigcup_{\alpha \in \Lambda_0} W_\alpha = f\left(f^{-1}\left(Y \setminus \bigcup_{\alpha \in \Lambda_0} W_\alpha\right)\right) \in f(I)\).

2) It is easy to see that \(J\) is an ideal in \(Y\). Suppose that \(\{W_\alpha\}_{\alpha \in \Lambda}\) is a family of open subsets of \(Y\) with \(Y \setminus \bigcup_{\alpha \in \Lambda} W_\alpha \in J\). Since \(X \setminus \bigcup_{\alpha \in \Lambda} f^{-1}(W_\alpha) = f^{-1}\left(Y \setminus \bigcup_{\alpha \in \Lambda} W_\alpha\right) \in I\), there is \(\Lambda_0 \subseteq \Lambda\), finite, with \(X \setminus \bigcup_{\alpha \in \Lambda_0} f^{-1}(W_\alpha) \in I\).

Given that \(f\) is continuous, \(X \setminus \bigcup_{\alpha \in \Lambda_0} f^{-1}(W_\alpha) \subseteq X \setminus \bigcup_{\alpha \in \Lambda_0} f^{-1}(W_\alpha)\), and so \(f^{-1}\left(Y \setminus \bigcup_{\alpha \in \Lambda_0} W_\alpha\right) = X \setminus \bigcup_{\alpha \in \Lambda_0} f^{-1}(W_\alpha) \in I\). Thus \(Y \setminus \bigcup_{\alpha \in \Lambda_0} W_\alpha \in J\).

3) Suppose that \(\{V_\alpha\}_{\alpha \in \Lambda}\) is a family of open subsets of \(X\), with \(X \setminus \bigcup_{\alpha \in \Lambda} V_\alpha \in f^{-1}(J)\). There exists \(J \in J\) such that \(X \setminus \bigcup_{\alpha \in \Lambda} V_\alpha = f^{-1}(J)\). Then \(Y \setminus \bigcup_{\alpha \in \Lambda} f(V_\alpha) = J\) and so there is \(\Lambda_0 \subseteq \Lambda\), finite, with \(Y \setminus \bigcup_{\alpha \in \Lambda_0} f(V_\alpha) \in J\). Given that \(f\) is open and biyective, \(f\) is closed, and so \(f(V_\alpha) \subseteq f(V_\alpha)\), for each \(\alpha \in \Lambda_0\). This implies that \(Y \setminus \bigcup_{\alpha \in \Lambda_0} f(V_\alpha) \in J\), and then \(X \setminus \bigcup_{\alpha \in \Lambda_0} V_\alpha \in f^{-1}(J)\).

We end this section by presenting a characterization of \(\rho I\)-QHC spaces in terms of pre-open and \(\alpha\)-open subsets. The proof is similar to that of Theorem 3.7.

**Theorem 2.4** If \((X, \tau, I)\) is an ideal space, the following statements are equivalents:

1) \((X, \tau, I)\) is \(\rho I\)-QHC.
2) For each family \( \{ V_\alpha \}_{\alpha \in \Lambda} \) of pre-open subsets, if \( X \setminus \bigcup_{\alpha \in \Lambda} V_\alpha \in \mathcal{I} \) then there exists \( \Lambda_0 \subseteq \Lambda \), finite, with \( X \setminus \bigcup_{\alpha \in \Lambda_0} V_\alpha \in \mathcal{I} \).

3) For each family \( \{ V_\alpha \}_{\alpha \in \Lambda} \) of \( \alpha \)-open subsets, if \( X \setminus \bigcup_{\alpha \in \Lambda} V_\alpha \in \mathcal{I} \) then there exists \( \Lambda_0 \subseteq \Lambda \), finite, with \( X \setminus \bigcup_{\alpha \in \Lambda_0} V_\alpha \in \mathcal{I} \).

### 3. \( \rho C(\mathcal{I}) \)-compact spaces

A topological space \( (X, \tau) \) is defined to be \( C \)-compact [13] if for each \( F \subseteq X \), closed, and each \( \tau \)-open cover \( \{ V_\alpha \}_{\alpha \in \Lambda} \) of \( F \), there exists \( \Lambda_0 \subseteq \Lambda \), finite, with \( F \subseteq \bigcup_{\alpha \in \Lambda_0} V_\alpha \).

An ideal space \( (X, \tau, \mathcal{I}) \) is said to be \( C(\mathcal{I}) \)-compact [2] if for each \( F \subseteq X \), closed, and each \( \tau \)-open cover \( \{ V_\alpha \}_{\alpha \in \Lambda} \) of \( F \), there exists \( \Lambda_0 \subseteq \Lambda \), finite, with \( F \setminus \bigcup_{\alpha \in \Lambda_0} V_\alpha \in \mathcal{I} \).

It is noted that \( C \)-compact \( \Rightarrow \) QHC, \( C(\mathcal{I}) \)-compact \( \Rightarrow \mathcal{I} \)-QHC and that if \( (X, \tau) \) is \( C \)-compact then \( (X, \tau, \mathcal{I}) \) is \( C(\mathcal{I}) \)-compact.

In this section we introduce and study the \( \rho C(\mathcal{I}) \)-compact spaces, which are stronger forms of \( C(\mathcal{I}) \)-compactness and \( \mathcal{I} \)-QHC. We present some of its properties and characterizations.

**Definition 3.1** The ideal space \( (X, \tau, \mathcal{I}) \) is said to be \( \rho C(\mathcal{I}) \)-compact if for each closed subset \( F \) of \( X \), and each family \( \{ V_\alpha \}_{\alpha \in \Lambda} \) of open subsets of \( X \) such that \( F \setminus \bigcup_{\alpha \in \Lambda} V_\alpha \in \mathcal{I} \), there exists \( \Lambda_0 \subseteq \Lambda \), finite, with \( F \setminus \bigcup_{\alpha \in \Lambda_0} V_\alpha \in \mathcal{I} \).

Note that if \( (X, \tau^*, \mathcal{I}) \) is \( \rho C(\mathcal{I}) \)-compact then \( (X, \tau, \mathcal{I}) \) is \( \rho C(\mathcal{I}) \)-compact.

It is also clear that:

1) \( (X, \tau) \) is \( C \)-compact \( \iff \) \( (X, \tau, \{ \emptyset \}) \) is \( \rho C(\{ \emptyset \}) \)-compact \( \iff \) \( (X, \tau, \{ \emptyset \}) \) is \( C(\{ \emptyset \}) \)-compact.

2) \( \rho C(\mathcal{I}) \)-compact \( \Rightarrow \rho \mathcal{I} \)-QHC.

3) \( \rho C(\mathcal{I}) \)-compact \( \Rightarrow \rho C(\mathcal{I}) \)-compact.

This implications are, in general, irreversible.

**Example 3.1** 1) We consider again the ideal space \( (\mathbb{Z}, \tau, \mathcal{I}) \) of Example 2.1, which is not \( \rho \mathcal{I} \)-QHC. We will demonstrated that \( (\mathbb{Z}, \tau) \) is \( C \)-compact, and so \( (\mathbb{Z}, \tau, \mathcal{I}) \) is \( C(\mathcal{I}) \)-compact.

Let \( F \) be a closed subset of \( \mathbb{Z} \) and \( \{ V_\alpha \}_{\alpha \in \Lambda} \) an open cover of \( F \).
(i) If $F \cap \{0, 1\} = \emptyset$ then $2\mathbb{Z} \cap F = \emptyset$ and $(2\mathbb{Z} + 1) \cap F = \emptyset$, and so $F = \emptyset$. If $\alpha_0 \in \Lambda$ then $F \subseteq \overline{V}_{\alpha_0}$.

(ii) If $F \cap \{0, 1\} = \{0, 1\}$ then there are $\alpha_0 \in \Lambda$ and $\alpha_1 \in \Lambda$ such that $0 \in V_{\alpha_0}$ and $1 \in V_{\alpha_1}$. This implies that $\overline{V}_{\alpha_0} \cup \overline{V}_{\alpha_1} = X$ and $F \subseteq \overline{V}_{\alpha_0} \cup \overline{V}_{\alpha_1}$.

(iii) If $F \cap \{0, 1\} = \{0\}$ then $(2\mathbb{Z} + 1) \cap F = \emptyset$, and there exists $\alpha_0 \in \Lambda$ with $0 \in V_{\alpha_0}$. Thus $F \subseteq 2\mathbb{Z} \subseteq \overline{V}_{\alpha_0} \subseteq F_{\alpha_0}$.

(iv) If $F \cap \{0, 1\} = \{1\}$ then $2\mathbb{Z} \cap F = \emptyset$, and there exists $\alpha_1 \in \Lambda$ with $1 \in V_{\alpha_1}$. Thus $F \subseteq 2\mathbb{Z} + 1 \subseteq \overline{V}_{\alpha_1}$.

Hence the space $(\mathbb{Z}, \tau, \mathcal{I})$ is $C(\mathcal{I})$-compact. However this space is not $\rho C(\mathcal{I})$-compact, because $(X, \tau, \mathcal{I})$ is not $\rho \mathcal{I}$-QHC.

2) Let $\mathcal{U}$ be the usual topology for $X = [0, 1]$. Let $F = \{1/n : n \in \mathbb{Z}^+\}$. We consider the topology $\mathcal{U}^*$ for $X$ generated by $\mathcal{U} \cup \{X \setminus F\}$. A base for $\mathcal{U}^*$ is $\mathcal{B} = \mathcal{U} \cup \{V \setminus F : V \in \mathcal{U}\}$.

We have that:

a) $F$ is closed and discrete in $(X, \mathcal{U}^*)$.

If $n \in \mathbb{Z}^+$, $r_n = \frac{1}{2} \left(\frac{1}{n} - \frac{1}{n+1}\right)$, and if $W_n = (\frac{1}{n} - r_n, \frac{1}{n} + r_n) \cap X$, then $W_n \in \mathcal{U} \subseteq \mathcal{U}^*$ and $W_n \cap F = \{\frac{1}{n}\}$.

b) The family $\{W_n\}_{n \in \mathbb{Z}^+}$ is a $\mathcal{U}^*$-open cover of $F$.

c) $F$ is nowhere dense in $(X, \mathcal{U}^*)$, because $\text{int}_{\mathcal{U}^*}(\text{adh}_{\mathcal{U}^*}(F)) = \text{int}_{\mathcal{U}^*}(F) = \emptyset$, since $\emptyset$ is the only element in $\mathcal{B}$ which is contained in $F$.

d) If $V \in \mathcal{U}^*$ then $\text{adh}_{\mathcal{U}^*}(V) = \text{adh}_{\mathcal{U}}(V)$.

It is clear that $\text{adh}_{\mathcal{U}^*}(V) \subseteq \text{adh}_{\mathcal{U}}(V)$. Suppose that $z \in \text{adh}_{\mathcal{U}}(V)$ and that $B \in \mathcal{B}$, with $z \in B$. If $B \in \mathcal{U}$ then $V \cap B \neq \emptyset$. If there exists $W \in \mathcal{U}$ such that $B = W \setminus F$ then $W \cap V \neq \emptyset$. Since $F$ is nowhere dense in $(X, \mathcal{U}^*)$, we have that $(W \cap V) \cap (X \setminus F) \neq \emptyset$, and so $V \cap B \neq \emptyset$. Thus $z \in \text{adh}_{\mathcal{U}^*}(V)$.

e) The space $(X, \mathcal{U}^*)$ is not C-compact, and then $(X, \mathcal{U}^*, \{\emptyset\})$ is not $\rho C(\{\emptyset\})$-compact.

If $n \in \mathbb{Z}^+$, $\text{adh}_{\mathcal{U}^*}(W_n) = \text{adh}_{\mathcal{U}}(W_n) = [\frac{1}{n} - r_n, \frac{1}{n} + r_n] \cap X$ and so $\text{adh}_{\mathcal{U}^*}(W_n) \cap F = \{\frac{1}{n}\}$. Hence $F \subseteq \bigcup_{n \in \mathbb{Z}^+} W_n$, but if $n_1, n_2, ..., n_r \in \mathbb{Z}^+$, $F \not\subseteq \bigcup_{j=1}^r \text{adh}_{\mathcal{U}^*}(W_{n_j})$.

f) The space $(X, \mathcal{U}^*)$ is QHC, and then $(X, \mathcal{U}^*, \{\emptyset\})$ is $\rho \{\emptyset\}$-QHC.

Let $\{V_\alpha\}_{\alpha \in \Lambda}$ be a $\mathcal{U}^*$-open cover of $X$. There exists $\alpha_0 \in \Lambda$ such that $0 \in V_{\alpha_0}$. Let $B$ be an element of $\mathcal{B}$ with $0 \in B \subseteq V_{\alpha_0}$.

(i) If $B \in \mathcal{U}$ then there exists $r \in (0, \frac{1}{2}) \setminus F$ with $[0, r) \subseteq B$. Then $[0, r] \subseteq \text{adh}_{\mathcal{U}^*}(B) \subseteq \text{adh}_{\mathcal{U}^*}(V_{\alpha_0})$. 
The set $F \setminus [0,r]$ is finite. Suppose that $F \setminus [0,r] = \{f_1, f_2, ..., f_n\}$, and that $f_1 < f_2 < \cdots < f_{n-1} = \frac{1}{2} < f_n = 1$. For all $j \in \{1,2,...,n\}$ there exists $\alpha_j \in \Lambda$ with $f_j \in V_{\alpha_j}$, and there exists $\epsilon_j > 0$ such that $(f_j - \epsilon_j, f_j + \epsilon_j) \subseteq V_{\alpha_j}$ if $j \in \{1,2,...,n-1\}$, $(f_n - \epsilon_n, f_n] \subseteq V_{\alpha_n}$ and $(f_j - \epsilon_j, f_j + \epsilon_j) \cap F = \{f_j\}$ for each $j \in \{1,2,...,n\}$.

Let $F_j = [f_j - \epsilon_j, f_j + \epsilon_j]$, $T_j = (f_j - \epsilon_j, f_j + \epsilon_j)$ if $j \in \{1,2,...,\ n-1\}$, $F_n = [f_n - \epsilon_n, f_n]$ and $T_n = (f_n - \epsilon_n, f_n]$.

Clearly $\{f_1, f_2, ..., f_n\} \subseteq \bigcup_{k=1}^{n} F_k \subseteq \bigcup_{k=1}^{n} \text{adh}_{U^*} (V_{\alpha_k})$.

Now, $[0,1] \setminus \left( [0,r) \cup \bigcup_{k=1}^{n} T_k \right)$ is a finite union of closed intervals, each of which disjoint of $F$. Suppose that $[0,1] \setminus \left( [0,r) \cup \bigcup_{k=1}^{n} T_k \right) = \bigcup_{i=1}^{m} [a_i, b_i]$. It is easy to see that for every $i \in \{1,2,...,m\}$ there exists $\Lambda_i \subseteq \Lambda$, finite, such that $[a_i, b_i] \subseteq \bigcup_{\alpha \in \Lambda_i} \text{adh}_{U^*} (V_{\alpha})$.

Therefore $X = \left( \bigcup_{k=0}^{n} \text{adh}_{U^*} (V_{\alpha_k}) \right) \cup \left( \bigcup_{i=1}^{m} \bigcup_{\alpha \in \Lambda_i} \text{adh}_{U^*} (V_{\alpha}) \right)$.

(ii) If there exists $V \in U$ with $B = V \setminus F$, then there exists $r \in (0, \frac{1}{2}) \setminus F$ such that $[0,r) \subseteq V$, and so $[0,r) = \text{adh}_{U^*} ([0,r) \setminus F) \subseteq \text{adh}_{U^*} (B) \subseteq \text{adh}_{U^*} (V_{\alpha_0})$. Now we proceed as in case (i).

In conclusion, the space $(X, U^*)$ is QHC.

**Theorem 3.1** If the space $(X, \tau, I)$ is $\rho I$-compact then $(X, \tau, I)$ is $\rho C(I)$-compact.

**Proof.** Suppose that $K$ is a closed subset of $X$, and that $\{V_{\alpha}\}_{\alpha \in \Lambda}$ is a family of open subsets of $X$ with $K \setminus \bigcup_{\alpha \in \Lambda} V_{\alpha} \in I$, this is, $X \setminus \left[ (X \setminus K) \cup \bigcup_{\alpha \in \Lambda} V_{\alpha} \right] \in I$. By hypothesis, there exists $\Lambda_0 \subseteq \Lambda$, finite, such that $X \setminus \left[ (X \setminus K) \cup \bigcup_{\alpha \in \Lambda_0} V_{\alpha} \right] \in I$, this is, $K \setminus \bigcup_{\alpha \in \Lambda_0} V_{\alpha} \in I$. Hence $K \setminus \bigcup_{\alpha \in \Lambda_0} \overline{V_{\alpha}} \in I$.

The converse of this theorem, in general, is not true.

**Example 3.2** If $X = [0, \infty)$, $\tau = \{ (r, \infty) : r \geq 0 \} \cup \{ \emptyset, X \}$, and $I = I_f$ then we know that $(X, \tau, I)$ is not $\rho I$-compact [9].

However, $(X, \tau, I)$ is $\rho C(I)$-compact, and so $\rho I$-QHC, since if $V \in \tau \setminus \{ \emptyset \}$, $\overline{V} = X$. 
The following example shows that QHC and $\rho I$-QHC are independent concepts, as well as C-compact and $\rho C(I)$-compact.

**Example 3.3**

1) The space $(Z, \tau, I)$ of Example 2.1 is not $\rho I$-QHC, but $(Z, \tau)$ is compact. This implies that Compact $\not\Rightarrow \rho I$-compact, C-compact $\not\Rightarrow \rho C(I)$-compact and QHC $\not\Rightarrow \rho I$-QHC.

2) If $U$ is the usual topology for $\mathbb{R}$, then clearly $(\mathbb{R}, U, \mathcal{P}(\mathbb{R}))$ is $\rho P(\mathbb{R})$-compact, but $(\mathbb{R}, U)$ is not QHC. This implies that $\rho I$-compact $\not\Rightarrow$ compact, $\rho C(I)$-compact $\not\Rightarrow$ C-compact and $\rho I$-QHC $\not\Rightarrow$ QHC.

Then we have the following diagram:

\[
\begin{array}{cccc}
\text{Compact} & \rightarrow & I - \text{Compact} & \leftarrow \rho I - \text{Compact} \\
\downarrow & & \downarrow & \\
C - \text{Compact} & \rightarrow & C(I) - \text{Compact} & \rightarrow \rho C(I) - \text{Compact} \\
\downarrow & & \downarrow & \\
QHC & \rightarrow & I - \text{QHC} & \leftarrow \rho I - \text{QHC}
\end{array}
\]

**Theorem 3.2**
The ideal space $(X, \tau, I)$ is $\rho C(I)$-compact if and only if, for each closed subset $F$ of $X$, and each open filter base $\Omega$ on $X$ such that $\{V \cap F : V \in \Omega\} \subseteq \mathcal{P}(X) \setminus I$, one has $\bigcap_{V \in \Omega} V \cap F \in I$.

**Proof.** ($\Rightarrow$) Suppose that $(X, \tau, I)$ is $\rho C(I)$-compact and that there are $F \subseteq X$, closed, and an open filter base $\Omega$ on $X$ such that $\{V \cap F : V \in \Omega\} \subseteq \mathcal{P}(X) \setminus I$ and $\bigcap_{V \in \Omega} V \cap F \in I$.

Since $F \setminus \bigcup_{V \in \Omega} (X \setminus V) \in I$, there is, $\{V_1, V_2, ..., V_n\} \subseteq \Omega$ with $F \setminus \bigcup_{i=1}^{n} X \setminus V_i \in I$, or equivalently, $F \setminus \bigcup_{i=1}^{n} (X \setminus V_i) \in I$.

Since $F \setminus \bigcup_{i=1}^{n} (X \setminus V_i) \subseteq F \setminus \bigcup_{i=1}^{n} (X \setminus V_i)$, we have that

$$\left(\bigcap_{i=1}^{n} V_i\right) \cap F = F \setminus \bigcup_{i=1}^{n} (X \setminus V_i) \in I.$$

But there exists $V \in \Omega$ with $V \subseteq \bigcap_{i=1}^{n} V_i$, and so $V \cap F \in I$. This contradicts that $\{V \cap F : V \in \Omega\} \subseteq \mathcal{P}(X) \setminus I$. 


Suppose that \((X, \tau, I)\) is not \(\rho C(I)\)-compact. There exist \(F \subseteq X\), closed, and a family \(\{V_\alpha\}_{\alpha \in \Lambda}\) of open subsets of \(X\), such that \(F \setminus \bigcup_{\alpha \in \Lambda} V_\alpha \in I\), but for each \(\Lambda_0 \subseteq \Lambda\), finite, \(F \setminus \bigcup_{\alpha \in \Lambda_0} V_\alpha \notin I\). In particular, for all \(\alpha \in \Lambda\), \(F \setminus V_\alpha \notin I\). We may assume that \(\{V_\alpha\}_{\alpha \in \Lambda}\) is closed for finite unions, because otherwise we can replace \(\{V_\alpha\}_{\alpha \in \Lambda}\) by the family of all finite unions of elements in \(\{V_\alpha\}_{\alpha \in \Lambda}\).

Then the set \(B = \{X \setminus V_\alpha : \alpha \in \Lambda\}\) is an open filter base on \(X\), and \(\{B \cap F : B \in B\} \subseteq \mathcal{P}(X) \setminus I\).

The hypothesis implies that \(\bigcap_{B \in B} B \cap F \notin I\), this is \(\bigcap_{\alpha \in \Lambda} X \setminus V_\alpha \cap F \notin I\). But for each \(\alpha \in \Lambda\), \(X \setminus V_\alpha = X \setminus V_i \subseteq X \setminus V_\alpha\), and so \(F \setminus \bigcup_{\alpha \in \Lambda} V_\alpha = \bigcap_{\alpha \in \Lambda} (X \setminus V_\alpha) \cap F \notin I\), contradiction.

Next we present other interesting characterizations of \(\rho C(I)\)-compactness.

**Theorem 3.3** For an ideal space \((X, \tau, I)\), the following statements are equivalents:

1) \((X, \tau, I)\) is \(\rho C(I)\)-compact.

2) For all closed subset \(F\) and all family \(\{F_\alpha\}_{\alpha \in \Lambda}\) of closed subsets of \(X\), if \(\bigcap_{\alpha \in \Lambda} (F \cap F_\alpha) \in I\), there is \(\Lambda_0 \subseteq \Lambda\), finite, such that \(\bigcap_{\alpha \in \Lambda_0} (F \cap F_\alpha) \in I\).

3) For each closed subset \(F\) and each family \(\{F_\alpha\}_{\alpha \in \Lambda}\) of closed subsets of \(X\), if \(\left\{F \cap F_\alpha : \alpha \in \Lambda\right\}\) has the finite-intersection property modulo \(I\), then \(\bigcap_{\alpha \in \Lambda} (F \cap F_\alpha) \notin I\).

4) For all closed subset \(F\) and all family \(\{V_\alpha\}_{\alpha \in \Lambda}\) of regular open subsets of \(X\), if \(F \setminus \bigcup_{\alpha \in \Lambda} V_\alpha \in I\), there is \(\Lambda_0 \subseteq \Lambda\), finite, such that \(F \setminus \bigcup_{\alpha \in \Lambda_0} V_\alpha \in I\).

5) For each closed subset \(F\) of \(X\) and each family \(\{F_\alpha\}_{\alpha \in \Lambda}\) of regular closed subsets of \(X\), if \(\bigcap_{\alpha \in \Lambda} (F \cap F_\alpha) \in I\), there is \(\Lambda_0 \subseteq \Lambda\), finite, such that \(\bigcap_{\alpha \in \Lambda_0} (F \setminus F_\alpha) \in I\).

6) For each closed subset \(F\) and each family \(\{F_\alpha\}_{\alpha \in \Lambda}\) of regular closed subsets of \(X\), if \(\left\{F \cap F_\alpha : \alpha \in \Lambda\right\}\) has the finite-intersection property modulo \(I\), then \(\bigcap_{\alpha \in \Lambda} (F \cap F_\alpha) \notin I\).
7) If \( F \subseteq X \) is closed, \( W \subseteq X \) is open with \( F \setminus W \in \mathcal{I} \), and if \( \{V_{\alpha}\}_{\alpha \in \Lambda} \) is a family of open subsets of \( X \), such that \( (X \setminus F) \setminus \bigcup_{\alpha \in \Lambda} V_{\alpha} \in \mathcal{I} \), then there exists \( \Lambda_0 \subseteq \Lambda \), finite, with \( X \setminus \left( W \cup \bigcup_{\alpha \in \Lambda_0} \overline{V_{\alpha}} \right) \in \mathcal{I} \).

**Proof.** The implications 1)\( \Rightarrow \)2), 2)\( \Rightarrow \)3), 5)\( \Rightarrow \)6) are easy to be established.

3)\( \Rightarrow \)4) Let \( F \) a closed subset of \( X \) and \( \{V_{\alpha}\}_{\alpha \in \Lambda} \) a family of regular open subsets of \( X \) with \( F \setminus \bigcup_{\alpha \in \Lambda} V_{\alpha} \in \mathcal{I} \), or equivalently, \( \cap (F \cap (X \setminus V_{\alpha})) \in \mathcal{I} \). Then the family \( \{F \cap \text{int} (X \setminus V_{\alpha}) : \alpha \in \Lambda\} \) has no the finite-intersection property modulo \( \mathcal{I} \), and so there exists \( \Lambda_0 \subseteq \Lambda \), finite, with \( \cap (F \cap \text{int}(X \setminus V_{\alpha})) \in \mathcal{I} \), or equivalently, \( F \setminus \bigcup_{\alpha \in \Lambda_0} \overline{V_{\alpha}} \in \mathcal{I} \).

4)\( \Rightarrow \)5) It is sufficient to note that the complement of a regular closed subset of \( X \) is regular open.

6)\( \Rightarrow \)1) Let \( F \) a closed subset of \( X \) and \( \{V_{\alpha}\}_{\alpha \in \Lambda} \) a family of open subsets of \( X \) with \( F \setminus \bigcup_{\alpha \in \Lambda} V_{\alpha} \in \mathcal{I} \), that is, \( \cap (F \setminus (X \setminus V_{\alpha})) \in \mathcal{I} \). Since \( \text{int} (X \setminus V_{\alpha}) \subseteq X \setminus V_{\alpha} \), for all \( \alpha \in \Lambda \), then \( \cap (F \setminus \text{int}(X \setminus V_{\alpha})) \in \mathcal{I} \). But \( \text{int} (X \setminus V_{\alpha}) \) is regular closed, for all \( \alpha \in \Lambda \). By the hypothesis there exists \( \Lambda_0 \subseteq \Lambda \), finite, such that \( \cap (F \setminus \text{int}(X \setminus V_{\alpha})) \in \mathcal{I} \), and so \( \cap (F \setminus \text{int}(X \setminus V_{\alpha})) \in \mathcal{I} \), that is, \( F \setminus \bigcup_{\alpha \in \Lambda_0} \overline{V_{\alpha}} \in \mathcal{I} \).

7)\( \Rightarrow \)1) Suppose that \( F \subseteq X \) is closed and that \( \{V_{\alpha}\}_{\alpha \in \Lambda} \) is a family of open subsets of \( X \), with \( F \setminus \bigcup_{\alpha \in \Lambda} V_{\alpha} \in \mathcal{I} \). Let \( W = \bigcup_{\alpha \in \Lambda} V_{\alpha} \) and \( K = X \setminus W \). We have that \( K \setminus (X \setminus F) = (X \setminus W) \setminus (X \setminus F) = F \setminus W \in \mathcal{I} \), and that \( (X \setminus K) \setminus \bigcup_{\alpha \in \Lambda} V_{\alpha} = \varnothing \in \mathcal{I} \). The hypothesis implies that there exists \( \Lambda_0 \subseteq \Lambda \), finite, such that \( X \setminus \left[ (X \setminus F) \cup \bigcup_{\alpha \in \Lambda_0} \overline{V_{\alpha}} \right] \in \mathcal{I} \), this is, \( F \setminus \bigcup_{\alpha \in \Lambda_0} \overline{V_{\alpha}} \in \mathcal{I} \).

1)\( \Rightarrow \)7) Suppose that \( F \subseteq X \) is closed, \( W \subseteq X \) is open with \( F \setminus W \in \mathcal{I} \), and that \( \{V_{\alpha}\}_{\alpha \in \Lambda} \) is a family of open subsets of \( X \), with \( (X \setminus F) \setminus \bigcup_{\alpha \in \Lambda} V_{\alpha} \in \mathcal{I} \).

Since \( (X \setminus W) \setminus \bigcup_{\alpha \in \Lambda} V_{\alpha} = X \setminus \left[ W \cup \bigcup_{\alpha \in \Lambda} V_{\alpha} \right] \subseteq (X \setminus F) \setminus \bigcup_{\alpha \in \Lambda} V_{\alpha} \cup (F \setminus W) \in \mathcal{I} \), and \( X \setminus W \) is closed, there exists \( \Lambda_0 \subseteq \Lambda \), finite, with \( (X \setminus W) \setminus \bigcup_{\alpha \in \Lambda_0} \overline{V_{\alpha}} \in \mathcal{I} \),
this is, $X \setminus \left( W \cup \bigcup_{\alpha \in \Lambda_0} V_\alpha \right) \in \mathcal{I}$.

In the following theorem we review the behavior of $\rho(C(I))$-compact spaces under continuous or open functions.

**Theorem 3.4** 1) If $(X, \tau, \mathcal{I})$ is $\rho(C(I))$-compact and if $f : (X, \tau) \to (Y, \beta)$ is a continuous biyective function, then $(Y, \beta, f(\mathcal{I}))$ is $\rho(C(f(\mathcal{I})))$-compact.

2) If $(X, \tau, \mathcal{I})$ is $\rho(C(I))$-compact, $f : (X, \tau) \to (Y, \beta)$ is a continuous function and if $\mathcal{J}$ is the ideal $\{ V \subseteq Y : f^{-1}(V) \in \mathcal{I} \}$, then $(Y, \beta, \mathcal{J})$ is $\rho(C(\mathcal{J}))$-compact.

3) If $(Y, \beta, \mathcal{J})$ is $\rho(C(\mathcal{J}))$-compact and if $f : (X, \tau) \to (Y, \beta)$ is an open and biyective function, then $(X, \tau, f^{-1}(\mathcal{J}))$ is $\rho(C(f^{-1}(\mathcal{J})))$-compact, where $f^{-1}(\mathcal{J})$ is the ideal $\{ f^{-1}(J) : J \in \mathcal{J} \}$.

**Proof.** 1) Let $B$ a closed subset of $Y$, and $\{ V_\alpha \}_{\alpha \in \Lambda}$ a family of open subsets of $Y$, with $B \setminus \bigcup_{\alpha \in \Lambda} V_\alpha \in f(\mathcal{I})$. There exists $I \in \mathcal{I}$ such that $B \setminus \bigcup_{\alpha \in \Lambda} V_\alpha = f(I)$.

Since $f^{-1}(B) \setminus \bigcup_{\alpha \in \Lambda} f^{-1}(V_\alpha) = f^{-1}(f(I)) = I \in \mathcal{I}$, there exists $\Lambda_0 \subseteq \Lambda$, finite, with $f^{-1}(B) \setminus \bigcup_{\alpha \in \Lambda_0} f^{-1}(V_\alpha) \in \mathcal{I}$.

But $f^{-1}(B) \setminus \bigcup_{\alpha \in \Lambda_0} f^{-1}(V_\alpha) \subseteq f^{-1}(B) \setminus \bigcup_{\alpha \in \Lambda_0} f^{-1}(V_\alpha)$, and this implies that $f^{-1}(B) \setminus \bigcup_{\alpha \in \Lambda_0} f^{-1}(V_\alpha) \in \mathcal{I}$.

Then $B \setminus \bigcup_{\alpha \in \Lambda_0} V_\alpha = f\left( f^{-1}(B) \setminus \bigcup_{\alpha \in \Lambda_0} f^{-1}(V_\alpha) \right) \in f(\mathcal{I})$.

2) Suppose that $B \subseteq Y$ is closed and that $\{ V_\alpha \}_{\alpha \in \Lambda}$ is a family of open subsets of $Y$, with $B \setminus \bigcup_{\alpha \in \Lambda} V_\alpha \in \mathcal{J}$.

Given that $f^{-1}(B) \setminus \bigcup_{\alpha \in \Lambda} f^{-1}(V_\alpha) = f^{-1}\left( B \setminus \bigcup_{\alpha \in \Lambda} V_\alpha \right) \in \mathcal{I}$, there exists $\Lambda_0 \subseteq \Lambda$, finite, such that $f^{-1}(B) \setminus \bigcup_{\alpha \in \Lambda_0} f^{-1}(V_\alpha) \in \mathcal{I}$, and given that for all $\alpha \in \Lambda_0$, $f^{-1}(V_\alpha) \subseteq f^{-1}(V_\alpha)$, we have that $f^{-1}(B) \setminus \bigcup_{\alpha \in \Lambda_0} f^{-1}(V_\alpha) \in \mathcal{I}$, this is, $f^{-1}\left( B \setminus \bigcup_{\alpha \in \Lambda_0} V_\alpha \right) \in \mathcal{I}$. Hence $B \setminus \bigcup_{\alpha \in \Lambda_0} V_\alpha \in \mathcal{J}$.

3) Note that since $f$ is biyective and open then $f$ is closed. Suppose that $A \subseteq X$ is closed and that $\{ V_\alpha \}_{\alpha \in \Lambda}$ is a family of open subsets of $X$, with
A \bigcup_{\alpha \in \Lambda} V_\alpha \in f^{-1}(J). There exists J \in \mathcal{J} such that A \bigcup_{\alpha \in \Lambda} V_\alpha = f^{-1}(J), and so
\begin{align*}
f(A) \setminus \bigcup_{\alpha \in \Lambda} f(V_\alpha) &= \left( A \setminus \bigcup_{\alpha \in \Lambda} V_\alpha \right) = f(f^{-1}(J)) = J \in \mathcal{J}. Since f(A) \text{ is closed in } Y, \text{ there is } \Lambda_0 \subseteq \Lambda, \text{ finite, with } f(A) \setminus \bigcup_{\alpha \in \Lambda_0} f(V_\alpha) \in \mathcal{J}. Given that f \text{ is closed,}
\end{align*}
\(f(V_\alpha) \subseteq f(\overline{V_\alpha})\) for all \(\alpha \in \Lambda_0\), and so
\begin{align*}
f\left( A \setminus \bigcup_{\alpha \in \Lambda_0} \overline{V_\alpha} \right) &= f(A) \setminus \bigcup_{\alpha \in \Lambda_0} f(\overline{V_\alpha}) \in \mathcal{J}. Then A \setminus \bigcup_{\alpha \in \Lambda_0} \overline{V_\alpha} = f^{-1}\left( f\left( A \setminus \bigcup_{\alpha \in \Lambda_0} \overline{V_\alpha} \right) \right) \in f^{-1}(J).
\end{align*}

Next we consider some special subsets of \(\rho C(\mathcal{I})\)-compact spaces.

**Definition 3.2** If \((X, \tau, \mathcal{I})\) is an ideal space and \(A \subseteq X\), \(A\) is said to be \(\rho C(\mathcal{I})\)-compact if for each \(F \subseteq A\), closed in \(A\), and for each family \(\{V_\alpha\}_{\alpha \in \Lambda}\) of open subsets of \(X\), if \(F \setminus \bigcup_{\alpha \in \Lambda} V_\alpha \in \mathcal{I}\), there exists \(\Lambda_0 \subseteq \Lambda\), finite, with \(F \setminus \bigcup_{\alpha \in \Lambda_0} V_\alpha \in \mathcal{I}\).

**Example 3.4** In the following ideal spaces, each subset it is \(\rho C(\mathcal{I})\)-compact.
1) \((X, \tau, \mathcal{P}(X))\), where \((X, \tau)\) is any topological space.
2) \((X, \beta, \mathcal{I})\), where \(X\) is an infinite set, \(\beta\) is the cofinite topology on \(X\), and \(\mathcal{I}\) is any ideal in \(X\).
3) \((\mathbb{Z}, \tau, \mathcal{I})\), where \(\mathcal{I} = \mathcal{P}(2\mathbb{Z} + 1)\) and \(\tau\) is the topology on \(\mathbb{Z}\) given by: \(V \in \tau \iff \text{for each } n \in \mathbb{Z}, \text{ if } n \in V \text{ then } [n]_2 \in V\). Here \([n]_2 = \left\{ \begin{array}{ll} 0 & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd} \end{array} \right.\).

**Theorem 3.5** 1) If \((X, \tau, \mathcal{I})\) is \(\rho C(\mathcal{I})\)-compact and \(A \subseteq X\) is closed, then \(A\) is \(\rho C(\mathcal{I})\)-compact.
2) If \((X, \tau, \mathcal{I})\) is an ideal space and \(A_1 \subseteq X\) and \(A_2 \subseteq X\) are \(\rho C(\mathcal{I})\)-compact, then \(A_1 \cup A_2\) is \(\rho C(\mathcal{I})\)-compact.

**Proof.** 1) It is clear because if \(B\) is closed in \(A\), then \(B\) is closed in \(X\).
2) Suppose that \(B\) is closed in \(A_1 \cup A_2\), and that \(\{V_\alpha\}_{\alpha \in \Lambda}\) is a family of open subsets of \(X\) with \(B \setminus \bigcup_{\alpha \in \Lambda} V_\alpha \in \mathcal{I}\). There exists \(G \subseteq X\), closed, such that
\begin{align*}
B &= (A_1 \cup A_2) \cap G = (A_1 \cap G) \cup (A_2 \cap G). Since A_i \cap G \text{ is closed in } A_i \text{ and } (A_i \cap G) \setminus \bigcup_{\alpha \in \Lambda} V_\alpha \in \mathcal{I}, \text{ for each } i \in \{1, 2\}, \text{ there exists } \Lambda_i \subseteq \Lambda, \text{ finite, with } (A_i \cap G) \setminus \bigcup_{\alpha \in \Lambda_i} \overline{V_\alpha} \in \mathcal{I}, \text{ for each } i \in \{1, 2\}.
\end{align*}
Thus \((A_1 \cap G) \setminus \bigcup_{\alpha \in \Lambda_1 \cup \Lambda_2} V_\alpha \in \mathcal{I}\), and \([ (A_1 \cup A_2) \cap G ] \setminus \bigcup_{\alpha \in \Lambda_1 \cup \Lambda_2} \overline{V}_\alpha \in \mathcal{I}\), this is \(B \setminus \bigcup_{\alpha \in \Lambda_1 \cup \Lambda_2} \overline{V}_\alpha \in \mathcal{I}\).

In the next result we present a new characterization of \(\rho C(\mathcal{I})\)-compactness, in terms of some special open subsets.

**Definition 3.3** If \((X, \tau, \mathcal{I})\) is an ideal space and \(Y \subseteq X\), then \(Y\) is closure \(\rho C(\mathcal{I})\)-compact if for all \(K \subseteq Y\), closed in \(Y\), and all family \(\{V_\alpha\}_{\alpha \in \Lambda}\) of open subsets of \(X\), if \(K \setminus \bigcup_{\alpha \in \Lambda} V_\alpha \in \mathcal{I}\), there exists \(\Lambda_0 \subseteq \Lambda\), finite, with \(K \setminus \bigcup_{\alpha \in \Lambda_0} \text{adh}_\tau (V_\alpha \cap Y) \in \mathcal{I}\).

**Example 3.5** Let \(\mathcal{U}\) the usual topology for \(X = [0, 1]\), \(Y = (0, 1]\) and \(K \subseteq Y\), closed in \(Y\).

(i) Suppose that \(\{V_\alpha\}_{\alpha \in \Lambda}\) is a \(\mathcal{U}\)-open cover of \(K\). Since \(K\) is compact in \(X\), there exists \(\Lambda_0 \subseteq \Lambda\), finite, with \(K \subseteq \bigcup_{\alpha \in \Lambda_0} V_\alpha\), and so \(K \subseteq \bigcup_{\alpha \in \Lambda_0} (V_\alpha \cap Y)\).

But, for all \(\alpha \in \Lambda_0\), \(\text{adh}_{\mathcal{U}} (V_\alpha \cap Y) = \overline{V}_\alpha \cap Y \cap Y = \overline{V}_\alpha \cap Y\), because \(Y\) is open. Therefore \(Y\) is closure \(\rho C(\{\emptyset\})\)-compact.

(ii) \(Y\) is not \(\rho C(\{\emptyset\})\)-compact, because \(Y \subseteq \bigcup_{0 < r_i < 1} (r, 1]\), but if \(0 < r_1 < r_2 < \cdots < r_n < 1\) then \(Y \notin \bigcup_{i=1}^{n} (r_i, 1] = \bigcup_{i=1}^{n} [r_i, 1] = [r_1, 1]\).

**Theorem 3.6** The ideal space \((X, \tau, \mathcal{I})\) is \(\rho C(\mathcal{I})\)-compact if and only if each \(Y \in \tau\) is closure \(\rho C(\mathcal{I})\)-compact.

**Proof.** \((\Rightarrow)\) Suppose that \((X, \tau, \mathcal{I})\) is \(\rho C(\mathcal{I})\)-compact and that \(Y \in \tau\).

Let \(K \subseteq Y\), closed in \(Y\), and \(\{V_\alpha\}_{\alpha \in \Lambda}\) a family of open subsets of \(X\) with \(K \setminus \bigcup_{\alpha \in \Lambda} V_\alpha \in \mathcal{I}\). Since \(K\) is closed in \(X\) and \((X, \tau, \mathcal{I})\) is \(\rho C(\mathcal{I})\)-compact, there exists \(\Lambda_0 \subseteq \Lambda\), finite, with \(K \setminus \bigcup_{\alpha \in \Lambda_0} V_\alpha \in \mathcal{I}\), and so \(K \setminus \bigcup_{\alpha \in \Lambda_0} \overline{V}_\alpha \in \mathcal{I}\). Given that \(Y\) is open in \(X\), \(\text{adh}_\tau (V_\alpha \cap Y) = \overline{V}_\alpha \cap Y\), for all \(\alpha \in \Lambda_0\).

But \(K \setminus \bigcup_{\alpha \in \Lambda_0} (V_\alpha \cap Y) = K \setminus \bigcup_{\alpha \in \Lambda_0} \overline{V}_\alpha \in \mathcal{I}\).

Thus \(K \setminus \bigcup_{\alpha \in \Lambda_0} \text{adh}_\tau (V_\alpha \cap Y) \in \mathcal{I}\).

\((\Leftarrow)\) Suppose that \(F\) is closed in \(X\), and that \(\{V_\alpha\}_{\alpha \in \Lambda}\) is a family of open subsets of \(X\) with \(F \setminus \bigcup_{\alpha \in \Lambda} V_\alpha \in \mathcal{I}\). Let \(\alpha_0 \in \Lambda\). The set \(Y = X \setminus \overline{V}_{\alpha_0}\) is open in \(X\) and \(F \cap Y\) is closed in \(Y\).
Since $F \cap Y \subseteq F$ we have that $F \cap Y \setminus \bigcup_{\alpha \in \Lambda} V_{\alpha} \subseteq F \setminus \bigcup_{\alpha \in \Lambda} V_{\alpha}$. Thus there exists $\Lambda_0 \subseteq \Lambda \setminus \{\alpha_0\}$, finite, such that $(F \cap Y) \setminus \bigcup_{\alpha \in \Lambda_0} \text{adh}_{\tau_Y} (V_{\alpha} \cap Y) \in I$.

Given that $Y \in \tau$, $\text{adh}_{\tau_Y} (V_{\alpha} \cap Y) = \overline{V_{\alpha}} \cap Y \subseteq \overline{V_{\alpha}}$, and so $(F \cap Y) \setminus \bigcup_{\alpha \in \Lambda_0} \overline{V_{\alpha}} \in I$.

Therefore $F \setminus \bigcup_{\alpha \in \Lambda_0 \cup \{\alpha_0\}} \overline{V_{\alpha}} \in I$.

Finally, we show an additional characterization of $\rho C(I)$- compactness, by means of pre-open and $\alpha$-open subsets.

**Theorem 3.7** If $(X, \tau, I)$ is an ideal space, the following statements are equivalents:

1) $(X, \tau, I)$ is $\rho C(I)$-compact.

2) For each $F \subseteq X$, closed, and each family $\{V_{\alpha}\}_{\alpha \in \Lambda}$ of pre-open subsets of $X$, if $F \setminus \bigcup_{\alpha \in \Lambda} V_{\alpha} \in I$ then there exists $\Lambda_0 \subseteq \Lambda$, finite, with $F \setminus \bigcup_{\alpha \in \Lambda_0} \overline{V_{\alpha}} \in I$.

3) For each $F \subseteq X$, closed, and each family $\{V_{\alpha}\}_{\alpha \in \Lambda}$ of $\alpha$-open subsets of $X$, if $F \setminus \bigcup_{\alpha \in \Lambda} V_{\alpha} \in I$ then there exists $\Lambda_0 \subseteq \Lambda$, finite, with $F \setminus \bigcup_{\alpha \in \Lambda_0} \overline{V_{\alpha}} \in I$.

**Proof.** It is sufficient to show that 1) $\Rightarrow$ 2), since open $\Rightarrow$ $\alpha$-open $\Rightarrow$ pre-open.

1) $\Rightarrow$ 2) Suppose that $F \subseteq X$ is closed and that $\{V_{\alpha}\}_{\alpha \in \Lambda}$ is a family of pre-open subsets of $X$, with $F \setminus \bigcup_{\alpha \in \Lambda} V_{\alpha} \in I$. Given that $V_{\alpha} \subseteq \overline{V_{\alpha}}$, for each $\alpha \in \Lambda$, we have that $F \setminus \bigcup_{\alpha \in \Lambda} \overline{V_{\alpha}} \subseteq F \setminus \bigcup_{\alpha \in \Lambda} \overline{V_{\alpha}} \in I$, and then there exists $\Lambda_0 \subseteq \Lambda$, finite, such that $F \setminus \bigcup_{\alpha \in \Lambda_0} \overline{V_{\alpha}} \in I$. But $\overline{V_{\alpha}} \subseteq \overline{V_{\alpha}}$, for all $\alpha \in \Lambda_0$. Thus $F \setminus \bigcup_{\alpha \in \Lambda_0} \overline{V_{\alpha}} \in I$.

**References**


