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A SIMPLE AND ACCURATE METHOD FOR DETERMINATION OF SOME GENERALIZED SEQUENCE OF NUMBERS

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Abstract: In this paper we show how the action of the symmetrizing endomorphism operator $\delta_{e_1e_2}^{-k}$ to the series $\sum\limits_{j=0}^{\infty} a_j e_1^j z^j$ allows us to obtain an alternative approach for the determination of some generalized sequence of numbers.

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1. Introduction and the Main Result

The second-order linear recurrence sequence $(U_n(a, b; p, q))_{n\geq 0}$, or briefly $(U_n)_{n\geq 0}$, is defined by

$$U_{n+2} = aU_{n+1} + bU_n,$$

where $U_0 = \alpha$, $U_1 = \beta$ and $n > 0$. This sequence was introduced in 1965 by Horadam [6, 7] which generalizes many sequences (see [8]). Some examples of such sequences are Lucas number sequences $(L_n)_{n\geq 0}$ for parameters $a = b = \beta = 1$, $\alpha = 2$; Pell number sequences $(P_n)_{n\geq 0}$ and Pell-Lucas number sequences

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$(Q_n)_{n\geq 0}$ for parameters $a = 2$, $b = \beta = 1$, $\alpha = 0$ and $a = \alpha = \beta = 2$, $b = 1$, respectively. In this paper, we show that the use of the action of the symmetric endomorphism operator $\delta_{e_1e_2}^{-k}$ [3] to the series $\sum\limits_{j=0}^{\infty} a_j e_1^j z^j$, gives an alternative approach for determining the generating functions of some sequences of numbers preceded cited.

Definition 1. [1] Let A and B be any two alphabets. Then we define $S_j(A-B)$ by the following form:

$$\frac{\Pi_{b\epsilon B}(1-zb)}{\Pi_{a\epsilon A}(1-za)} = \sum_{j=0}^{\infty} S_j(A-B)z^j, \qquad (1.1)$$

with the condition $S_j(A-B) = 0$ for $j < 0$ (see [1]).

Corollary 2. *[3] By taking $A = 0$ in (1.1), we obtain*

$$\Pi_{b\epsilon B}(1-zb) = \sum_{j=0}^{\infty} S_j(-B)z^j. \qquad (1.2)$$

Definition 3. [11] Given a function g on $\mathbb{R}^n$, the divided difference operator is defined as follows:

$$\partial_{x_ix_{i+1}}(g) = \frac{g(x_1,\cdots,x_i,x_{i+1},\cdots x_n) - g(x_1,\cdots x_{i-1},x_{i+1,}x_i,x_{i+2}\cdots x_n)}{x_i - x_{i+1}}.$$

Definition 4. [2] Given an alphabet $E = \{e_1, e_2\}$,the symmetrizing operator $\delta_{e_1e_2}^k$ is defined by

$$\delta_{e_1e_2}^k(e_1^j) = \frac{e_1^{k+j} - e_2^{k+j}}{e_1 - e_2} = S_{k+j-1}(e_1+e_2), \text{ for all } k, j \in \mathbb{N}.$$

Definition 5. [3] Given a function f, the symmetrizing operator $\delta_{e_1e_2}^{-k}$ is defined by

$$\delta_{e_1e_2}^{-k} f(e_1) = \frac{e_2^k f(e_1) - e_1^k f(e_2)}{(e_1e_2)^k(e_1-e_2)}, \text{ for all } k \in \mathbb{N}.$$

Lemma 6. *[3] Let $E = \{e_1, e_2\}$, we define the operator $\delta_{e_1e_2}^{-k}$ as follows:*

$$\delta_{e_1e_2}^{-k} f\ (e_1) = \frac{-S_{k-1}(E)}{e_1^k e_2^k} f(e_1) + \frac{e_1^k}{e_1^k e_2^k}\partial_{e_1e_2} f(e_1), \textit{ for all } k \in \mathbb{N}.$$

In the next Theorem, we will combine all these results in a unified way such that all these obtained results can be treated as special case of the following Theorem.

Theorem 7. *Given an alphabet* $E = \{e_1, e_2\}$*, two sequences* $\sum_{j=0}^{+\infty} a_j z^j$, $\sum_{j=0}^{+\infty} b_j z^j$ *such that* $\left(\sum_{j=0}^{+\infty} a_j z^j\right)\left(\sum_{j=0}^{+\infty} b_j z^j\right) = 1$*, then*

$$\frac{\sum\limits_{j=0}^{\infty} b_j \; \delta_{e_1 e_2}^{k}(e_1^j) z^j}{\left(\sum\limits_{j=0}^{\infty} b_j e_1^j z^j\right)\left(\sum\limits_{j=0}^{\infty} b_j e_2^j z^j\right)} = \sum_{j=0}^{k-1} a_j e_1^j e_2^j \delta_{e_1 e_2}^{k-j}(e_1) z^j - e_1^k e_2^k z^{k+1} \sum_{j=0}^{\infty} a_{j+k+1} \delta_{e_1 e_2}^{j}(e_1) z^j. \quad (1.3)$$

The paper is organized as follows. In Section 2 we give the proof of Theorem 7 and in Section 3 we give some applications for Theorem 7.

2. Proofs

In this section, we present a proof of Theorem 7.

On one hand, since $f(e_1) = \frac{1}{\sum\limits_{j=0}^{\infty} b_j e_1^j z^j}$, we have

$$\begin{aligned}
\partial_{e_1 e_2} f(e_1) =& \frac{1}{e_1 - e_2}\left(\frac{1}{\sum\limits_{j=0}^{\infty} b_j e_1^j z^j} - \frac{1}{\sum\limits_{j=0}^{\infty} b_j e_2^j z^j}\right) \\
=& \frac{\sum\limits_{j=0}^{\infty} b_j e_2^j z^j - \sum\limits_{j=0}^{\infty} b_j e_1^j z^j}{(e_1 - e_2)\left(\sum\limits_{j=0}^{\infty} b_j e_1^j z^j\right)\left(\sum\limits_{j=0}^{\infty} b_j e_2^j z^j\right)} \\
=& -\frac{\sum\limits_{j=0}^{\infty} b_j \frac{e_1^j - e_2^j}{e_1 - e_2} z^j}{\left(\sum\limits_{j=0}^{\infty} b_j e_1^j z^j\right)\left(\sum\limits_{j=0}^{\infty} b_j e_2^j z^j\right)}.
\end{aligned}$$

By Lemm 6, it follows that

$$\frac{-S_{k-1}(E)}{e_1^k e_2^k} f(e_1) + \frac{e_1^k}{e_1^k e_2^k} \partial_{e_1 e_2} f(e_1)$$

$$= \frac{-S_{k-1}(E)}{e_1^k e_2^k} \frac{1}{\sum\limits_{j=0}^{\infty} b_j e_1^j z^j} + \frac{e_1^k}{e_1^k e_2^k} \partial_{e_1 e_2} \frac{1}{\sum\limits_{j=0}^{\infty} b_j e_1^j z^j}$$

$$= \frac{1}{e_1^k e_2^k} \left(\frac{-\sum\limits_{j=0}^{\infty} b_j e_2^j S_{k-1}(E) z^j + \frac{e_1^k}{e_1 - e_2} \sum\limits_{j=0}^{\infty} b_j \left(e_2^j - e_1^j\right) z^j}{\left(\sum\limits_{j=0}^{\infty} b_j e_1^j z^j\right)\left(\sum\limits_{j=0}^{\infty} b_j e_2^j z^j\right)} \right)$$

$$= \frac{-1}{e_1^k e_2^k} \left(\frac{\sum\limits_{j=0}^{\infty} b_j \left(e_2^j \frac{e_1^k - e_2^k}{e_1 - e_2} + e_1^k \frac{e_1^j - e_2^j}{e_1 - e_2}\right) z^j}{\left(\sum\limits_{j=0}^{\infty} b_j e_1^j z^j\right)\left(\sum\limits_{j=0}^{\infty} b_j e_2^j z^j\right)} \right)$$

$$= \frac{-1}{e_1^k e_2^k} \left(\frac{\sum\limits_{j=0}^{\infty} b_j \left(\frac{e_1^{k+j} - e_2^{k+j}}{e_1 - e_2}\right) z^j}{\left(\sum\limits_{j=0}^{\infty} b_j e_1^j z^j\right)\left(\sum\limits_{j=0}^{\infty} b_j e_2^j z^j\right)} \right)$$

$$= \frac{-1}{e_1^k e_2^k} \left(\frac{\sum\limits_{j=0}^{\infty} b_j \ \delta_{e_1 e_2}^k (e_1^j) z^j}{\left(\sum\limits_{j=0}^{\infty} b_j e_1^j z^j\right)\left(\sum\limits_{j=0}^{\infty} b_j e_2^j z^j\right)} \right),$$

which is the left hand side of (1.3).

On the other hand, since $f(e_1) = \sum\limits_{j=0}^{\infty} a_j e_1^j z^j$, we have that

$$\begin{aligned} \delta_{e_1 e_2}^{-k} f(e_1) &= \delta_{e_1 e_2}^{-k} \left(\sum_{j=0}^{\infty} a_j e_1^j z^j \right) \\ &= \frac{e_2^k \sum\limits_{j=0}^{\infty} a_j e_1^j z^j - e_1^k \sum\limits_{j=0}^{\infty} a_j e_2^j z^j}{e_1^k e_2^k (e_1 - e_2)} \end{aligned}$$

$$= \frac{1}{e_1^k e_2^k} \left(\sum_{j=0}^{\infty} a_j \frac{e_2^k e_1^j - e_1^k e_2^j}{e_1 - e_2} z^j \right).$$

Hence, we have that

$$\begin{aligned} \delta_{e_1e_2}^{-k} f(e_1) &= \frac{1}{e_1^k e_2^k} \left(\sum_{j=0}^{k-1} a_j \frac{e_2^k e_1^j - e_1^k e_2^j}{e_1 - e_2} z^j + \sum_{j=k+1}^{\infty} a_j \frac{e_2^k e_1^j - e_1^k e_2^j}{e_1 - e_2} z^j \right) \\ &= \frac{-1}{e_1^k e_2^k} \left(\sum_{j=0}^{k-1} a_j e_1^j e_2^j \delta_{e_1e_2}^{k-j}(e_1) z^j - e_1^k e_2^k z^{k+1} \sum_{j=0}^{\infty} a_{j+k+1} \delta_{e_1e_2}^{j}(e_1) z^j \right). \end{aligned}$$

This completes the proof.

3. Applications

In this part, we derive the new generating functions of some knowns polynomials. Indeed, we consider Theorem7 in order to derive Fibonacci numbers, Pell-Lucas numbers and Tchebychev polynomials of second kind with $k = 1$, $k = 2$ and $k = 3$, for the case $\frac{1}{1-z} = \sum_{j=0}^{\infty} z^j$.

Corollary 8. *Given an alphabet $E = \{e_1, e_2\}$, we have*

$$\sum_{j=0}^{\infty} S_j(e_1 + e_2) z^j = \frac{1}{(1 - e_1 z)(1 - e_2 z)} \qquad \text{(see [5])}. \tag{3.1}$$

Corollary 9. *Given an alphabet $E = \{e_1, e_2\}$, we have*

$$\sum_{j=0}^{\infty} S_{j+1}(e_1 + e_2) z^j = \frac{e_1 + e_2 - e_1 e_2 z}{(1 - e_1 z)(1 - e_2 z)} \qquad \text{(see [4])} \tag{3.2}$$

Corollary 10. *Given an alphabet $E = \{e_1, e_2\}$, we have*

$$\sum_{j=0}^{\infty} S_{j+2}(e_1 + e_2) z^j = \frac{(e_1 + e_2)^2 - e_1 e_2 - e_1 e_2 (e_1 + e_2) z}{(1 - z e_1)(1 - z e_2)}. \tag{3.3}$$

This case consists of two related parts. Firstly, replacing e_2 by $(-e_2)$ in (3.1), (3.2) and (3.3), we obtain

$$\sum_{j=0}^{\infty} S_j(e_1+[-e_2])z^j = \frac{1}{1-(e_1-e_2)z-e_1e_2z^2}, \tag{3.4}$$

$$\sum_{j=0}^{\infty} S_{j+1}(e_1+[-e_2])z^j = \frac{e_1-e_2+e_1e_2z}{1-(e_1-e_2)z-e_1e_2z^2}, \tag{3.5}$$

$$\sum_{j=0}^{\infty} S_{j+2}(e_1+[-e_2])z^j = \frac{(e_1-e_2)^2+e_1e_2+e_1e_2(e_1-e_2)z}{1-(e_1-e_2)z-e_1e_2z^2}. \tag{3.6}$$

Choosing e_1 and e_2 such that $\begin{cases} e_1e_2=1, \\ e_1-e_2=1, \end{cases}$ and substituting in (3.4), (3.5) and (3.6) we get

$$\sum_{j=0}^{\infty} S_j(e_1+[-e_2])z^j = \sum_{j=0}^{\infty} F_jz^j = \frac{1}{1-z-z^2}, \tag{3.7}$$

$$\sum_{j=0}^{\infty} S_{j+1}(e_1+[-e_2])z^j = (1+z)\sum_{j=0}^{\infty} F_jz^j = \frac{1+z}{1-z-z^2}, \tag{3.8}$$

$$\sum_{j=0}^{\infty} S_{j+2}(e_1+[-e_2])z^j = (2+z)\sum_{j=0}^{\infty} F_jz^j = \frac{2+z}{1-z-z^2}, \tag{3.9}$$

which is given by Boussayoud et al. [2, 4]. Formula (3.9) represents a new generating function.

Corollary 11. *For all $j \in \mathbb{N}$ we have*

$$\begin{aligned} S_{j+1}(e_1+[-e_2]) &= (1+z)S_j(e_1+[-e_2]), \\ S_{j+2}(e_1+[-e_2]) &= (2+z)S_j(e_1+[-e_2]). \end{aligned}$$

Choosing e_1 and e_2 such that $\begin{cases} e_1e_2=1, \\ e_1-e_2=2, \end{cases}$ and substituting in (3.4), (3.5) and (3.6) we obtain three new generating functions given by

$$\sum_{j=0}^{\infty} S_j(e_1+[-e_2])z^j = \frac{1}{1-2z-z^2}, \tag{3.10}$$

$$\sum_{j=0}^{\infty} S_{j+1}(e_1+[-e_2])z^j = \frac{2+z}{1-2z-z^2}, \tag{3.11}$$

$$\sum_{j=0}^{\infty} S_{j+2}\left(e_1+[-e_2]\right) z^j = \frac{5+2z}{1-2z-z^2}. \tag{3.12}$$

Multiplying the equation (3.7) by (3) and subtract it from (3.8) we obtain

$$\sum_{j=0}^{\infty} \left(3S_j(e_1+[-e_2]) - S_{j+1}(e_1+[-e_2])\right) z^j = \sum_{j=0}^{\infty} L_j z^j = \frac{2-z}{1-z-z^2},$$

which represents a generating function for Lucas numbers [13].

Corollary 12. *For all $j \in \mathbb{N}$, we have*

$$L_j = 3S_j(e_1+[-e_2]) - S_{j+1}(e_1+[-e_2]), \text{ with } e_1 = \frac{1+\sqrt{5}}{2}, \; e_2 = \frac{1+\sqrt{5}}{2}$$

Multiplying the equation (3.10) by (-2) and added to (3.11) we obtain

$$\sum_{j=0}^{\infty} \left(S_{j+1}(e_1+[-e_2]) - 2S_j(e_1+[-e_2])\right) z^j = \frac{z}{1-2z-z^2},$$

which represents a generating function for Pell numbers [13].

Corollary 13. *For all $j \in \mathbb{N}$, we have*

$$P_j = S_{j+1}(e_1+[-e_2]) - 2S_j(e_1+[-e_2]), \text{ with } e_1 = 1+\sqrt{2}, \; e_2 = 1-\sqrt{2}$$

Multiplying the equation (3.10) by 6 and added to (3.11) by (-2), we obtain

$$\sum_{j=0}^{\infty} \left(6S_j(e_1+[-e_2]) - 2S_{j+1}(e_1+[-e_2])\right) z^j = \frac{2-2z}{1-2z-z^2},$$

which represents a generating function for Pell-Lucas numbers [12].

Corollary 14. *For all $j \in \mathbb{N}$, we have*

$$Q_j = 6S_j(e_1+[-e_2]) - 2S_{j+1}(e_1+[-e_2]), \text{ with } e_1 = 1+\sqrt{2}, \; e_2 = 1-\sqrt{2}$$

Finally, replacing e_1 by $2e_1$ and e_2 by $(-2e_2)$ in (3.4), (3.5) and (3.6), under the condition $4e_1e_2 = -1$, we obtain, for $x = e_1 - e_2$ the following

$$\sum_{j=0}^{\infty} S_j(2e_1+[-2e_2]) z^j = \frac{1}{1-2xz+z^2}, \text{ such that } U_j(x) = S_j(2e_1+[-2e_2]).$$

$$\sum_{j=0}^{+\infty} S_{j+1}(2e_1 + [-2e_2])z^j = \frac{2x - z}{1 - 2xz + z^2}, \tag{3.12}$$

which represents a generating function [4] such that

$$S_{j+1}(2e_1 + [-2e_2]) = (2x - z)U_j(e_1 - e_2),$$

$$\sum_{j=0}^{\infty} S_{j+2}\left(e_1 + [-e_2]\right) z^j = \frac{4x^2 - 1 - 2xz}{1 - 2xz + z^2}, \tag{3.13}$$

which represents a new generating function.

Moreover, from (3.12) and (3.13) we deduce that

$$\sum_{j=0}^{+\infty} \left[S_{j+1}(2e_1 + [-2e_2]) - xS_j(2e_1 + [-2e_2])\right] z^j = \frac{x - z}{1 - 2xz + z^2}, \tag{3.14}$$

$$\sum_{j=0}^{+\infty} \left[S_{j+2}(2e_1 + [-2e_2]) - xS_{j+1}(2e_1 + [-2e_2])\right] z^j = \frac{3x^2 - 1 - xz}{1 - 2xz + z^2}, \tag{3.15}$$

which represent two new generating functions.

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