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SUZUKI TYPE UNIQUE COMMON TRIPLED FIXED POINT THEOREM FOR WEAK ϕ-CONTRACTION IN MULTIPLICATIVE METRIC SPACES

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Abstract: In this paper, we obtain a Suzuki type unique common tripled fixed point theorem for weak ϕ-contraction in multiplicative metric spaces.

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1. Introduction

The notion of multiplicative metric was introduced by Bashirov et al. [3] in 2008. After that Ozavsar and Cevikel (see [8]) investigated its topological properties and proved some fixed point theorems. For more works on fixed, common fixed point theorems in Multiplicative metric spaces, we refer to [5, 2, 6, 9].

The coupled fixed point is introduced by Bhaskar and Lakshmikantham, see [4]. Later some of authors proved coupled fixed and coupled common fixed point theorems (see [1, 7, 10, 11, 16]).

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In 2010, a fixed point of order $N \geq 3$ was established by Samet and Vetro, see [12]. Later several authors obtained coincidence and common tripled fixed point theorem in various spaces, for example refer to [13, 14, 15]

The aim of this paper is to prove Suzuki type unique common tripled fixed point theorem for weak ϕ-contraction in Multiplicative metric spaces.

First we give the following theorem of Suzuki [17].

Theorem 1.1. (see [17]) *Let (X, d) be a complete metric space and let T be a mapping on X. Define a non-increasing function $\theta : [0, 1) \to \left(\frac{1}{2}, 1\right]$ by*

$$\theta(r) = \begin{cases} 1 & \text{if } 0 \leq r \leq \frac{(\sqrt{5}-1)}{2}, \\ (1-r)r^{-2} & \text{if } \frac{(\sqrt{5}-1)}{2} \leq r \leq 2^{-\frac{1}{2}}, \\ (1+r)^{-1} & \text{if } 2^{-\frac{1}{2}} \leq r < 1. \end{cases}$$

Assume that there exists $r \in [0, 1)$ such that

$$\theta(r)d(x, Tx) \leq d(x, y) \Rightarrow d(Tx, Ty) \leq rd(x, y)$$

for all $x, y \in X$. Then there exists a unique fixed point z of T. Moreover $\lim_{n\to\infty} T^n x = z$ for all $x \in X$.

Now we recall some basic definitions and examples in multiplicative metric spaces.

Definition 1.2. (see [3]) Multiplicative metric on a nonempty set X is a mapping $d : X \times X \to R^+$ satisfying the following conditions:

(d_1) $d(x, y) \geq 1$ for all $x, y \in X$

(d_2) $d(x, y) = 1$ if and only if $x = y$,

(d_3) $d(x, y) = d(y, x)$,

(d_4) $d(x, y) \leq d(x, z).d(z, y)$ for all $x, y, z \in X$.

The pair (X, d) is called a multiplicative metric space.

Example 1.3. (see [9]) Let R^n_+ be the collection of all n - tupples of positive real numbers. And let

$$d^* : R^n_+ \times R^n_+ \to R$$

be defined as

$$d^*(x, y) = \left|\frac{x_1}{y_1}\right|^* . \left|\frac{x_2}{y_2}\right|^* \cdots \left|\frac{x_n}{y_n}\right|^*$$

where $x = (x_1, x_2 \cdots x_n), y = (y_1, y_2 \cdots y_n) \in R_+^n$ and $|.|^* : R_+ \to R_+$ is defined as follows

$$|a|^* = \begin{cases} a & if \quad a \geq 1 \\ \frac{1}{a} & if \quad a < 1 \end{cases}$$

Then clearly $d^*(x, y)$ is a multiplicative metric.

Example 1.4. (see [9]) Let $d : R \times R \to [1, \infty)$ be defined as $d(x, y) = a^{|x-y|}$ where $x, y \in R$ and $a > 1$. Then $d(x, y)$ is multiplicative metric.

Definition 1.5. (see [8]) Let (X, d) be a multiplicative metric space, x_0 an arbitrary point in X, and $\epsilon > 1$. A multiplicative open ball $B(x_0, \epsilon)$ of radius ϵ centered at x_0 is the set $\{z \in X : d(z, x_0) < \epsilon\}$.

A sequence $\{x_n\}$ in a multiplicative metric space (X, d) is said to be multiplicative convergent to some point $x \in X$ if, for any given $\epsilon > 1$, then there exists $n_0 \in N$ such that $x_n \in B(x, \epsilon)$ for all $n \geq n_0$. If $\{x_n\}$ converges to x, we write $x_n \to x$ as $n \to \infty$.

Definition 1.6. (see [8]) A sequence $\{x_n\}$ in a multiplicative metric space (X, d) is said to be multiplicative convergent to x in X if and only if $d(x_n, x) \to 1$ as $n \to \infty$.

Definition 1.7. (see [8]) Let (X, d_X) and (Y, d_Y) be two multiplicative metric space and x_0 an arbitrary but fixed element of X. A mapping $f : X \to Y$ is said to be multiplicative continuous at x_0 if and only if $x_n \to x_0$ in (X, d_X) implies that $f(x_n) \to f(x_0)$ in (Y, d_Y). That is, given arbitrary $\epsilon > 1$, then there exists $\delta > 1$ which depends on x_0 and ϵ such that $d_Y(fx, fx_0) < \epsilon$ for all those x in X for which $d_X(x, x_0) < \delta$.

Definition 1.8. (see [8]) A sequence $\{x_n\}$ in a multiplicative metric space (X, d) is said to be multiplicative Cauchy sequence if, for any $\epsilon > 1$, there exists $n_0 \in N$ such that $d(x_m, x_n) < \epsilon$ for all $m, n \geq n_0$.

Definition 1.9. (see [8]) A multiplicative metric space (X, d) is said to be complete if every multiplicative Cauchy sequence $\{x_n\}$ in X is multiplicative convergent in X.

Definition 1.10. [4] An element $(x, y) \in X \times X$ is called a coupled fixed point of mapping $F : X \times X \to X$ if $x = F(x, y)$ and $y = F(y, x)$.

Definition 1.11. [1] An element $(x, y) \in X \times X$ is called (g_1) a coupled coincident point of mappings $F : X \times X \to X$ and $f : X \to X$ if $fx = F(x, y)$ and $fy = F(y, x)$.

(g_2) a common coupled fixed point of mappings $F : X \times X \to X$ and $f : X \to X$ if $x = fx = F(x, y)$ and $y = fy = F(y, x)$.

Definition 1.12. [1] The mappings $F : X \times X \to X$ and $f : X \to X$ are called w - compatible if $f(F(x, y)) = F(fx, fy)$ and $f(F(y, x)) = F(fy, fx)$ whenever $fx = F(x, y)$ and $fy = F(y, x)$.

Definition 1.13. [14] An element $(x, y, z) \in X \times X \times X$ is called

(i) a tripled coincident point of $F : X \times X \times X \to X$ and $f : X \to X$ if

$$fx = F(x, y, z),\ fy = F(y, z, x) \text{ and } fz = F(z, x, y),$$

(ii) a common tripled fixed point of $F : X \times X \times X \to X$ and $f : X \to X$ if

$$x = fx = F(x, y, z),\ y = fy = F(y, z, x) \text{ and } z = fz = F(z, x, y).$$

Definition 1.14. [14] The mappings $F : X \times X \times X \to X$ and $f : X \to X$ are called w - compatible if $f(F(x, y, z)) = F(fx, fy, fz)$, $f(F(y, z, x)) = F(fy, fz, fx)$ and$f(F(z, x, y)) = F(fz, fx, fy)$ whenever $fx = F(x, y, z)$, $fy = F(y, z, x)$ and $fz = F(z, x, y)$.

Let Φ be set of all continuous functions $\phi : [1, \infty) \to [1, \infty)$ such that

(i) ϕ is non - decreasing with $\phi(t) < t$

(ii) $\lim_{t \to r^+} \phi(t) < r, t > 1$

Now we prove our main result.

2. Main Result

Theorem 2.1. *Let (X, d) be multiplicative metric space and let $T : X \times X \times X \to X$ and $g : X \to X$ be mappings satisfying*

(2.1.1)

$$\frac{1}{2} \min \left\{ \begin{array}{c} d(gx, T(x, y, z)), \\ d(gy, T(y, z, x)), \\ d(gz, T(z, x, y)) \end{array} \right\} \leq \max \{d(gx, gu), d(gy, gv), d(gz, gw)\}$$

implies that

$$d\left(T(x, y, z), T(u, v, w)\right) \leq$$

$$\phi\left(\max\left\{\begin{array}{c} d(gx,gu),d(gy,gv),d(gz,gw), \\ d(gx,T(x,y,z)),d(gy,T(y,z,x)),d(gz,T(z,x,y)), \\ d(gu,T(u,v,w)),d(gv,T(v,w,u)),d(gw,T(w,u,v)), \\ \sqrt{d(gx,T(u,v,w))},\sqrt{d(gy,T(v,w,u))},\sqrt{d(gz,T(w,u,v))} \end{array}\right\}\right),$$

for all x,y,z,u,v,w in X and $\phi\in\Phi$,

(2.1.2) $T(X\times X\times X)\subseteq g(X)$ and $g(X)$ is a complete subspaces of X,

(2.1.3) the pair (T,g) is w - compatible.

Then T and g have a unique common tripled fixed point in $X\times X\times X$.

Proof. Let x_0, y_0, z_0 be arbitrary points in X.From (2.1.2), there exist sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\}$, $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ in X such that

$$\begin{array}{rcl} gx_{n+1} &=& T(x_n,y_n,z_n), \\ gy_{n+1} &=& T(y_n,z_n,x_n), \\ gz_{n+1} &=& T(z_n,x_n,y_n), \qquad n=0,1,2,\cdots \end{array}$$

Case (a) : If $gx_m = gx_{m+1}$, $gy_m = gy_{m+1}$ and $gz_m = gz_{m+1}$ for some m.

Then (x_m, y_m, z_m) is tripled coincidence point in $X\times X\times X$.

Case (b): Assume $gx_n \neq gx_{n+1}$ or $gy_n \neq gy_{n+1}$ or $gz_n \neq gz_{n+1}$ for all n.

Clearly we have

$$\begin{aligned} \tfrac{1}{2}d(gx_n,T(x_n,y_n,z_n)) &\le d(gx_n,gx_{n+1}) \\ &\le \max\left\{\begin{array}{c} d(gx_n,gx_{n+1}), \\ d(gy_n,gy_{n+1}), \\ d(gz_n,gz_{n+1}) \end{array}\right\}. \end{aligned}$$

Thus

$$\tfrac{1}{2}\min\left\{\begin{array}{c} d(gx_n,T(x_n,y_n,z_n)), \\ d(gy_n,T(y_n,z_n,x_n)), \\ d(gz_n,T(z_n,x_n,y_n)) \end{array}\right\} \le \max\left\{\begin{array}{c} d(gx_n,gx_{n+1}), \\ d(gy_n,gy_{n+1}), \\ d(gz_n,gz_{n+1}) \end{array}\right\}.$$

From (2.1.1) , we get

$$
\begin{aligned}
&d(T(x_n,y_n,z_n),T(x_{n+1},y_{n+1},z_{n+1}))\\
&\le \phi\left(\max\left\{\begin{array}{c}
d(gx_n,gx_{n+1}),d(gy_n,gy_{n+1}),d(gz_n,gz_{n+1}),\\
d(gx_n,T(x_n,y_n,z_n)),d(gy_n,T(y_n,z_n,x_n)),\\
d(gz_n,T(z_n,x_n,y_n)),\\
d(gx_{n+1},T(x_{n+1},y_{n+1},z_{n+1})),\\
d(gy_{n+1},T(y_{n+1},z_{n+1},x_{n+1})),\\
d(gz_{n+1},T(z_{n+1},x_{n+1},y_{n+1})),\\
\sqrt{d(gx_n,T(x_{n+1},y_{n+1},z_{n+1}))},\\
\sqrt{d(gy_n,T(y_{n+1},z_{n+1},x_{n+1}))},\\
\sqrt{d(gz_n,T(z_{n+1},x_{n+1},y_{n+1}))}
\end{array}\right\}\right),\\
&= \phi\left(\max\left\{\begin{array}{c}
d(gx_n,gx_{n+1}),d(gy_n,gy_{n+1}),\\
d(gz_n,gz_{n+1}),d(gx_n,gx_{n+1}),d(gy_n,gy_{n+1}),\\
d(gz_n,gz_{n+1}),d(gx_{n+1},gx_{n+2}),d(gy_{n+1},gy_{n+2}),\\
d(gz_{n+1},gz_{n+2}),\sqrt{d(gx_n,gx_{n+2})},\\
\sqrt{d(gy_n,gy_{n+2})},\sqrt{d(gz_n,gz_{n+2})}
\end{array}\right\}\right),\\
&\le \phi\left(\max\left\{\begin{array}{c}
d(gx_n,gx_{n+1}),d(gy_n,gy_{n+1}),\\
d(gz_n,gz_{n+1}),d(gx_{n+1},gx_{n+2}),d(gy_{n+1},gy_{n+2}),\\
d(gz_{n+1},gz_{n+2}),\sqrt{d(gx_n,gx_{n+2})},\\
\sqrt{d(gy_n,gy_{n+2})},\sqrt{d(gz_n,gz_{n+2})}
\end{array}\right\}\right).
\end{aligned}
$$

But

$$
\begin{aligned}
\sqrt{d(gx_n,gx_{n+2})} &\le \sqrt{d(gx_n,gx_{n+1}).d(gx_{n+1},gx_{n+2})}\\
&\le \max\{d(gx_n,gx_{n+1}),d(gx_{n+1},gx_{n+2})\}.
\end{aligned}
$$

Similarly

$$\sqrt{d(gy_n,gy_{n+2})} \le \max\{d(gy_n,gy_{n+1}),d(gy_{n+1},gy_{n+2})\}$$

and

$$\sqrt{d(gz_n,gz_{n+2})} \le \max\{d(gz_n,gz_{n+1}),d(gz_{n+1},gz_{n+2})\}.$$

Therefore

$$
\begin{aligned}
&d(gx_{n+1},gx_{n+2})\\
&\quad\le \phi\left(\max\left\{\begin{array}{c}
d(gx_n,gx_{n+1}),d(gy_n,gy_{n+1}),d(gz_n,gz_{n+1}),\\
d(gx_{n+1},gx_{n+2}),d(gy_{n+1},gy_{n+2}),d(gz_{n+1},gz_{n+2})
\end{array}\right\}\right).
\end{aligned}
$$

Put $R_n=\max\{d(gx_n,gx_{n+1}),d(gy_n,gy_{n+1}),d(gz_n,gz_{n+1})\}$.

Hence

$$d(gx_{n+1}, gx_{n+2}) \quad \leq \phi\left(\max\{R_n, R_{n+1}\}\right).$$

Similarly

$$d(gy_{n+1}, gy_{n+2}) \quad \leq \phi\left(\max\{R_n, R_{n+1}\}\right)$$

and

$$d(gz_{n+1}, gz_{n+2}) \quad \leq \phi\left(\max\{R_n, R_{n+1}\}\right)$$

Now

$$\begin{aligned} R_{n+1} &= \max\{d(gx_{n+1}, gx_{n+2}), d(gy_{n+1}, gy_{n+2}), d(gz_{n+1}, gz_{n+2})\} \\ &\leq \phi\left(\max\left\{\begin{array}{c} d(gx_n, gx_{n+1}), d(gy_n, gy_{n+1}), d(gz_n, gz_{n+1}), \\ d(gx_{n+1}, gx_{n+2}), d(gy_{n+1}, gy_{n+2}), d(gz_{n+1}, gz_{n+2}) \end{array}\right\}\right) \\ &\leq \phi\left(\max\{R_n, R_{n+1}\}\right). \end{aligned}$$

If R_{n+1} is maximum, we get

$$R_{n+1} \leq \phi\left(R_{n+1}\right) < R_{n+1}$$

which is a contradiction.
Hence R_n is a maximum

$$\begin{aligned} R_{n+1} &\leq \phi\left(R_n\right) \\ &< R_n. \end{aligned} \tag{1}$$

Similarly

$$R_{n+2} < R_{n+1}.$$

Thus $\{R_n\}$ is decreasing sequence of positive real number and must converges to a real number $\delta \geq 1$ (say).

Suppose $\delta > 1$.

Letting $n \to \infty$ in (1), we get

$$\begin{aligned} \delta &\leq \lim_{n\to\infty} \phi\left(R_n\right) \\ &= \lim_{t\to\delta^+} \phi\left(t\right), where \quad t = \lim_{n\to\infty} R_n \\ &< \delta \end{aligned}$$

which is a contradiction.

Hence $\delta = 1$. Thus

$$\lim_{n\to\infty} \max\{d(gx_n, gx_{n+1}), d(gy_n, gy_{n+1}), d(gz_n, gz_{n+1})\} = 1$$

which implies that

$$\lim_{n\to\infty} d(gx_n, gx_{n+1}) = \lim_{n\to\infty} d(gy_n, gy_{n+1}) = \lim_{n\to\infty} d(gz_n, gz_{n+1}) = 1. \quad (2)$$

Now we prove that $\{gx_n\}$, $\{gy_n\}$ and $\{gz_n\}$ are multiplicative Cauchy sequences.

On contrary suppose that $\{gx_n\}$ or $\{gy_n\}$ or $\{gz_n\}$ are not multiplicative Cauchy sequences.

Then there exist an $\epsilon > 1$ and monotone increasing sequence of natural numbers $\{m_k\}$ amd $\{n_k\}$ such that $n_k > m_k > k$,

$$\max\{d(gx_{m_k}, gx_{n_k}), d(gy_{m_k}, gy_{n_k}), d(gz_{m_k}, gz_{n_k})\} \geq \epsilon \quad (3)$$

and

$$\max\{d(gx_{m_k}, gx_{n_{k-1}}), d(gy_{m_k}, gy_{n_{k-1}}), d(gz_{m_k}, gz_{n_{k-1}})\} < \epsilon. \quad (4)$$

From (3) and (4), we have

$$\begin{aligned} \epsilon &\leq \max\{d(gx_{m_k}, gx_{n_k}), d(gy_{m_k}, gy_{n_k}), d(gz_{m_k}, gz_{n_k})\} \\ &\leq \max \left\{ \begin{array}{c} d(gx_{m_k}, gx_{n_{k-1}}).d(gx_{n_{k-1}}, gx_{n_k}), \\ d(gy_{m_k}, gy_{n_{k-1}}).d(gy_{n_{k-1}}, gy_{n_k}), \\ d(gz_{m_k}, gz_{n_{k-1}}).d(gz_{n_{k-1}}, gz_{n_k}) \end{array} \right\}. \end{aligned}$$

Letting $k \to \infty$, we have

$$\epsilon \leq \lim_{k\to\infty} \max\{d(gx_{m_k}, gx_{n_k}), d(gy_{m_k}, gy_{n_k}), d(gz_{m_k}, gz_{n_k})\} \leq \epsilon.$$

Thus

$$\lim_{k\to\infty} \max\{d(gx_{m_k}, gx_{n_k}), d(gy_{m_k}, gy_{n_k}), d(gz_{m_k}, gz_{n_k})\} = \epsilon. \quad (5)$$

Form (3), we have

$$\begin{aligned} \epsilon &\leq \max\{d(gx_{m_k}, gx_{n_k}), d(gy_{m_k}, gy_{n_k}), d(gz_{m_k}, gz_{n_k})\} \\ &\leq \max \left\{ \begin{array}{c} d(gx_{m_k}, gx_{n_{k+1}}).d(gx_{n_{k+1}}, gx_{n_k}), \\ d(gy_{m_k}, gy_{n_{k+1}}).d(gy_{n_{k+1}}, gy_{n_k}), \\ d(gz_{m_k}, gz_{n_{k+1}}).d(gz_{n_{k+1}}, gz_{n_k}) \end{array} \right\} \\ &\leq \max \left\{ \begin{array}{c} d(gx_{m_k}, gx_{n_k}).d(gx_{n_k}, gx_{n_{k+1}}).d(gx_{n_{k+1}}, gx_{n_k}), \\ d(gy_{m_k}, gy_{n_k}).d(gy_{n_k}, gy_{n_{k+1}}).d(gy_{n_{k+1}}, gy_{n_k}), \\ d(gz_{m_k}, gz_{n_k}).d(gz_{n_k}, gz_{n_{k+1}}).d(gz_{n_{k+1}}, gz_{n_k}) \end{array} \right\}. \end{aligned}$$

Letting $k \to \infty$, we have

$$\lim_{k\to\infty} \max \left\{ \begin{array}{c} d(gx_{m_k}, gx_{n_{k+1}}), \\ d(gy_{m_k}, gy_{n_{k+1}}), \\ d(gz_{m_k}, gz_{n_{k+1}}) \end{array} \right\} = \epsilon. \tag{6}$$

From (3), we have

$$\begin{aligned} \epsilon &\leq \max\{d(gx_{m_k}, gx_{n_k}), d(gy_{m_k}, gy_{n_k}), d(gz_{m_k}, gz_{n_k})\} \\ &\leq \max \left\{ \begin{array}{c} d(gx_{m_k}, gx_{m_{k+1}}).d(gx_{m_{k+1}}, gx_{n_k}), \\ d(gy_{m_k}, gy_{m_{k+1}}).d(gy_{m_{k+1}}, gy_{n_k}), \\ d(gz_{m_k}, gz_{m_{k+1}}).d(gz_{m_{k+1}}, gz_{n_k}) \end{array} \right\} \\ &\leq \max \left\{ \begin{array}{c} d(gx_{m_k}, gx_{m_{k+1}}).d(gx_{m_{k+1}}, gx_{n_{k+2}}).d(gx_{n_{k+2}}, gx_{n_k}), \\ d(gy_{m_k}, gy_{m_{k+1}}).d(gy_{m_{k+1}}, gy_{n_{k+2}}).d(gy_{n_{k+2}}, gy_{n_k}), \\ d(gz_{m_k}, gz_{m_{k+1}}).d(gz_{m_{k+1}}, gz_{n_{k+2}}).d(gz_{n_{k+2}}, gz_{n_k}) \end{array} \right\} \\ &\leq \max \left\{ \begin{array}{c} d(gx_{m_k}, gx_{m_{k+1}}).d(gx_{m_{k+1}}, gx_{n_{k+2}}).d(gx_{n_{k+2}}, gx_{n_{k+1}}).d(gx_{n_{k+1}}, gx_{n_k}), \\ d(gy_{m_k}, gy_{m_{k+1}}).d(gy_{m_{k+1}}, gy_{n_{k+2}}).d(gy_{n_{k+2}}, gy_{n_{k+1}}).d(gy_{n_{k+1}}, gy_{n_k}), \\ d(gz_{m_k}, gz_{m_{k+1}}).d(gz_{m_{k+1}}, gz_{n_{k+2}}).d(gz_{n_{k+2}}, gz_{n_{k+1}}).d(gz_{n_{k+1}}, gz_{n_k}) \end{array} \right\} \quad (7) \\ &\leq \max \left\{ \begin{array}{c} (d(gx_{m_k}, gx_{m_{k+1}}))^2.d(gx_{m_k}, gx_{n_k}).(d(gx_{n_{k+2}}, gx_{n_{k+1}}))^2.(d(gx_{n_{k+1}}, gx_{n_k}))^2, \\ (d(gy_{m_k}, gy_{m_{k+1}}))^2.d(gy_{m_k}, gy_{n_k}).(d(gy_{n_{k+2}}, gy_{n_{k+1}}))^2.(d(gy_{n_{k+1}}, gy_{n_k}))^2, \\ (d(gz_{m_k}, gz_{m_{k+1}}))^2.d(gz_{m_k}, gz_{n_k}).(d(gz_{n_{k+2}}, gz_{n_{k+1}}))^2.(d(gz_{n_{k+1}}, gz_{n_k}))^2 \end{array} \right\}. \end{aligned}$$

Letting $k \to \infty$, using (2), (5) and (7), we have

$$\lim_{k\to\infty} \max\{d(gx_{m_k+1}, gx_{n_{k+2}}), d(gy_{m_k+1}, gy_{n_{k+2}}), d(gz_{m_k+1}, gz_{n_{k+2}})\} = \epsilon. \tag{8}$$

Also from (3), we have

$$\begin{aligned} \epsilon &\leq \max\{d(gx_{m_k}, gx_{n_k}), d(gy_{m_k}, gy_{n_k}), d(gz_{m_k}, gz_{n_k})\} \\ &\leq \max \left\{ \begin{array}{c} d(gx_{m_k}, gx_{n_{k+2}}).d(gx_{n_{k+2}}, gx_{n_k}), \\ d(gy_{m_k}, gy_{n_{k+2}}).d(gy_{n_{k+2}}, gy_{n_k}), \\ d(gz_{m_k}, gz_{n_{k+2}}).d(gz_{n_{k+2}}, gz_{n_k}), \end{array} \right\} \\ &\leq \max \left\{ \begin{array}{c} d(gx_{m_k}, gx_{n_{k+2}}).d(gx_{n_k+2}, gx_{n_{k+1}}).d(gx_{n_{k+1}}, gx_{n_k}), \\ d(gy_{m_k}, gy_{n_{k+2}}).d(gy_{n_k+2}, gy_{n_{k+1}}).d(gy_{n_{k+1}}, gy_{n_k}), \\ d(gz_{m_k}, gz_{n_{k+2}}).d(gz_{n_k+2}, gz_{n_{k+1}}).d(gz_{n_{k+1}}, gz_{n_k}) \end{array} \right\} \quad (9) \\ &\leq \max \left\{ \begin{array}{c} d(gx_{m_k}, gx_{n_k}).(d(gx_{n_k+2}, gx_{n_{k+1}}))^2.(d(gx_{n_{k+1}}, gx_{n_k}))^2, \\ d(gy_{m_k}, gy_{n_k}).(d(gy_{n_k+2}, gy_{n_{k+1}}))^2.(d(gy_{n_{k+1}}, gy_{n_k}))^2, \\ d(gz_{m_k}, gz_{n_k}).(d(gz_{n_k+2}, gz_{n_{k+1}}))^2.(d(gz_{n_{k+1}}, gz_{n_k}))^2 \end{array} \right\}. \end{aligned}$$

Letting $k \to \infty$, from (2),(5) and (9)

$$\lim_{k\to\infty} \max\{d(gx_{m_k}, gx_{n_{k+2}}), d(gy_{m_k}, gy_{n_{k+2}}), d(gz_{m_k}, gz_{n_{k+2}})\} = \epsilon. \tag{10}$$

Again from (3), we have

$$\begin{aligned}\epsilon &\leq \max\{d(gx_{m_k},gx_{n_k}),d(gy_{m_k},gy_{n_k}),d(gz_{m_k},gz_{n_k})\}\\
&\leq \max\left\{\begin{array}{c} d(gx_{m_k},gx_{m_{k+1}}).d(gx_{m_{k+1}},gx_{n_k}),\\ d(gy_{m_k},gy_{m_{k+1}}).d(gy_{n_{k+1}},gy_{n_k}),\\ d(gz_{m_k},gz_{m_{k+1}}).d(gz_{n_{k+1}},gz_{n_k})\end{array}\right\}\\
&\leq \max\left\{\begin{array}{c} d(gx_{m_k},gx_{m_{k+1}}).d(gx_{m_k+1},gx_{n_{k+1}}).d(gx_{n_{k+1}},gx_{n_k}),\\ d(gy_{m_k},gy_{m_{k+1}}).d(gy_{m_k+1},gy_{n_{k+1}}).d(gy_{n_{k+1}},gy_{n_k}),\\ d(gz_{m_k},gz_{m_{k+1}}).d(gz_{m_k+1},gz_{n_{k+1}}).d(gz_{n_{k+1}},gz_{n_k})\end{array}\right\} \quad (11)\\
&\leq \max\left\{\begin{array}{c} (d(gx_{m_k},gx_{m_{k+1}}))^2.d(gx_{m_k},gx_{n_k}).(d(gx_{n_{k+1}},gx_{n_k}))^2,\\ (d(gy_{m_k},gy_{m_{k+1}}))^2.d(gy_{m_k},gy_{n_k}).(d(gy_{n_{k+1}},gy_{n_k}))^2,\\ (d(gz_{m_k},gz_{m_{k+1}}))^2.d(gz_{m_k},gz_{n_k}).(d(gz_{n_{k+1}},gz_{n_k}))^2\end{array}\right\}.\end{aligned}$$

Letting $k \to \infty$, from (2),(5) and (11), we have that

$$\lim_{k\to\infty}\max\{d(gx_{m_k+1},gx_{n_{k+1}}),d(gy_{m_k+1},gy_{n_{k+1}}),d(gz_{m_k+1},gz_{n_{k+1}})\}=\epsilon. \quad (12)$$

Now from (3), we get

$$\begin{aligned}&\epsilon \leq \max\{d(gx_{m_k},gx_{n_k}),d(gy_{m_k},gy_{n_k}),d(gz_{m_k},gz_{n_k})\}\\
&\leq \max\left\{\begin{array}{c} d(gx_{m_k},gx_{n_{k+1}}).d(gx_{n_{k+1}},gx_{n_k}),\\ d(gy_{m_k},gy_{n_{k+1}}).d(gy_{n_{k+1}},gy_{n_k}),\\ d(gz_{m_k},gz_{n_{k+1}}).d(gz_{n_{k+1}},gz_{n_k}),\end{array}\right\}\\
&\leq \max\left\{\begin{array}{c} d(gx_{m_k},gx_{m_{k+1}}).d(gx_{m_k+1},gx_{n_{k+2}}).d(gx_{n_{k+2}},gx_{n_{k+1}}).d(gx_{n_{k+1}},gx_{n_k}),\\ d(gy_{m_k},gy_{m_{k+1}}).d(gy_{m_k+1},gy_{n_{k+2}}).d(gy_{n_{k+2}},gy_{n_{k+1}}).d(gy_{n_{k+1}},gy_{n_k}),\\ d(gz_{m_k},gz_{m_{k+1}}).d(gz_{m_k+1},gz_{n_{k+2}}).d(gz_{n_{k+2}},gz_{n_{k+1}}).d(gz_{n_{k+1}},gz_{n_k})\end{array}\right\}.\end{aligned}$$

Letting $k \to \infty$, using (2), we obtain that

$$\epsilon \leq \lim_{k\to\infty}\max\{d(gx_{m_{k+1}},gx_{n_{k+2}}),d(gy_{m_{k+1}},gy_{n_{k+2}}),d(gz_{m_{k+1}},gz_{n_{k+2}})\}. \quad (13)$$

Now we will show that

$$\frac{1}{2}\min\left\{\begin{array}{c} d(gx_{m_k},T(x_{m_k},y_{m_k},z_{m_k})),\\ d(gy_{m_k},T(y_{m_k},z_{m_k},x_{m_k})),\\ d(gz_{m_k},T(z_{m_k},x_{m_k},y_{m_k}))\end{array}\right\} \leq \max\left\{d(gx_{m_k},gx_{n_k+1}),d(gy_{m_k},gx_{n_k+1}),d(gz_{m_k},gz_{n_k+1})\right\}.$$

On contrary suppose that

$$\max\left\{d(gx_{m_k},gx_{n_k+1}),d(gy_{m_k},gx_{n_k+1}),d(gz_{m_k},gz_{n_k+1})\right\}$$

$$< \frac{1}{2}\min\left\{\begin{array}{c} d(gx_{m_k}, T(x_{m_k}, y_{m_k}, z_{m_k})), \\ d(gy_{m_k}, T(y_{m_k}, z_{m_k}, x_{m_k})), \\ d(gz_{m_k}, T(z_{m_k}, x_{m_k}, y_{m_k})) \end{array}\right\}.$$

Letting $k \to \infty$, we have

$$\epsilon \leq \frac{1}{2}\min\{1,1,1\} = \frac{1}{2}$$

which is a contradiction.

Hence

$$\frac{1}{2}\min\left\{\begin{array}{c} d(gx_{m_k}, T(x_{m_k}, y_{m_k}, z_{m_k})), \\ d(gy_{m_k}, T(y_{m_k}, z_{m_k}, x_{m_k})), \\ d(gz_{m_k}, T(z_{m_k}, x_{m_k}, y_{m_k})) \end{array}\right\}$$
$$\leq \max\left\{d(gx_{m_k}, gx_{n_k+1}), d(gy_{m_k}, gx_{n_k+1}), d(gz_{m_k}, gz_{n_k+1})\right\}.$$

Now from (2.1.1), we get

$$d(T(x_{m_k}, y_{m_k}, z_{m_k}), T(x_{n_k+1}, y_{n_k+1}, z_{n_k+1}))$$
$$\leq \phi\left(\max\left\{\begin{array}{c} d(gx_{m_k}, gx_{n_k+1}), d(gy_{m_k}, gy_{n_k+1}), \\ d(gz_{m_k}, gz_{n_k+1}), \\ d(gx_{m_k}, T(x_{m_k}, y_{m_k}, z_{m_k})), \\ d(gy_{m_k}, T(y_{m_k}, z_{m_k}, x_{m_k})), \\ d(gz_{m_k}, T(z_{m_k}, x_{m_k}, y_{m_k})), \\ d(gx_{n_k+1}, T(x_{n_k+1}, y_{n_k+1}, z_{n_k+1})), \\ d(gy_{n_k+1}, T(y_{n_k+1}, z_{n_k+1}, x_{n_k+1})), \\ \sqrt{d(gx_{m_k}, T(x_{n_k+1}, y_{n_k+1}, z_{n_k+1}))}, \\ \sqrt{d(gy_{m_k}, T(y_{n_k+1}, z_{n_k+1}, x_{n_k+1}))}, \\ \sqrt{d(gz_{m_k}, T(z_{n_k+1}, x_{n_k+1}, y_{n_k+1}))} \end{array}\right\}\right),$$

$$= \phi\left(\max\left\{\begin{array}{c} d(gx_{m_k}, gx_{n_k+1}), d(gy_{m_k}, gy_{n_k+1}), \\ d(gz_{m_k}, gz_{n_k+1}), d(gx_{m_k}, gx_{m_k+1}), \\ d(gy_{m_k}, gy_{m_k+1}), d(gz_{m_k}, gz_{m_k+1}), \\ d(gx_{n_k+1}, gx_{n_k+2}), d(gy_{n_k+1}, gy_{n_k+2}), \\ d(gz_{n_k+1}, gz_{n_k+2}), \sqrt{d(gx_{m_k}, gx_{n_k+2})}, \\ \sqrt{d(gy_{m_k}, gy_{n_k+2})}, \sqrt{d(gz_{m_k}, gz_{n_k+2})} \end{array}\right\}\right).$$

Thus

$$d(gx_{m_k+1}, gx_{n_k+2}) \leq \phi\left(\max\left\{\begin{array}{c} d(gx_{m_k}, gx_{n_{k+1}}), d(gy_{m_k}, gy_{n_{k+1}}), \\ d(gz_{m_k}, gz_{n_{k+1}}), d(gx_{m_k}, gx_{m_k+1}), \\ d(gy_{m_k}, gy_{m_k+1}), d(gz_{m_k}, gz_{m_k+1}), \\ d(gx_{n_{k+1}}, gx_{n_{k+2}}), d(gy_{n_{k+1}}, gy_{n_{k+2}}), \\ d(gz_{n_{k+1}}, gz_{n_{k+2}}), \sqrt{d(gx_{m_k}, gx_{n_{k+2}})}, \\ \sqrt{d(gy_{m_k}, gy_{n_{k+2}})}, \sqrt{d(gz_{m_k}, gz_{n_{k+2}})} \end{array}\right\}\right).$$

Similarly, we have

$$d(gy_{m_k+1}, gy_{n_k+2}) \leq \phi\left(\max\left\{\begin{array}{c} d(gx_{m_k}, gx_{n_{k+1}}), d(gy_{m_k}, gy_{n_{k+1}}), \\ d(gz_{m_k}, gz_{n_{k+1}}), d(gx_{m_k}, gx_{m_k+1}), \\ d(gy_{m_k}, gy_{m_k+1}), d(gz_{m_k}, gz_{m_k+1}), \\ d(gx_{n_{k+1}}, gx_{n_{k+2}}), d(gy_{n_{k+1}}, gy_{n_{k+2}}), \\ d(gz_{n_{k+1}}, gz_{n_{k+2}}), \sqrt{d(gx_{m_k}, gx_{n_{k+2}})}, \\ \sqrt{d(gy_{m_k}, gy_{n_{k+2}})}, \sqrt{d(gz_{m_k}, gz_{n_{k+2}})} \end{array}\right\}\right)$$

and

$$d(gz_{m_k+1}, gz_{n_k+2}) \leq \phi\left(\max\left\{\begin{array}{c} d(gx_{m_k}, gx_{n_{k+1}}), d(gy_{m_k}, gy_{n_{k+1}}), \\ d(gz_{m_k}, gz_{n_{k+1}}), d(gx_{m_k}, gx_{m_k+1}), \\ d(gy_{m_k}, gy_{m_k+1}), d(gz_{m_k}, gz_{m_k+1}), \\ d(gx_{n_{k+1}}, gx_{n_{k+2}}), d(gy_{n_{k+1}}, gy_{n_{k+2}}), \\ d(gz_{n_{k+1}}, gz_{n_{k+2}}), \sqrt{d(gx_{m_k}, gx_{n_{k+2}})}, \\ \sqrt{d(gy_{m_k}, gy_{n_{k+2}})}, \sqrt{d(gz_{m_k}, gz_{n_{k+2}})} \end{array}\right\}\right).$$

Thus

$$\max\left\{\begin{array}{c} d(gx_{m_k+1}, gx_{n_k+2}), \\ d(gy_{m_k+1}, gy_{n_k+2}), \\ d(gz_{m_k+1}, gz_{n_k+2}) \end{array}\right\} \leq \phi\left(\max\left\{\begin{array}{c} d(gx_{m_k}, gx_{n_{k+1}}), d(gy_{m_k}, gy_{n_{k+1}}), \\ d(gz_{m_k}, gz_{n_{k+1}}), d(gx_{m_k}, gx_{m_k+1}), \\ d(gy_{m_k}, gy_{m_k+1}), d(gz_{m_k}, gz_{m_k+1}), \\ d(gx_{n_{k+1}}, gx_{n_{k+2}}), d(gy_{n_{k+1}}, gy_{n_{k+2}}), \\ d(gz_{n_{k+1}}, gz_{n_{k+2}}), \sqrt{d(gx_{m_k}, gx_{n_{k+2}})}, \\ \sqrt{d(gy_{m_k}, gy_{n_{k+2}})}, \sqrt{d(gz_{m_k}, gz_{n_{k+2}})} \end{array}\right\}\right).$$

From (13), we have

$$
\begin{aligned}
\epsilon &\leq \lim_{k\to\infty} \max\left\{\begin{array}{c} d(gx_{m_k+1}, gx_{n_k+2}), \\ d(gy_{m_k+1}, gy_{n_k+2}), \\ d(gz_{m_k+1}, gz_{n_k+2}) \end{array}\right\} \\
&\leq \lim_{k\to\infty} \phi\left(\max\left\{\begin{array}{c} d(gx_{m_k}, gx_{n_{k+1}}), d(gy_{m_k}, gy_{n_{k+1}}), \\ d(gz_{m_k}, gz_{n_{k+1}}), d(gx_{m_k}, gx_{m_k+1}), \\ d(gy_{m_k}, gy_{m_k+1}), d(gz_{m_k}, gz_{m_k+1}), \\ d(gx_{n_{k+1}}, gx_{n_{k+2}}), d(gy_{n_{k+1}}, gy_{n_{k+2}}), \\ d(gz_{n_{k+1}}, gz_{n_{k+2}}), \sqrt{d(gx_{m_k}, gx_{n_{k+2}})}, \\ \sqrt{d(gy_{m_k}, gy_{n_{k+2}})}, \sqrt{d(gz_{m_k}, gz_{n_{k+2}})} \end{array}\right\}\right) \\
&= \lim_{t\to\epsilon^+} \phi(t)
\end{aligned}
$$

where

$$
\begin{aligned}
t &= \lim_{k\to\infty} \max\left\{\begin{array}{c} d(gx_{m_k}, gx_{n_{k+1}}), d(gy_{m_k}, gy_{n_{k+1}}), \\ d(gz_{m_k}, gz_{n_{k+1}}), d(gx_{m_k}, gx_{m_k+1}), \\ d(gy_{m_k}, gy_{m_k+1}), d(gz_{m_k}, gz_{m_k+1}), \\ d(gx_{n_{k+1}}, gx_{n_{k+2}}), d(gy_{n_{k+1}}, gy_{n_{k+2}}), \\ d(gz_{n_{k+1}}, gz_{n_{k+2}}), \sqrt{d(gx_{m_k}, gx_{n_{k+2}})}, \\ \sqrt{d(gy_{m_k}, gy_{n_{k+2}})}, \sqrt{d(gz_{m_k}, gz_{n_{k+2}})} \end{array}\right\} > 1 \\
&< \epsilon
\end{aligned}
$$

which is a contradiction.

Hence $\{gx_n\}$, $\{gy_n\}$ and $\{gz_n\}$ are Cauchy sequences in the multiplicative metric space (X, d).

Hence we have

$$\lim_{n,m\to\infty} d(gx_n, gx_m) = \lim_{n,m\to\infty} d(gy_n, gy_m) = \lim_{n,m\to\infty} d(gz_n, gz_m) = 1.$$

Suppose $g(X)$ is a complete subspace of X.

Since $\{gx_n\}$, $\{gy_n\}$ and $\{gz_n\}$ are Cauchy sequence in the multiplicative metric space $(g(X), d)$.

It follows that $\{gx_n\}$, $\{gy_n\}$ and $\{gz_n\}$ convergent to some r, s and t in $g(X)$.

Thus

$$\lim_{n\to\infty} d(gx_n, r) = \lim_{n\to\infty} d(gy_n, s) = \lim_{n\to\infty} d(gz_n, t) = 1.$$

Since $r, s, t \in g(X)$, there exist $a, b, c \in X$ such that $r = ga$, $s = gb$ and $t = gc$.

Now we claim that for each $n \geq 1$, at least one of the following holds

$$\frac{1}{2}\min\left\{\begin{array}{c} d(gx_n, gx_{n+1}), \\ d(gy_n, gy_{n+1}), \\ d(gz_n, gz_{n+1}) \end{array}\right\} \leq \max\left\{\ d(gx_n, ga), d(gy_n, gb), d(gz_n, gc)\ \right\}$$

or

$$\frac{1}{2}\min\left\{\begin{array}{c} d(gx_{n+1}, gx_{n+2}), \\ d(gy_{n+1}, gy_{n+2}), \\ d(gz_{n+1}, gz_{n+2}) \end{array}\right\} \leq \max\left\{\ d(gx_{n+1}, ga), d(gy_{n+1}, gb), d(gz_{n+1}, gc)\ \right\}.$$

On contrary suppose that

$$\frac{1}{2}\min\left\{\begin{array}{c} d(gx_n, gx_{n+1}), \\ d(gy_{n+1}, gy_{n+2}), \\ d(gz_{n+1}, gz_{n+2}) \end{array}\right\} > \max\left\{\ d(gx_n, ga), d(gy_n, gb), d(gz_{n+1}, gc)\ \right\}$$

and

$$\frac{1}{2}\min\left\{\begin{array}{c} d(gx_{n+1}, gx_{n+2}), \\ d(gy_{n+1}, gy_{n+2}), \\ d(gz_{n+1}, gz_{n+2}) \end{array}\right\} > \max\left\{\ d(gx_{n+1}, ga), d(gy_{n+1}, gb), d(gz_{n+1}, gb)\ \right\}.$$

Now

$$\begin{aligned} d(gx_n, gx_{n+1}) &\leq d(gx_n, ga).d(ga, gx_{n+1}) \\ &< \frac{1}{2}d(gx_n, gx_{n+1}).\frac{1}{2}d(gx_{n+1}, gx_{n+2}) \\ &= \frac{1}{4}d(gx_n, gx_{n+1}).d(gx_{n+1}, gx_{n+2}). \end{aligned}$$

Letting $n \to \infty$ and using (2), we have $1 \leq \frac{1}{4}$ which is a contradiction.

Hence claim holds.

Sub case (i): Suppose

$$\frac{1}{2}\min\left\{\begin{array}{c} d(gx_n, gx_{n+1}), \\ d(gy_n, gy_{n+1}), \\ d(gz_n, gz_{n+1}) \end{array}\right\} \leq \max\left\{\ d(gx_n, ga), d(gy_n, gb), d(gz_n, gc)\ \right\}.$$

Now we prove that $T(a,b,c) = r$, $T(b,c,a) = s$ and $T(c,a,b) = t$.

From (2.1.1), we get that

$$d\left(T(x_n,y_n,z_n),T(a,b,c)\right) \leq \phi\left(\max\left\{\begin{array}{c} d(gx_n,ga),d(gy_n,gb),d(gz_n,gc), \\ d(gx_n,T(x_n,y_n,z_n)),d(gy_n,T(y_n,z_n,x_n)), \\ d(gz_n,T(z_n,x_n,y_n)),d(ga,T(a,b,c)), \\ d(gb,T(b,c,a)),d(gc,T(c,a,b)), \\ \sqrt{d(gx_n,T(a,b,c))},\sqrt{d(gy_n,T(b,c,a))},\sqrt{d(gz_n,T(c,a,b))} \end{array}\right\}\right).$$

Letting $n \to \infty$, we have

$$\begin{aligned} &d(ga,T(a,b,c)) \\ &\leq \phi\left(\max\left\{\begin{array}{c} d(ga,ga),d(gb,gb),d(gc,gc), \\ d(ga,ga),d(gb,gb), \\ d(gc,gc),d(ga,T(a,b,c)), \\ d(gb,T(b,c,a)),d(gc,T(c,a,b)), \\ \sqrt{d(ga,T(a,b,c))},\sqrt{d(gb,T(b,c,a))},\sqrt{d(gc,T(c,a,b))} \end{array}\right\}\right) \\ &\leq \phi\left(\max\left\{\ d(ga,T(a,b,c)),d(gb,T(b,c,a)),d(gc,T(c,a,b))\ \right\}\right). \end{aligned}$$

Similarly we can show that

$$\begin{aligned} &d(gb,T(b,c,a)) \\ &\qquad \leq \phi\left(\max\left\{\ d(ga,T(a,b,c)),d(gb,T(b,c,a)),d(gc,T(c,a,b))\ \right\}\right), \end{aligned}$$

and

$$\begin{aligned} &d(gc,T(c,a,b)) \\ &\qquad \leq \phi\left(\max\left\{\ d(ga,T(a,b,c)),d(gb,T(b,c,a)),d(gc,T(c,a,b))\ \right\}\right). \end{aligned}$$

Thus

$$\begin{aligned} &\max\left\{\begin{array}{c} d(ga,T(a,b,c)), \\ d(gb,T(b,c,a)), \\ d(gc,T(c,a,b)) \end{array}\right\} \\ &\leq \phi\left(\max\left\{\ d(ga,T(a,b,c)),d(gb,T(b,c,a)),d(gc,T(c,a,b))\ \right\}\right) \\ &< \max\left\{\ d(ga,T(a,b,c)),d(gb,T(b,c,a)),d(gc,T(c,a,b))\ \right\}, \end{aligned}$$

which is a contradiction.

Hence $r = ga = T(a,b,c)$, $s = gb = T(b,c,a)$ and $t = gc = T(c,a,b)$.

Since (T,g) is w- compatible, we have $T(r,s,t) = gr$, $T(s,t,r) = gs$ and $T(t,r,s) = gt$.

Now we show that $T(r,s,t) = gr = r$, $T(s,t,r) = gs = s$ and $T(t,r,s) = gt = t$.

Let us suppose that $T(r,s,t) \neq r$ or $T(s,t,r) \neq s$ or $T(t,r,s) \neq t$.

Clearly

$$\frac{1}{2}\min\left\{\begin{array}{c} d(gr,T(r,s,t)),d(gs,T(s,t,r)),\\ d(gt,T(t,r,s)) \end{array}\right\} \leq \max\left\{d(gr,ga),d(gs,gb),d(gt,gc)\right\}$$

implies that

$$\begin{aligned} d(T(r,s,t),r) &= d\left(T(r,s,t),T(a,b,c)\right)\\ &\leq \phi\left(\max\left\{\begin{array}{c} d(gr,ga),d(gs,gb),d(gt,gc),\\ d(gr,T(r,s,t)),d(gs,T(s,t,r)),d(gt,T(t,r,s)),\\ d(ga,T(a,b,c)),d(gb,T(b,c,a)),\\ d(gc,T(c,a,b)),\sqrt{d(gr,T(a,b,c))},\\ \sqrt{d(gs,T(b,c,a))},\sqrt{d(gt,T(c,a,b))} \end{array}\right\}\right)\\ &\leq \phi\left(\max\left\{\begin{array}{c} d(r,T(r,s,t)),\\ d(s,T(s,t,r)),\\ d(t,T(t,r,s)) \end{array}\right\}\right). \end{aligned}$$

Similarly we can show that

$$d(T(s,t,r),s) \leq \phi\left(\max\left\{\begin{array}{c} d(r,T(r,s,t)),\\ d(s,T(s,t,r)),\\ d(t,T(t,r,s)) \end{array}\right\}\right)$$

and

$$d(T(t,r,s),s) \leq \phi\left(\max\left\{\begin{array}{c} d(r,T(r,s,t)),\\ d(s,T(s,t,r)),\\ d(t,T(t,r,s)) \end{array}\right\}\right).$$

Thus

$$\begin{aligned} \max\left\{\begin{array}{c} d(r,T(r,s,t)),\\ d(s,T(s,t,r)),\\ d(t,T(t,r,s)) \end{array}\right\} &\leq \phi\left(\max\left\{\begin{array}{c} d(r,T(r,s,t)),\\ d(s,T(s,t,r)),\\ d(t,T(t,r,s)) \end{array}\right\}\right)\\ &< \max\left\{\begin{array}{c} d(r,T(r,s,t)),\\ d(s,T(s,t,r)),\\ d(t,T(t,r,s)) \end{array}\right\} \end{aligned}$$

is a contradiction.

Hence(r,s,t) is common tripled fixed point of T and g.

Let (r^*,s^*,t^*) be another common tripled fixed point of T and g.

Clearly

$$\frac{1}{2}\min\left\{\begin{array}{c} d(gr,T(r,s,t)),\\ d(gs,T(s,t,r)),\\ d(gt,T(t,r,s))\end{array}\right\} \le \max\{d(gr,gr^*),d(gs,gs^*),d(gt,gt^*)\}$$

implies that

$$\begin{aligned} d(r,r^*) &= d\left(T(r,s,t),T(r^*,s^*,t^*)\right)\\ &\le \phi\left(\max\left\{\begin{array}{c} d(gr,gr^*),d(gs,gs^*),d(gt,gt^*),\\ d(gr,T(r,s,t)),d(gs,T(s,t,r)),d(gt,T(t,r,s)),\\ d(gr^*,T(r^*,s^*,t^*)),d(gs^*,T(s^*,t^*,r^*)),\\ d(gt^*,T(t^*,r^*,s^*)),\sqrt{d(gr,T(r^*,s^*,t^*))},\\ \sqrt{d(gs,T(s^*,t^*,r^*))},\sqrt{d(gt,T(t^*,r^*,s^*))}\end{array}\right\}\right)\\ &\le \phi\left(\max\left\{\ d(r,r^*),d(s,s^*),d(t,t^*)\ \right\}\right). \end{aligned}$$

Similarly we can show that

$$d(s,s^*) \quad \le \phi\left(\max\left\{\ d(r,r^*),d(s,s^*),d(t,t^*)\ \right\}\right)$$

and

$$d(t,t^*) \quad \le \phi\left(\max\left\{\ d(r,r^*),d(s,s^*),d(t,t^*)\ \right\}\right).$$

Thus

$$\begin{aligned} \max\left\{\ d(r,r^*),d(s,s^*),d(t,t^*)\ \right\} &\le \phi\left(\max\left\{\ d(r,r^*),d(s,s^*),d(t,t^*)\ \right\}\right)\\ &< \max\left\{\ d(r,r^*),d(s,s^*),d(t,t^*)\ \right\} \end{aligned}$$

is a contradiction.

Hence $r=r^*$,$s=s^*$ and $t=t^*$.

Hence (r,s,t) is a unique common tripled fixed point of T and g.

Now we claim that $r\neq s$ or $s\neq t$ or $t\neq r$.

Clearly

$$\frac{1}{2}\min\left\{\begin{array}{c} d(gr,T(r,s,t)),\\ d(gs,T(s,t,r)),\\ d(gt,T(t,r,s))\end{array}\right\} \le \max\{d(gr,gs),d(gs,gt),d(gt,gr)\}$$

implies that

$$d(r,s) = d\left(T(r,s,t),T(s,t,r)\right)$$

$$\leq \phi\left(\max\left\{\begin{array}{c} d(gr,gs),d(gs,gt),d(gt,gr),\\ d(gr,T(r,s,t)),d(gs,T(s,t,r)),d(gt,T(t,r,s)),\\ d(gs,T(s,t,r)),d(gt,T(t,r,s)),\\ d(gr,T(r,s,t)),\sqrt{d(gr,T(s,t,r))},\\ \sqrt{d(gs,T(t,r,s))},\sqrt{d(gt,T(r,s,t))} \end{array}\right\}\right)$$
$$\leq \phi\left(\max\left\{\ d(r,s),d(s,t),d(t,r)\ \right\}\right).$$

Similarly we can show that

$$d(s,t) \quad \leq \phi\left(\max\left\{\ d(r,s),d(s,t),d(t,r)\ \right\}\right)$$

and

$$d(t,r) \quad \leq \phi\left(\max\left\{\ d(r,s),d(s,t),d(t,r)\ \right\}\right).$$

Thus

$$\begin{aligned}\max\left\{\ d(r,s),d(s,t),d(t,r)\ \right\} &\leq \phi\left(\max\left\{\ d(r,s),d(s,t),d(t,r)\ \right\}\right)\\ &< \max\left\{\ d(r,s),d(s,t),d(t,r)\ \right\}\end{aligned}$$

is a contradiction.

Thus $r = s = t$.

Hence (r, r, r) is unique common tripled fixed point of T and g.

There exist a unique common tripled fixed point of T and g when

$$\frac{1}{2}\min\left\{\begin{array}{c} d(gx_{n+1},gx_{n+2}),\\ d(gy_{n+1},gy_{n+2}),\\ d(gz_{n+1},gz_{n+2}) \end{array}\right\}$$
$$\leq \max\left\{\ d(gx_{n+1},ga),d(gy_{n+1},gb),d(gz_{n+1},gc)\ \right\}$$

holds. □

Example 2.2. Let $X = [0, 1]$ and $d(x, y) = e^{|x-y|}$ for all $x, y \in X$ then (X, d) is a complete multiplicative metric space. Let $T : X \times X \times X \to X$ and $g : X \to X$ be defined by $T(x, y, z) = \frac{x+y+z}{4}$ and $g(x) = \frac{x}{2}$.Let $\phi : [1, \infty) \to [1, \infty)$ be defined by $\phi(t) = t^{\frac{1}{2}}$. Then T and g satisfies all the conditions of Theorem 2.1.

Clearly (0, 0 ,0) is the unique common tripled fixed point of T and g.

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