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SOME RESULTS ON SEMIGROUP IDEALS IN PRIME RING WITH DERIVATIONS

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Abstract: Let R be a prime ring, I be a nonzero semigroup ideal of R, d, g, h be derivations of R and $a, b \in R$. It is proved that if $d(x) = ag(x)+h(x)b$ for all $x \in I$ and a, b are not in $Z(R)$ then there exists for some $\lambda \in C$ such that $h(x) = \lambda [a, x]$, $g(x) = \lambda [b, x]$ and $d(x) = \lambda [ab, x]$ for all $x \in I$.

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1. Introduction

R denotes an associative ring with center $Z(R)$. A ring R is called prime if $xRy = 0$ either $x = 0$ or $y = 0$. A nonempty subset I of R is called right semigroup ideal (resp. left semigroup ideal) if $IR \subseteq I$ (resp. $RI \subseteq I$); and if I is both right and left semigroup ideal then I said to be a semigroup ideal. An additive map $d : R \longrightarrow R$ is called derivation if $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. For any $x, y \in R$, $xy - yx$ is denoted by $[x, y]$. For fixed $a \in R$, the mapping $I_a : R \longrightarrow R$ given by $I_a(x) = [a, x]$ is a derivation called inner derivation determined by a. In [3], I. N. Herstein proved that if R is a prime ring with characteristic 2, d is a nonzero derivation of R and $a \in R$ such that $ad(x) - d(x)a = 0$ for all $x \in R$, then $a^2 \in Z(R)$. Moreover, if $a \notin Z(R)$ then there exists $\lambda \in C$ such that $d(x) = \lambda [a, x]$ for all $x \in R$. Later in [1], Bresar

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showed that if g and h are derivations of a prime ring R and a, b in R but not in $Z(R)$ such that $ag(x) + h(x)b = 0$ for all $x \in R$, then there exists $\lambda \in C$ such that $g(x) = \lambda [b, x]$ and $h(x) = \lambda [a, x]$ for all $x \in R$. Moreover, $g \neq 0$ then $ab \in Z(R)$. It is clear that Bresar generalized Herstein's result. In this paper, our aim is to generalized Bresar's result for a nonzero semigroup ideal of R.

In all present paper R is a prime ring, $Q_r(R)$ is the Martindale Quotient Ring of R. The center of $Q_r(R)$ is denoted by C and called the extended centroid of R.

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2. Results

Lemma 1. *[2, Lemma 1.3.2] If $a, b \in R$ are such that $axb = bxa$ for all $x \in R$, and if $a \neq 0$ then $b = \lambda a$ for some λ in the extended centroid of R.*

Lemma 2. *Let I be a nonzero right (left) semigroup ideal of R and d be a derivation of R. If $d(x) = 0$ for all $x \in I$ then $d = 0$.*

Proof. Taking xr with $r \in R$ instead of x in the hypothesis, it holds that for all $x \in I, r \in R$

$$xd(r) = 0.$$

Replacing x by xs with $s \in R$ in the last equation, it implies that $xsd(r) = 0$ for all $x \in I$ and $r, s \in R$. Thus,

$$xRd(r) = (0), \ \forall x \in I, r \in R.$$

Since R is a prime ring and I is a nonzero right semigroup ideal of R, it implies that d is zero. □

Lemma 3. *Let I be a nonzero semigroup ideal of R, d be a derivation of R and $a \in R$. If $ad(x) = 0$ for all $x \in I$ then $a = 0$ or $d = 0$.*

Proof. Replacing x by xr with $r \in R$ in the hypothesis, it holds that $axd(r) = 0$ for all $x \in I$, $r \in R$. Taking sx with $s \in R$ instead of x in the last equation, it holds that $axsd(r) = 0$ for all $x \in I$ and $s, r \in R$. Thus,

$$axRd(r) = (0), \ \forall x \in I, r \in R.$$

By the primeness of R, it follows that $ax = 0$ for all $x \in I$ or $d(r) = 0$ for all $r \in R$. Hence,

$$aI = (0) \text{ or } d = 0.$$

Assume that $d \neq 0$. It yields that $ax = 0$ for all $x \in I$. Replacing x by rx with $r \in R$ in the last equation, we have that $arx = 0$ for all $x \in I$, $r \in R$. Thus,

$$aRx = (0),\ \forall x \in I.$$

Since R is a prime ring and I is a nonzero semigroup ideal of R, a is zero. □

Lemma 4. *Let I be a nonzero semigroup ideal of R and a, b be elements of R. If $aIb = 0$ then $a = 0$ or $b = 0$.*

Proof. The hypothesis yields $axb = 0$ for all $x \in I$. Replacing x by xr with $r \in R$ in the last equation, it implies that $axrb = 0$ for all $x \in I$ and $r \in R$. Thus,

$$axRb = (0),\ \forall x \in I.$$

Thus, the primeness of R forces that $ax = 0$ for all $x \in I$ or $b = 0$. Suppose that $b \neq 0$. It means that $ax = 0$ for all $x \in I$. Taking rx with $r \in R$ instead of x in the last equation, it holds that $arx = 0$ for $x \in I$, $r \in R$. Thus,

$$aRx = (0),\ \forall x \in I.$$

Since R is prime ring and I is a nonzero ideal of R, we get $a = 0$. □

Lemma 5. *Let I be a nonzero semigroup ideal of R, d, g be derivations of R and*

$$d(x)g(y) = g(x)d(y),\ \forall x, y \in I.$$

If d is nonzero then there exists some $\lambda \in C$ such that $g(x) = \lambda d(x)$ for all $x \in I$.

Proof. Replacing y by zy with $z \in I$ in the hypothesis, it implies that $d(x)\,g(z)\,y + d(x)\,zg(y) = g(x)\,d(z)\,y + g(x)\,zd(y)$ for all $x, y, z \in I$. By using the hypothesis, this equation reduces to

$$d(x)\,zg(y) = g(x)\,zd(y)\,,\ \forall\ x, y, z \in I. \tag{1}$$

Taking zr with $r \in R$ instead of z in (1), it forces that

$$d(x)zrg(y) = g(x)zrd(y),\ \forall x, y, z \in I, r \in R. \tag{2}$$

Since d is a nonzero derivation of R, there exists $a \in I$ such that $d(a) \neq 0$ from Lemma 2. Assume that $d(a)z = 0$ for all $z \in I$. According to Lemma 4, it yields $d(a) = 0$ which is contradiction. Thus, there exists $t \in I$ such that $d(a)t \neq 0$. Replacing x and y by a in (2), it holds that $d(a)zrg(a) = g(a)zrd(a)$

for all $z \in I$, $r \in R$. Taking t instead of z in the last equation, it follows that $d(a)trg(a) = g(a)trd(a)$ for all $r \in R$. Right multiplying the last equation by t

$$d(a)trg(a)t = g(a)trd(a)t, \ \forall r \in R. \tag{3}$$

is obtained. Because of $d(a)t \neq 0$, from Lemma 1, there exists $\lambda_{a,t} \in C$ such that

$$g(a)t = \lambda_{a,t} d(a)t. \tag{4}$$

Suppose that $d(a)k \neq 0$ for some $k \in I$. From (4), there exists $\lambda_{a,k} \in C$ such that

$$g(a)k = \lambda_{a,k} d(a)t. \tag{5}$$

Replacing z by t and multiplying from right by k in (2), it holds that

$$d(x)trg(y)k = g(x)trd(y)k,$$

for all $x, y \in I$, $r \in R$. Taking a instead of x and y in the last equation, it follows that

$$d(a)trg(a)k = g(a)trd(a)k, \ \forall r \in R.$$

By using (4) and (5),

$$d(a)tr\lambda_{a,k} d(a)k = \lambda_{a,t} d(a)trd(a)k, \ \forall r \in R$$

is obtained. That is,

$$(\lambda_{a,k} - \lambda_{a,t})d(a)trd(a)k = 0, \forall r \in R.$$

The last equation multiplies from left by $q \in Q_r(R)$, it follows that $(\lambda_{a,k} - \lambda_{a,t})qd(a)trd(a)k = 0$ for all $r \in R$, $q \in Q_r(R)$. Thus,

$$(\lambda_{a,k} - \lambda_{a,t})Q_r(R)d(a)trd(a)k = (0), \ \forall r \in R.$$

From the primeness of $Q_r(R)$, it implies that $\lambda_{a,k} = \lambda_{a,t}$ or $d(a)trd(a)k = 0$ for all $r \in R$. If $\lambda_{a,k} \neq \lambda_{a,t}$, it holds that $d(a)trd(a)k = 0$ for all $r \in R$. Since R is a prime ring, it yields that $d(a)t = 0$ or $d(a)k = 0$. This is a contradiction because of $d(a)t \neq 0$ and $d(a)k \neq 0$. That is,

$$\lambda_{a,k} = \lambda_{a,t} = \lambda_a$$

Hence, for all $t \in I$ satisfying $d(a)t \neq 0$

$$g(a)t = \lambda_a d(a)t. \tag{6}$$

Suppose that $d(a)t = 0$ for some $t \in I$. Replacing x by a and z by t in (2), it holds that $d(a)trg(y) = g(a)trd(y)$ for all $y \in I$, $r \in R$. Because of $d(a)t = 0$, it follows that

$$g(a)trd(y) = 0, \ \forall y \in I, r \in R.$$

Since R is a prime ring and d is a nonzero derivation of R, it implies that $g(a)t = 0$. That is, for all $t \in I$ satisfying $d(a)t = 0$

$$g(a)t = \lambda_a d(a)t \tag{7}$$

From (6) and (7), for all $t \in I$ and for all $a \in I$ satisfying $d(a) \neq 0$

$$g(a)t = \lambda_a d(a)t$$

is obtained. This means that $(g(a) - \lambda_a d(a))t = 0$ for all $t \in I$. From Lemma 4, it forces for all $a \in I$ satisfying $d(a) \neq 0$

$$g(a) = \lambda_a d(a). \tag{8}$$

Suppose that $d(b) \neq 0$ for some $b \in I$. In this case, there exists λ_b such that

$$g(b) = \lambda_b d(b) \tag{9}$$

Taking a, b instead of x, y in (1) respectively, it follows that

$$d(a)zg(b) = g(a)zd(b), \forall z \in I.$$

By using (8) and (9),

$$d(a)z\lambda_b d(b) = \lambda_a d(a)zd(b), \ \forall z \in I$$

is obtained. That is,

$$(\lambda_b - \lambda_a)d(a)zd(b) = 0, \ \forall z \in I.$$

It follows that

$$(\lambda_b - \lambda_a)Q_r(R)d(a)zd(b) = (0), \forall z \in I.$$

Since $Q_r(R)$ is a prime ring and $d(a)$, $d(b)$ are nonzero, it implies that

$$\lambda_a = \lambda_b = \lambda$$

Thus, for all $a \in I$ satisfying $d(a) \neq 0$

$$g(a) = \lambda d(a). \tag{10}$$

Assume that $d(a) = 0$ for some $a \in I$. Replacing x by a in (1) and by using $d(a) = 0$,

$$g(a)zd(y) = 0, \forall y, z \in I$$

is obtained. From Lemma 4, it yields that either $g(a) = 0$ or $d(y) = 0$ for all $y \in I$. Since d is a nonzero derivation, it yields $g(a) = 0$. So if $d(a) = 0$ for some $a \in I$ then $g(a) = 0$. Thus, for all $a \in I$ satisfying $d(a) = 0$

$$g(a) = \lambda d(a) \tag{11}$$

From (10) and (11), there exists some $\lambda \in C$ such that

$$g(x) = \lambda d(x), \ \forall x \in I.$$

□

Lemma 6. *Let I be nonzero semigroup ideal of R, d, g, h, f be derivations of R and*

$$d(x)\, g(y) = h(x)\, f(y) \quad \forall x, y, z \in I.$$

If d and f are nonzero then there exists some $\lambda \in C$ such that $g(x) = \lambda f(x)$ and $h(x) = \lambda d(x)$ for all $x \in I$.

Proof. Replacing x by xz with $z \in I$ in the hypothesis, it holds that

$$d(x)zg(y) + xd(z)g(y) = h(x)zf(y) + xh(z)f(y) \ \ \forall x, y, z \in I. \tag{12}$$

Taking z instead of x in the hypothesis, it follows that $d(z)g(y) = h(z)f(y)$ for all $z, y \in I$. This equation multiplies from left by x, it follows that $xd(z)g(y) = xh(z)f(y)$ $\forall x, y, z \in I$. Hence, (12) reduces to

$$d(x)zg(y) = h(x)zf(y), \ \forall x, y, z \in I. \tag{13}$$

Taking z by $zf(w)$ with $w \in I$ in (13), it holds that

$$d(x)z(f(w)g(y) - g(w)f(y) = 0, \ \forall x, y, z, w \in I.$$

From Lemma 4, it follows that either $f(w)g(y) = g(w)f(y)$ for all $w, y \in I$ or $d(x) = 0$ for all $x \in I$. Since d is a nonzero derivation of R from Lemma 2, it hold that $f(w)g(y) = g(w)f(y)$ for all $w, y \in I$. So, by Lemma 5 , there exists some $\lambda \in C$ such that $g(x) = \lambda d(x)$ for all $x \in I$. Replacing $g(y)$ by $\lambda f(y)$ in (13), it holds that $(\lambda d(x) - h(x))zf(y) = 0$ for all $x, z, y \in I$. According to Lemma 4, it implies that either $h(x) = \lambda d(x)$ for all $x \in I$ or $f(y) = 0$ for all $y \in I$. Because f is a nonzero derivation of R from Lemma 2, it follows that $h(x) = \lambda d(x)$ for all $x \in I$. □

Theorem 7. *Let I be a nonzero semigroup ideal of R and d, g, h be derivations of R. Suppose that $a, b \in R$ such that*

$$d(x) = ag(x) + h(x)b, \ \forall x \in I.$$

If $a, b \notin Z(R)$ then there exists some $\lambda \in C$ such that $h(x) = \lambda[a, x]$, $g(x) = \lambda[b, x]$ and $d(x) = \lambda[ab, x]$ for all $x \in I$.

Proof. Replacing x by xy with $y \in I$ in the hypothesis, it follows that

$$h(x)(by - yb) = (ax - xa)g(y), \ \forall x, y \in I.$$

Let I_a, I_b be inner derivation determined by a, b respectively. Hence, the last equation yields

$$I_a(x)g(y) = h(x)I_b(y) \ \forall x, y \in I.$$

Since a and b are not elements of $Z(R)$, I_a and I_b are nonzero inner derivations of R. Thus, from Lemma 6, there exists some $\lambda \in C$ such that $h(x) = \lambda[a, x]$, $g(x) = \lambda[b, x]$. Hence, it follows that $d(x) = \lambda[ab, x]$ for all $x \in I$. □

Corollary 8. *Let I be a nonzero semigroup ideals of R, g, h be derivations of R and $a, b \in R$ such that*

$$ag(x) + h(x)b = 0, \ \forall x \in I.$$

If $a, b \notin Z(R)$ then there exists some $\lambda \in C$ such that $g(x) = \lambda[b, x]$ and $h(x) = \lambda[b, x]$ for all $x \in I$. Moreover, if $g \neq 0$ then $ab \in Z(R)$.

Proof. Since 0 is a derivation of R, there exists some $\lambda \in C$ such that $g(x) = \lambda[b, x]$, $h(x) = \lambda[b, x]$ and $0(x) = \lambda[ab, x]$ for all $x \in I$ from Theorem 7. Assume that $g \neq 0$. In this case, $\lambda \neq 0$.By using $0 = \lambda[ab, x]$ for all $x \in I$, it follows that $0 = \lambda q[ab, x]$ for all $x \in I$, $q \in Q_r(R)$. Thus,

$$\lambda Q_r(R)[ab, x] = (0), \ \forall x \in I.$$

By the primeness of $Q_r(R)$, either $\lambda = 0$ or $[ab, x] = 0$ for all $x \in I$. Because of $\lambda \neq 0$,

$$[ab, x] = 0, \ \forall x \in I$$

is obtained. Taking rx with $r \in R$ instead of x in this equation, it holds that $[ab, r]x = 0$ for all $x \in I, r \in R$. From Lemma 4, it implies that $[ab, r] = 0$ for all $r \in R$. Thus, $ab \in Z(R)$. □

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