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DIFFERENTIAL EQUATIONS ASSOCIATED WITH DEGENERATE BELL POLYNOMIALS

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Abstract: The degenerate Bell numbers and polynomials, recently introduced by Kim and Kim, turned out to be very useful for studying special numbers and polynomials and mathematical physics. In this paper, we derive a family of linear differential equations satisfied by the generating function of the degenerate Bell polynomials and give some explicit and interesting identities for those polynomials arising from the linear differential equations.

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1. Introduction

As is well known, the Bell polynomials (also called the exponential polynomials)

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are given by the generating function

$$e^{x\left(e^{t}-1\right)}=\sum_{n=0}^{\infty}\mathrm{Bel}_{n}\left(x\right)\frac{t^{n}}{n!},\quad\left(\text{see }[1,\ 4,\ 3,\ 2,\ 9,\ 8,\ 13]\right).\tag{1}$$

From (1), we note that $\mathrm{Bel}_0(x)=1$, $\mathrm{Bel}_1(x)=x$, $\mathrm{Bel}_2(x)=x^2+x$, $\mathrm{Bel}_3(x)=x^3+3x^2+x$, $\mathrm{Bel}_4(x)=x^4+6x^3+7x^2+x$, ...

The Stirling numbers of the first kind are defined by

$$\begin{aligned}(x)_n&=x(x-1)\cdots(x-n+1)\\&=\sum_{l=0}^{n}S_1(n,l)\,x^l,\quad(n\geq 0),\end{aligned}\tag{2}$$

and the Stirlings number of the second kind are given by

$$x^n=\sum_{l=0}^{n}S_2(n,l)\,(x)_l\,,\quad(n\geq 0),$$

$$\left(\text{see }[1,\ 4,\ 3,\ 2,\ 5,\ 6,\ 7,\ 9,\ 8,\ 10,\ 11,\ 12,\ 13]\right)\tag{3}$$

From (2) and (3), we note that the generating functions of Stirling numbers are given by

$$\left(e^t-1\right)^n=n!\sum_{l=n}^{\infty}S_2(l,n)\frac{t^l}{l!},\quad(n\geq 0),\tag{4}$$

and

$$\left(\log(1+t)\right)^n=n!\sum_{l=n}^{\infty}S_1(n,l)\frac{t^l}{l!},\quad\left(\text{see }[8,\ 10,\ 11,\ 12,\ 13]\right).\tag{5}$$

By (1), we get

$$\mathrm{Bel}_k(x)=e^{-x}\sum_{l=0}^{\infty}\frac{x^l}{l!}l^k,\quad(k\geq 0),\tag{6}$$

and

$$\mathrm{Bel}_n(x)=\sum_{l=0}^{n}S_2(n,l)\,x^l,\quad(n\geq 0).$$

In [4], L. Carlitz introduced the degenerate Bernoulli and Euler polynomials. Since then, various versions of degenerate special numbers and polynomials have been studied by the several authors (see [1, 4, 3, 2, 5, 6, 7, 9, 8, 10, 11, 12, 13]). The degenerate Bell numbers and polynomials, recently introduced by Kim and Kim (see [9]), turned out to be very useful for studying special numbers and polynomials and mathematical physics. In this paper, we derive a family of linear differential equations satisfied by the generating function of the degenerate Bell polynomials and give some explicit and interesting identities for those polynomials arising from the linear differential equations.

2. Differential Equations Associated with Bell Polynomials

Let

$$\begin{aligned} F &= F\left(t;x,\lambda\right) \\ &= \left(1+\lambda\right)^{\frac{x}{\lambda}\left(\left(1+\lambda t\right)^{\frac{1}{\lambda}}-1\right)} \\ &= e^{\frac{\log(1+\lambda)}{\lambda}x\left(\left(1+\lambda t\right)^{\frac{1}{\lambda}}-1\right)} \\ &= e^{\alpha x\left(\left(1+\lambda t\right)^{\frac{1}{\lambda}}-1\right)}, \end{aligned} \tag{7}$$

where $\alpha = \frac{\log(1+\lambda)}{\lambda}$.

From (1), we note that

$$F^{(1)} = \frac{d}{dt}F\left(t;x,\lambda\right) = \alpha x\left(1+\lambda t\right)^{\frac{1}{\lambda}-1}F, \tag{8}$$

$$F^{(2)} = \frac{d}{dt}F^{(1)} = \left(\alpha x\left(1-\lambda\right)\left(1+\lambda t\right)^{\frac{1}{\lambda}-2} + \left(\alpha x\right)^2\left(1+\lambda t\right)^{\frac{2}{\lambda}-2}\right)F, \tag{9}$$

and

$$\begin{aligned} F^{(3)} =& \frac{d}{dt}F^{(2)} \\ =& \Big(\alpha x\left(1-\lambda\right)\left(1-2\lambda\right)\left(1+\lambda t\right)^{\frac{1}{\lambda}-3} + 3\left(\alpha x\right)^2\left(1-\lambda\right)\left(1+\lambda t\right)^{\frac{2}{\lambda}-3} \\ &+ \left(\alpha x\right)^3\left(1+\lambda t\right)^{\frac{3}{\lambda}-3}\Big)F. \end{aligned} \tag{10}$$

Continuing this process, we can set

$$\begin{aligned} F^{(N)} &= \left(\frac{d}{dt}\right)^N F\left(t;x,\lambda\right) \\ &= \left(\sum_{i=1}^{N} a_i\left(N,\lambda\right)\left(\alpha x\right)^i\left(1+\lambda t\right)^{\frac{i}{\lambda}-N}\right)F, \end{aligned} \tag{11}$$

where $N = 1, 2, 3, \ldots$.

Let us take the derivative with respect to t of (11). Then we have

$$\begin{aligned} F^{(N+1)} &= \frac{dF^{(N)}}{dt} \\ &= \left(\sum_{i=1}^{N} (i - N\lambda)\, a_i(N,\lambda)\,(\alpha x)^i (1+\lambda t)^{\frac{i}{\lambda}-N-1} \right) F \\ &\quad + \left(\sum_{i=2}^{N+1} a_{i-1}(N,\lambda)\,(\alpha x)^i (1+\lambda t)^{\frac{i}{\lambda}-N-1} \right) F. \end{aligned} \tag{12}$$

On the other hand, by replacing N by $N+1$ in (11), we get

$$F^{(N+1)} = \left(\sum_{i=1}^{N+1} a_i(N+1,\lambda)\,(\alpha x)^i (1+\lambda t)^{\frac{i}{\lambda}-N-1} \right) F. \tag{13}$$

Comparing the coefficients on both sides of (12) and (13), we have

$$a_1(N+1,\lambda) = (1 - N\lambda)\, a_1(N,\lambda), \tag{14}$$

$$a_{N+1}(N+1,\lambda) = a_N(N,\lambda), \tag{15}$$

and

$$a_i(N+1,\lambda) = a_{i-1}(N,\lambda) + (i - N\lambda)\, a_i(N,\lambda), \tag{16}$$

where $2 \le i \le N$.

In addition,

$$\alpha x (1+\lambda t)^{\frac{1}{\lambda}-1} F = F^{(1)} = a_1(1,\lambda)\,(\alpha x)\,(1+\lambda t)^{\frac{1}{\lambda}} F. \tag{17}$$

Thus, by comparing the coefficients on both sides of (17), we get

$$a_1(1,\lambda) = 1. \tag{18}$$

From (14), we note that

$$\begin{aligned} a_1(N+1,\lambda) &= (1-N\lambda)\, a_1(N,\lambda) \\ &= (1-N\lambda)\,(1-(N-1)\lambda)\, a_1(N-1,\lambda) \\ &\vdots \\ &= (1-N\lambda)\,(1-(N-1)\lambda)\cdots(1-\lambda)\, a_1(1,\lambda) \\ &= \langle 1 - N\lambda; \lambda \rangle_N, \end{aligned} \tag{19}$$

where

$$\langle x;\alpha\rangle_N = x\,(x+\alpha)\cdots(x+(N-1)\,\alpha)\,, \quad (N \geq 1)\,,$$
$$\langle x;\alpha\rangle_0 = 1.$$

By (15), we easily get

$$a_{N+1}(N+1,\lambda) = a_N(N,\lambda) = \cdots = a_1(1,\lambda) = 1. \tag{20}$$

For $i = 2, 3, 4$ in (16), we have

$$a_2(N+1,\lambda) = \sum_{k=0}^{N-1} \langle 2 - N\lambda;\lambda\rangle_k\, a_1(N-k,\lambda)\,, \tag{21}$$

$$a_3(N+1,\lambda) = \sum_{k=0}^{N-2} \langle 3 - N\lambda;\lambda\rangle_k\, a_2(N-k,\lambda)\,, \tag{22}$$

and

$$a_4(N+1,\lambda) = \sum_{k=0}^{N-3} \langle 4 - N\lambda;\lambda\rangle_k\, a_3(N-k,\lambda)\,. \tag{23}$$

Continuing this process, we have

$$a_i(N+1,\lambda) = \sum_{k=0}^{N-i+1} \langle i - N\lambda;\lambda\rangle_k\, a_{i-1}(N-k,\lambda)\,, \tag{24}$$

for $2 \leq i \leq N$.

Now, we give explicit expressions for $a_i(N+1,\lambda)$, for $2 \leq i \leq N$.

From (19) and (21), we note that

$$\begin{aligned} a_2(N+1,\lambda) &= \sum_{k_1=0}^{N-1} \langle 2 - N\lambda;\lambda\rangle_{k_1}\, a_1(N-k_1,\lambda) \\ &= \sum_{k_1=0}^{N-1} \langle 2 - N\lambda;\lambda\rangle_{k_1} \langle 1-(N-k_1-1)\,\lambda;\lambda\rangle_{N-k_1-1}\,, \end{aligned} \tag{25}$$

$$a_3(N+1,\lambda) = \sum_{k_2=0}^{N-2} \langle 3 - N\lambda;\lambda\rangle_{k_2}\, a_2(N-k_2,\lambda) \tag{26}$$

$$= \sum_{k_2=0}^{N-2} \langle 3 - N\lambda; \lambda\rangle_{k_2}$$

$$\times \sum_{k_1=0}^{N-2-k_2} \langle 2 - (N - k_2 - 1)\,\lambda; \lambda\rangle_{k_1}$$

$$\times \langle 1 - (N - k_2 - k_1 - 2)\,\lambda; \lambda\rangle_{N-k_2-k_1-2}$$

$$= \sum_{k_2=0}^{N-2} \sum_{k_1=0}^{N-2-k_2} \langle 3 - N\lambda; k_2\rangle \langle 2 - (N - k_2 - 1)\,\lambda; \lambda\rangle_{k_1}$$

$$\times \langle 1 - (N - k_2 - k_1 - 2)\,\lambda; \lambda\rangle_{N-k_2-k_1-2}$$

and

$$a_4\,(N+1,\lambda) = \sum_{k_3=0}^{N-3} \langle 4 - N\lambda; \lambda\rangle_{k_3}\, a_3\,(N - k_3, \lambda) \tag{27}$$

$$= \sum_{k_3=0}^{N-3} \sum_{k_2=0}^{N-3-k_3} \sum_{k_1=0}^{N-3-k_3-k_2} \langle 4 - N\lambda; \lambda\rangle_{k_3}$$

$$\times \langle 3 - (N - k_3 - 1)\,\lambda; \lambda\rangle_{k_2} \langle 2 - (N - k_3 - k_2 - 2)\,\lambda; \lambda\rangle_{k_1}$$

$$\times \langle 1 - (N - k_3 - k_2 - k_1 - 3)\,\lambda; \lambda\rangle_{N-k_3-k_2-k_1-3}\,.$$

So, we can deduce that, for $2 \le i \le N$,

$$a_i\,(N+1,\lambda) = \sum_{k_{i-1}=0}^{N-i+1} \sum_{k_{i-2}=0}^{N-i+1-k_{i-1}} \cdots \sum_{k_1=0}^{N-i+1-k_{i-1}-\cdots-k_2} \tag{28}$$

$$\times \prod_{l=2}^{i} \left\langle l - \left(N - \sum_{j=l}^{i-1} k_j - i + l\right) \lambda; \lambda \right\rangle_{k_{l-1}}$$

$$\times \left\langle 1 - \left(N - \sum_{j=1}^{i-1} k_j - i + 1\right) \lambda; \lambda \right\rangle_{N-\sum_{j=1}^{i-1} k_j - i + 1}.$$

Remark. (28) also holds for $i = N+1$.

Therefore, by (28), we obtain the following theorem.

Theorem 1. *The family of differential equations*

$$F^{(N)} = \left(\sum_{i=1}^{N} a_i\,(N,\lambda) \left(\frac{\log\,(1+\lambda)}{\lambda}\right)^i x^i\,(1+\lambda t)^{\frac{i}{\lambda}-N}\right) F$$

have a solution

$$F = F(t; x, \lambda) = (1+\lambda)^{\frac{x}{\lambda}\left((1+\lambda t)^{\frac{1}{\lambda}}-1\right)},$$

where $a_1(N,\lambda) = \langle 1-(N-1)\lambda;\lambda\rangle_{N-1}$,

$$\begin{aligned} a_i(N,\lambda) = & \sum_{k_{i-1}=0}^{N-i} \sum_{k_{i-2}=0}^{N-i-k_{i-1}} \cdots \sum_{k_1=0}^{N-i-k_{i-1}-\cdots-k_2} \\ & \times \prod_{l=2}^{i} \left\langle l - \left(N - \sum_{j=l}^{i-1} k_j - i - 1 + l\right)\lambda;\lambda\right\rangle_{k_{l-1}} \\ & \times \left\langle 1 - \left(N - \sum_{j=1}^{i-1} k_j - i\right)\lambda;\lambda\right\rangle_{N-\sum_{j=1}^{i-1} k_j - i}, \\ & \text{(for } 2 \le i \le N). \end{aligned}$$

Recall that the degenerate Bell polynomials $\mathrm{Bel}_{n,\lambda}(x)$ are given by the generating function

$$\begin{aligned} F &= F(t;x,\lambda) \\ &= (1+\lambda)^{\frac{x}{\lambda}\left((1+\lambda t)^{\frac{1}{\lambda}}-1\right)} \\ &= \sum_{n=0}^{\infty} \mathrm{Bel}_{n,\lambda}(x)\frac{t^n}{n!}, \quad (\text{see [9]}). \end{aligned} \tag{29}$$

Thus, by (29), we get

$$\begin{aligned} \sum_{n=0}^{\infty} \mathrm{Bel}_{n+N,\lambda}(x)\frac{t^n}{n!} =& F^{(N)} \\ =& \left(\sum_{i=1}^{N} a_i(N,\lambda)\left(\frac{\log(1+\lambda)}{\lambda}x\right)^i (1+\lambda t)^{\frac{i}{\lambda}-N}\right) F \\ =& \sum_{i=1}^{N} a_i(N,\lambda)\left(\frac{\log(1+\lambda)}{\lambda}x\right)^i \left(\sum_{l=0}^{\infty}\left(\frac{i}{\lambda}-N\right)_l \lambda^l \frac{t^l}{l!}\right) \\ & \times \left(\sum_{n=0}^{\infty} \mathrm{Bel}_{m,\lambda}(x)\frac{t^m}{m!}\right) \end{aligned} \tag{30}$$

$$
\begin{aligned}
&= \sum_{i=1}^{N} a_i \left(N, \lambda\right) \left(\frac{\log\left(1+\lambda\right)}{\lambda} x \right)^i \\
&\quad \times \sum_{n=0}^{\infty} \sum_{l=0}^{n} \binom{n}{l} \left(\frac{i}{\lambda} - N \right)_l \lambda^l \mathrm{Bel}_{n-l,\lambda}\left(x\right) \frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty} \left\{ \sum_{i=1}^{N} \sum_{l=0}^{n} \binom{n}{l} \left(\frac{i}{\lambda} - N \right)_l \lambda^l \left(\frac{\log\left(1+\lambda\right)}{\lambda} \right)^i \right. \\
&\quad \left. \times\ a_i\left(N, \lambda\right) x^i \mathrm{Bel}_{n-l,\lambda}\left(x\right) \right\} \frac{t^n}{n!}.
\end{aligned}
$$

Therefore, by comparing the coefficients on the both sides of (30), we obtain the following theorem.

Theorem 2. *For $n = 0, 1, 2, \ldots$, and $N = 1, 2, 3, \ldots$, we have*

$$
\begin{aligned}
\mathrm{Bel}_{n+N,\lambda}\left(x\right) = \sum_{i=1}^{N} \sum_{l=0}^{n} \binom{n}{l} \left(\frac{i}{\lambda} - N \right)_l \lambda^l \left(\frac{\log\left(1+\lambda\right)}{\lambda} \right)^i \\
\times\ a_i\left(N, \lambda\right) x^i \mathrm{Bel}_{n-l,\lambda}\left(x\right),
\end{aligned}
$$

where $a_i\left(N, \lambda\right)$'s are as in Theorem 1.

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