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ON THE MITTAG-LEFFLER STABILITY OF IMPULSIVE FRACTIONAL NEURAL NETWORKS WITH FINITE DELAYS

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Abstract: This paper deals with Mittag-Leffler stability properties of impulsive Caputo fractional neural networks with dynamical thresholds and finite delays. The impulses are realized at fixed moments of time and can be considered as a control.

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In Memory of our teacher and friend
Professor Drumi Bainov

1. Introduction

Fractional differential equations are generalizations of integer order differential equations. These generalizations are valuable tools in modelling many processes and phenomena in various fields of engineering, physics, biology and economics. Since a fractional-order derivative is nonlocal and has weakly singular kernels, it provides an excellent instrument for the description of memory and hered-

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itary states of dynamical processes, which allows a greater flexibility in the system and it is the main advantage of fractional models in comparison with the classical integer-order counterparts.

The rapid development in the field of fractional-order equations during the last two decades invoked an increasing interest of scientists in the area of fractional-order neural networks (see [7], [18]).

Besides delay effects, impulsive effects are also likely to exist in the neural networks. Indeed, the state of electronic networks is often subject to instantaneous perturbations and experience abrupt changes at certain instants, which may be caused by switching phenomenon, frequency change or other sudden noise, that exhibit impulsive effects. In applied mathematics, it is now recognized that real-world phenomena that are subject to short-term perturbations whose duration is negligible in comparison with the duration of the process are more accurately described using impulsive differential equations. A great progress in studying of such equations and their applications has been made by Bainov and his collaborators (see, for example, the monographs [3], [11] and cited references therein). Accordingly, the study of impulsive integer-order neural networks is of a great importance and has gained considerable popularity in recent years (see [13]).

In this paper, we study the global Mittag-Leffler stability (global asymptotic stability) of impulsive fractional neural networks with dynamical thresholds. The Mittag-Leffler stability proposed in [12] generalizes the asymptotic stability notion for fractional-order systems and is correspondent to the notion of the exponential stability for integer-order dierential equations (see [12]). It is noted that for the integer-order neural networks the exponential stability gives a fast convergence rate (see [13]).

2. Preliminaries

Let $\mathbb{R}_+ = [0, \infty)$ and $t_0 \in \mathbb{R}_+$. Let the sequence of points $\{t_k\}_{k=1}^{\infty}$: $t_k \in \mathbb{R}_+,\ t_{k-1} \leq t_k.\ k = 2, 3, \ldots$ be given such that $\lim_{k\to\infty} t_k = \infty$.

Definition 1. (see [14]) For any $t \geq t_0$, the *Caputo fractional derivative of order* α, $0 < \alpha < 1$ with the lower limit t_0 for a function $l \in C^1[[t_0, b], \mathbb{R}]$, $b > t_0$, is defined as

$$ {}_{t_0}^{c}D_t^q l(t) = \frac{1}{\Gamma(1-q)} \int_{t_0}^{t} \frac{l'(\theta)}{(t-\theta)^q} d\theta. $$

where Γ denotes the Gamma function defined by the integral

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt.$$

Definition 2. ([8]) We say $m \in C^q([\alpha, \beta], \mathbb{R})$ if $m(t)$ is differentiable (i.e. $m'(t)$ exists), the Caputo derivative ${}_{\alpha}^{c}D^q m(t)$ exists for $t \in [\alpha, \beta]$.

Remark 1. Definition 2 could be extended to any interval $J \subset \mathbb{R}_+$.

In our further study we will use the following result

Lemma 1. *(Lemma 1 [2]). Let $x \in \mathbb{R}$ be a continuous and derivable function. Then, for any $t \geq t_0$*

$${}_{t_0}^{c}D_t^q \Big(x(t)\Big)^2 \leq 2x(t)\ {}_{t_0}^{c}D_t^q x(t).$$

Consider the following Caputo fractional impulsive delayed cellular neural network with dynamical thresholds

$$\begin{aligned} &{}_0^c D_t^q x(t) = -x(t) + af\left(x(t) - bx(t-\tau(t)) - c\right),\ t \neq t_k,\ t \geq 0, \\ &\Delta x(t_k) = -\sigma_k(x(t_k - 0) - C),\ k = 1, 2, ..., \end{aligned} \tag{1}$$

where $x \in \mathbb{R}$, $0 < q < 1$, $\Delta x(t_k) = x(t_k + 0) - x(t_k - 0)$, $f : \mathbb{R} \to \mathbb{R}$, $\tau(t)$ corresponds to the transmission delay and $\tau :\ [0, \infty) \to [0, T]$ with $T = const > 0$, $\lim_{t\to\infty}(t - \tau(t)) = \infty$, $\sigma_k = const$, $k = 1, 2, ...$, a, b, C and c are constants: $a, b > 0, C \neq 0$.

Note that for $q = 1$ the equation (1) is investigated in [15], and generalizes numerous non-impulsive neural network models.

Let $J \subset \mathbb{R}$ be a given interval and $J_{imp} = \{t \in J :\ t \neq t_k,\ k = 1, 2, \dots\}$ and introduce the following classes of functions

$$\begin{aligned} C^q(J_{imp}, \mathbb{R}) &= \bigcup_{k=0}^{\infty} C^q((t_k, t_{k+1}), \mathbb{R}), \\ C(J_{imp}, \mathbb{R}) &= \bigcup_{k=0}^{\infty} C((t_k, t_{k+1}), \Delta), \\ PC(J, \mathbb{R}) &= \{u \in C(J_{imp}, \mathbb{R}) :\ u(t_k) = \lim_{t\uparrow t_k} u(t) < \infty, \\ &\qquad u(t_k + 0) = \lim_{t\downarrow t_k} u(t) < \infty \text{ for all } k :\ t_k \in J\}, \\ PC^q(J, \mathbb{R}) &= \{u \in C^q(J_{imp}, \mathbb{R}) :\ u(t_k) = \lim_{t\uparrow t_k} u(t) < \infty, \end{aligned}$$

$$u(t_k+0)=\lim_{t\downarrow t_k}u(t)<\infty$$
$$u'(t_k)=\lim_{t\uparrow t_k}u'(t)<\infty,\ u'(t_k+0)=\lim_{t\downarrow t_k}u'(t)<\infty$$
$$\text{for all } k:\ t_k\in J\},$$
$$\begin{aligned}CB(J,\mathbb{R}) \;=\; &\{\sigma: J\to\mathbb{R}:\sigma(t)\text{ is bounded on } J\\ &\text{and it is continuous except at some points } \tau \text{ at which}\\ &\sigma(\tau+0)=\lim_{t\downarrow\tau}\sigma(t)<\infty,\ \sigma(\tau-0)=\lim_{t\uparrow\tau}\sigma(t)=\sigma(\tau)\}.\end{aligned}$$

Remark 2. From the above any solution of (1) is from the class $PC^q([t_0,b),\mathbb{R}^n)$, $b\le\infty$ (see [1]) .

Let $\varphi_0\in CB([-T,0],\mathbb{R})$. Denote by $x(t)=x(t;0,\varphi_0)$, $x\in\mathbb{R}$ the solution of impulsive fractional differential equation (1), satisfying the initial conditions

$$\begin{aligned}&x(t;0,\varphi_0)=\varphi_0(t),\ -T\ \le\ t\le\ 0,\\ &x(0^+;0,\varphi_0)=\varphi_0(0).\end{aligned}\tag{2}$$

We now look at the concept of a solution to Caputo fractional differential equations with impulses (1)(for detailed explanations see [1]), Denote $F(x,y)=-x+af\,(x-by-c)$ and $I_k(x)=-\sigma_k(x-C)$. There are mainly two viewpoints:

(V1): working in each subinterval $(t_k,t_{k+1}]$ (see, for examplle, [4], [5], [6]). Since the Caputo fractional derivative depends significantly on the initial point it leads to a change of the equation on each interval (t_k,t_{k+1}). Then the IVP for IFrDE (1) is equivalent to the following integral equation

$$x(t)=\begin{cases}x_0+\frac{1}{\Gamma(q)}\int_0^t(t-s)^{q-1}F(x(s),x(s-\tau(s))ds\\ \qquad\qquad\qquad\text{for}\ \ t\in[0,t_1],\\ x_0+\frac{1}{\Gamma(q)}\sum_{i=1}^k\int_{t_{i-1}}^{t_i}(t_i-s)^{q-1}F(x(s),x(s-\tau(s))ds\\ \quad+\frac{1}{\Gamma(q)}\int_{t_k}^t(t-s)^{q-1}F(x(s),x(s-\tau(s))ds\\ \quad+\sum_{i=1}^k I_i(x(t_i-0))\quad\text{for}\ \ t\in(t_k,t_{k+1}].\end{cases}$$

Using approach (V1) the solution $x(t;t_0,x_0)$ of (1) is given by

$$x(t;t_0,x_0)=\begin{cases}X_0(t;0,x_0)\quad\text{for}\ \ t\in[0,t_1],\\ X_1(t;t_1,\Phi_1(X_0(t_1;0,x_0)))\quad\text{for}\ \ t\in(t_1,t_2],\\ X_2(t;t_2,\Phi_2(X_1(t_2;t_1,\Phi_2(X_0(t_1;0,x_0)))\\ \qquad\qquad\qquad\text{for}\ \ t\in(t_2,t_3],\\ \dots\dots\dots\end{cases}$$

where $\Phi_k(x) = x + I_k(x)$ and

— $X_0(t; 0, x_0)$ is the solution of IVP for FrDE

$$\begin{aligned} &{}_{\tau_0}^{c}D_t^q x(t) = -x(t) + af\left(x(t) - bx(t - \tau(t)) - c\right) \\ &x(\tau_0) = x_0 \end{aligned} \tag{3}$$

with $\tau_0 = 0$,

— $X_1(t; t_1, \Phi_1(X_0(t_1; t_0, x_0)))$ is the solution of IVP for FrDE (3) with $\tau_0 = t_1,\ x_0 = \Phi_1(X_0(t_1; t_0, x_0))$,

— $X_2(t; t_2, \Phi_2(X_1(t_2; t_1, \Phi_1(X_0(t_1; t_0, x_0)))$ is the solution of IVP for the FrDE (3) with $\tau_0 = t_2$ and
$x_0 = \Phi_2(X_1(t_2; t_1, \Phi_1(X_0(t_1; t_0, x_0))$,

and so on.

Example 1. Consider the initial value problem for the scalar impulsive fractional differential equation with a Caputo derivative for $0 < q < 1$,

$$\begin{aligned} &{}_0^cD^q u = Au, \text{ for } t \geq 0,\ t \neq t_i, \\ &u(t_i + 0) = a_i u(t_i - 0) \text{ for } \quad i = 1, 2, \dots, \\ &u(0) = u_0, \end{aligned} \tag{4}$$

where $u \in \mathbb{R}$, $a_i = const \neq 1$, $i = 1, 2, \dots$, $t_0 = 0$.

The solution of (4) is

$$\begin{aligned} u(t; 0, u_0) = u_0 \Big(\prod_{i=1}^{k} a_i E_q(A(t_i - t_{i-1})^q)\Big) E_q(A(t - t_k)^q) \\ \text{for} \quad t \in (t_k, t_{k+1}],\ k = 0, 1, 2, 3, \dots, \end{aligned} \tag{5}$$

where E_q is the Mittag-Leffler function (with one parameter), defined by $E_q(z) = \sum\limits_{k=0}^{\infty} \frac{z^k}{\Gamma(qk+1)}$. □

(V2): keeping the lower limit 0 of the Caputo derivative for all $t \geq 0$ but considering different initial conditions on each interval (t_k, t_{k+1}) (see, for example, [9], [16], [17]). Then (1) is equivalent to the following integral equation

$$x(t) = \begin{cases} x_0 + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} F(x(s), x(s-\tau(s))) ds \\ \qquad \text{for } \ t \in [0, t_1], \\ x_0 + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} F(x(s), x(s-\tau(s))) ds \\ \quad + \sum_{i=1}^k I_i(x(t_i)) \quad \text{for } \ t \in (t_k, t_{k+1}], \ k = 1, 2, 3, \dots \end{cases}$$

Example 2. Consider the initial value problem for the scalar IFrDE with a Caputo derivative (4). Applying (V2), using $\frac{\lambda}{\Gamma(q)} \int_0^t \frac{E_q(\lambda s^q)}{(t-s)^q} ds = E_q(\lambda t^q) - 1$ we get the solution of (4), namely

$$u(t; 0, u_0) = u_0 \Big(E_q(At^q) + \sum_{i=1}^k E_q(At_i^q)(a_i - 1) \prod_{j=i+1}^k a_j \Big), \quad \text{for} \quad t \in (t_k, t_{k+1}], \ k = 0, 1, 2, 3, \dots \tag{6}$$

It looks like (5) is closer to the ordinary case ($q = 1$). □

In this paper we will use approach (V1).

For any function $\varphi \in CB([-T, 0], \mathbb{R})$ we will use the following norm $|\varphi|_T = \sup_{\theta \in [-T,0]} |\varphi(\theta)|$.

Definition 3. We say $x^* \in \mathbb{R}$ is an equilibrium of the equation (1) if $x(t) \equiv x^*$ is a solution of (1), i.e. $0 = -x^* + af(x^* - bx^* - c)$ and $0 = \sigma_i(x^* - C)$, $i = 1, 2, \dots$.

We introduce the following conditions:

H1. There exists a constant $L > 0$ such that

$$|f(u) - f(v)| \le L|u - v| \quad \text{for} \ \ u, v \in \mathbb{R}.$$

H2. $a > 0$, $b \ge 0$, $a(1 - b) < 1$ and $La(1 + b) < 1$.

H3. The constant C is such that $C = af(C - bC - c)$.

Remark 3. Note condition (H3) guaranties the existence of an equilibrium $x^* = C$ of the equation (1).

We shall introduce the global Mittag-Leffler stability notion of the equilibrium of (1) by the following definition, which is analogous to the definition given in [12].

Definition 4. The equilibrium x^* of (1) is said to be *globally Mittag-Leffler stable*, if for $\varphi_0 \in CB[[-T,0],\mathbb{R}]$ there exist constants $\nu > 0$ and $d > 0$ such that

$$|x(t) - x^*| \leq \{M(|\varphi_0 - x^*|_T)E_q(-dt^q)\}^\nu, \; t \geq 0,$$

where $x(t)$ is a solution of the IVP for the Caputo impulsive fractional differential equation (1), (2), E_q is the Mittag-Leffler function (with one parameter), $M : \mathbb{R}_+ \to \mathbb{R}_+ : \; M(0) = 0$, and $M(.)$ is Lipschitzian.

Remark 4. Global Mittag-Leffler stability implies global asymptotic stability (see [12]).

In our further study we will use the following result:

Lemma 2. *([10]). Let $m \in C([t_0 - \tau, \infty), \mathbb{R})$ and satisfies the Caputo fractional differential inequality ${}^c_{t_0}D^q_t m(t) \leq g(t, |m_t|_0), \; t > t_0$ with $q \in (0,1)$, where $g \in C([t_0,\infty) \times \mathbb{R}_+, \mathbb{R}_+$ and $|m_t|_0 = \max_{-\tau \leq s \leq 0} |m(t+s)|$. Assume that $\eta(t)$ is the maximal solution of the initial value problem for the scalar Caputo fractional differential equation*

$${}^c_{t_0}D^q_t u(t) = g(t,u) \text{ for } t \geq t_0, \qquad u(t_0) = u_0 \geq 0$$

existing on $[t_0, \infty)$.

Then if $|m_{t_0}|_0 \leq u_0$ we have $m(t) \leq \eta(t), \;\; t \geq t_0$.

Based on the above result we obtain the following comparison result for Caputo impulsive fractional differential equation:

Lemma 3. *Let the function $m \in PC^q(\mathbb{R}_+, \mathbb{R}) \bigcup PC([-T,0], \mathbb{R})$ and satisfies the Caputo fractional impulsive differential inequalities for $0 < q < 1$*

$$\begin{aligned} &{}^c_0D^q_t m(t) \leq A|m_t|_0, \text{ for } t \geq 0, \; t \neq t_i, \\ &m(t_i + 0) \leq a_i m(t_i - 0) \text{ for } \; i = 1,2,\ldots \end{aligned} \tag{7}$$

where $|m_t|_0 = \sup_{-T \leq s \leq 0} m(t+s)$, $A = const < 0$, $a_i = const < \infty :$

$$a_i \geq \frac{E_q(A(t_i - T - t_{i-1})^q)}{E_q(A(t_i - t_{i-1})^q)}, \; i = 1,2,\ldots, \quad \lim_{k\to\infty} \prod_{i=1}^k a_i = B < \infty,$$

$t_0 = 0$, $x_0 \geq 0$.

Then if $|m_0|_0 \leq u_0$ we have

$$\begin{aligned} m(t) \leq u_0 \Big(\prod_{i=1}^k a_i E_q(A(t_i - t_{i-1})^q)\Big) E_q(A(t-t_k)^q) \\ \text{for } \;\; t \in (t_k, t_{k+1}], \; k = 0,1,2,3,\ldots. \end{aligned} \tag{8}$$

P r o o f: We use an induction w.r.t. the interval of continuity.

Note the fractional equation (4) is the corresponding to the fractional impulsive differential inequalities (7) with a solution $\eta(t)$ given by (5).

Let $t \in [0, t_1]$. Then according to Lemma 2 applied to the interval $[0, t_1]$ with $g(t, u) = Au$, $t_0 = 0$ and $\eta(t) = u_0 E_q(At^q)$ we obtain $m(t) \le u_0 E_q(At^q)$.

Let $t = t_1$. Then $m(t_1 + 0) \le a_1 m(t_1 - 0) \le a_1 u_0 E_q(At_1^q) = \eta(t_1 + 0)$.

Let $t \in (t_1, t_2]$. Since $A < 0$ then

$$\sup_{-T \le s \le 0} m(t_1 + s) \le u_0 \sup_{-T \le s \le 0} E_q(A(t_1 + s)^q) = u_0 E_q(A(t_1 - T)^q).$$

From the condition for a_1 we obtain

$$|m_{t_1}|_0 \le a_1 u_0 E_q(A(t_1)^q) = \eta(t_1 + 0).$$

According to Lemma 2 applied to the interval $[t_1 + 0, t_2]$ with $g(t, u) = Au$, $t_0 = t_1 + 0$, $u_0 = \eta(t_1 + 0)$ we get

$$m(t) \le \eta(t) = \eta(t_1 + 0) E_q(A(t - t_1)^q) = u_0 a_1 E_q(At_1^q) E_q(A(t - t_1)^q).$$

Let $t = t_2$. Then

$$m(t_2 + 0) \le a_2 m(t_2 - 0) \le a_2 u_0 a_1 E_q(At_1^q) E_q(A(t_2 - t_1)^q) = \eta(t_2 + 0).$$

Let $t \in (t_2, t_3]$. Then

$$\begin{aligned} &\sup_{-T \le s \le 0} m(t_2 + s) \le u_0 a_1 E_q(At_1^q) \sup_{-T \le s \le 0} E_q(A(t_2 + s - t_1)^q) \\ &\le u_0 a_1 E_q(At_1^q) E_q(A(t_2 - T - t_1)^q). \end{aligned} \tag{9}$$

From the condition for a_2 we obtain

$$|m_{t_2}|_0 \le u_0 a_1 E_q(At_1^q) a_2 E_q(A(t_2 - t_1)^q) = \eta(t_2 + 0).$$

According to Lemma 2 applied for the interval $[t_2 + 0, t_3]$ with $g(t, u) = Au$, $t_0 = t_2 + 0$, $u_0 = \eta(t_2 + 0)$ we get

$$\begin{aligned} &m(t) \le \eta(t) = \eta(t_2 + 0) E_q(A(t - t_2)^q) \\ &= u_0 a_1 a_2 E_q(At_1^q) E_q(A(t_2 - t_1)^q) E_q(A(t - t_2)^q), \quad t \in (t_2, t_3]. \end{aligned}$$

Continue this process we prove the claim of Lemma 3. □

Remark 5. If $a_i = e^{p^i} > 1$ with $p \in (0,1)$ then $\lim_{k\to\infty} \prod_{i=1}^{k} a_i < \infty$, i.e. these a_i are admissible for Lemma 3.

Corollary 1. *Let the conditions of Lemma 3 are satisfied. Then* $m(t) \leq u_0 B E_q(At^q), \quad t \geq 0.$

P r o o f: The claim of Corollary 1 follows from the inequalities $a_i \geq 1, \ i = 1, 2, \dots,$

$$\Big(\prod_{i=1}^{k} E_q(A(t_i - t_{i-1})^q)\Big) E_q(A(t-t_k)^q) \leq E_q(At^q),$$

and

$$\prod_{i=1}^{k} a_i \leq B.$$

3. Main Results

In this section, a sufficient condition for global Mittag-Leffler stability which implies global asymptotic stability of the equilibrium of the impulsive fractional-order neural network system (1) will be derived.

Let $x(t)$ be a solution of (1), (2) defined on $[-T, \infty)$. Consider the substitution $y(t) = x(t) - c$, $t \in [-T, \infty)$. Then the function $y(t)$ is a solution of the following impulsive fractional differential equation

$$\begin{aligned} &{}_0^cD_t^q y(t) = G(y(t), y(t-\tau(t))), \ t \neq t_k, \ t \geq 0, \\ &\Delta y(t_k) = J_k(y(t_k)), \ k = 1, 2, ..., \end{aligned} \tag{10}$$

where $G(y, v) = -y - c + af(y - bv - bc)$ and $J_k(y) = -\sigma_k(y + c - C), \ k = 1, 2, \dots.$

It is easy to check that if $x(t) \equiv C$ (i.e. consider the equilibrium of (1)), then $y^* = C - c$ is the equilibrium of (10).

Set $u(t) = y(t) - y^*$, (or $u(t) = x(t) - x^*$) for $t \in [-T, \infty)$. Then the function $u(t)$ is a solution of the following impulsive fractional differential equation

$$\begin{aligned} &{}_0^cD_t^\alpha u(t) = F(u(t), u(t-\tau(t))), \ t \neq t_k, \ t \geq 0, \\ &\Delta u(t_k) = P_k(u(t_k)), \ k = 1, 2, ..., \end{aligned} \tag{11}$$

where $F(u, v) = G(u + y^*, v + y^*) = -u - C + af\,(u - bv - (b-1)C - c)$ and $P_k(u) = J_k(u + y^*) = -\sigma_k u$, $k = 1, 2, ...$

It is easy to check that if $x(t) \equiv C$ (i.e. consider the equilibrium of (1)), then $u^* = 0$ is the equilibrium of (11).

Theorem 1. *Let the following conditions be satisfied:*

1. *Conditions H1-H3 hold.*
2. *There exists a constant $K < \infty$ such that*

$$\lim_{k\to\infty}\prod_{i=1}^{k}|1-\sigma_i| = K \tag{12}$$

and

$$(1-\sigma_i)^2 \geq \frac{E_q(A(t_i - T - t_{i-1})^q)}{E_q(A(t_i - t_{i-1})^q)},\ i = 1, 2, \dots . \tag{13}$$

Then the equilibrium $x^* = C$ of (1) is globally Mittag–Leffler stable.

P r o o f: Let $\varphi_0 \in CB([-T, 0], \mathbb{R})$ be an arbitrary function and $x(t)$ be a solution of the IVP for the Caputo impulsive fractional differential equation (1), (2).

Consider the function $m(t) = \Big(u(t)\Big)^2 = \Big(x(t) - x^*\Big)^2$ for $t \in [-T, \infty)$.

Let $t = t_k,\ k = 1, 2, \dots$. From the impulsive condition in (11) we get

$$m(t_k + 0) = \Big(u(t_k + 0)\Big)^2 = \big(u(t_k - 0)\big)^2\Big(1 - \sigma_k\Big)^2 = (1 - \sigma_k)^2 m(t_k - 0). \tag{14}$$

Let $t \geq 0$ and $t \in (t_k, t_{k+1}]$. Then $u(t) \in C^1((t_k, t_{k+1}], \mathbb{R})$ and according to Lemma 1 we derive the estimate

$$\begin{aligned}
{}^{c}_{t_k}D^q_t m(t) &= {}^{c}_{t_k}D^q_t (u(t))^2 \leq 2u(t)\big({}^{c}_{t_k}D^q_t u(t)\big)\\
&\leq 2u(t)\Big(- u(t) + af\,(u(t) - bu(t-\tau(t)) - bC - c)\Big)\\
&= -2\Big(u(t)\Big)^2 + 2au(t)\Big(- f((1-b)C - c)\\
&\qquad + f\,(u(t) - bu(t-\tau(t)) + (1-b)C - c)\Big)\\
&\leq -2\Big(u(t)\Big)^2 + 2aL|u(t)|.|u(t) - b(u(t-\tau(t))|\\
&\leq -2(1-aL)m(t) + 2aLb|u(t)|.|u(t-\tau(t))|\\
&\leq -2(1-aL)m(t) + 2aLb|m_t|_0\\
&\leq -d|m_t|_0, \quad t \in (t_k, t_{k+1}],
\end{aligned} \tag{15}$$

where $d = 2(1 - aL(1 + b)) > 0$ according to condition H2.

Therefore, the function $m(t)$ satisfies the impulsive fractional differential inequalities (7) with $A = -d$, $a_i = (1-\sigma_i)^2$ and $u_0 = \sup_{-T\leq s\leq 0}\Big(\varphi_0(s) - x^*\Big)^2$. According to Corollary 1 we have

$$m(t) \leq K^2 \sup_{-T\leq s\leq 0}\Big(\varphi_0(s) - x^*\Big)^2 E_q(-dt^q),\ t \geq 0,$$

or

$$|u(t)| \leq K|\varphi_0(s) - x^*|_T\big\{E_q(-d\ t^q)\big\}^{1/2},\ t \geq 0.$$

Thus, the zero solution of the impulsive fractional differential equation (11) is globally Mittag–Leffler stable, and therefore the equilibrium x^* of (1) is globally Mittag-Leffler stable (and hence, globally asymptotically stable). □

Example 3. Consider the following Caputo fractional impulsive delayed cellular neural network with dynamical thresholds

$$\begin{aligned} {}_0^cD_t^{0.2}x(t) &= -x(t) + 0.5\cos\Big(x(t) - 0.5(t - 0.5\sin(t)) - 1\Big), \\ &\qquad\qquad\qquad \text{for}\ \ t \neq k,\ t \geq 0, \\ \Delta x(k) &= -(2e^{0.5^k} - 1)(x(k-0) - C),\ k = 1, 2, ..., \end{aligned} \tag{16}$$

where $x \in \mathbb{R}$, $t_k = k,\ k = 1, 2, \ldots,\ \sigma_i = 2e^{0.5^i} - 1,\ \tau(t) = 0.5\sin(t) \leq 0.5 = T$, $f(u) = cos(u),\ L = 1,\ c = 1,\ a = 0.5,\ b = 0.5,\ d = 2(1 - aL(1+b)) = 0.5$, $C \approx 0.221904$ is the solution of the equation $-C + 0.5\cos(0.5C + 1)$. Then all conditions H1,H2,H3 are satisfied. Also,

$$\lim_{k\to\infty}\prod_{i=1}^{k}|1-\sigma_i| = 2\lim_{k\to\infty}\prod_{i=1}^{k}e^{0.5^i} = 2\lim_{k\to\infty}e^{\sum_{i=1}^{k}0.5^i} = 2e^2 = K$$

and

$$\begin{aligned} (1-\sigma_i)^2 = 4e^{2(0.5^i)} > 4 > \frac{E_q(A(t_i - T - t_{i-1})^q)}{E_q(A(t_i - t_{i-1})^q)} \\ = \frac{E_{0.2}(-(1-0.5)^0.2)}{E_{0.2}(-1)} \\ \approx 1.07472, \quad i = 1, 2, \ldots \end{aligned} \tag{17}$$

According to Theorem 1 the equilibrium C of (16) is globally Mittag-Leffler stable (and hence, globally asymptotically stable) and

$$|x(t) - C| \leq 2e^2|\varphi_0(s) - C|_{0.5}\big\{E_{0.2}(-0.5\ t^{0.2})\big\}^{1/2},\ t \geq 0.$$

□

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