International Journal of Pure and Applied Mathematics
Volume 109 No. 4 2016, 789-797
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: 10.12732/ijpam.v109i4.3

ijpam.eu

ON SOME MISCONCEPTIONS AND CHATTERJEE-TYPE G-CONTRACTION

T. Phaneendra[1 §], S. Saravanan[2]
[1,2]Department of Mathematics
School of Advanced Sciences,
VIT University
Vellore-632014, Tamil Nadu, INDIA

Abstract: Let (X, d) be a G-metric space, f, a self-map on X and $x_0 \in X$. Some misconceptions are brought about in findings of Mustafa et al [2], and a fixed point theorem for a Chatterjee-type G-contraction on a complete G-metric space is proved. More over, the unique fixed point p will be its contractive fixed point, in the sense that for each each $x_0 \in X$, the f-iterates $x_0, fx_0, ..., f^n x_0, ...$ converge to p.

AMS Subject Classification: 54H25
Key Words: G-metric space, Chatterjee-type G-contraction, fixed point, G-contractive fixed point

1. Introduction

Let X be a nonempty set and $G : X \times X \times X \to \mathbb{R}$ such that

(G1) $G(x, y, z) \geq 0$ for all $x, y, z \in X$ with $G(x, y, z) = 0$ if $x = y = z$,

(G2) $G(x, x, y) > 0$ for all $x, y \in X$ with $x \neq y$,

(G3) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $z \neq y$,

Received: Jule 2, 2016
Revised: September 13, 2016
Published: October 5, 2016

§Correspondence author

(G4) $G(x,y,z) = G(x,z,y) = G(y,x,z) = G(z,x,y)$
$= G(y,z,x) = G(z,y,x)$ for all $x,y,z \in X$

(G5) $G(x,y,z) \leq G(x,w,w) + G(w,y,z)$ for all $x,y,z,w \in X$

Then G is called a G-metric on X and the pair (X,G), a G-metric space. Axiom (G4) reveals that G is symmetric in the three variables x, y and z, and Axiom (G5) is referred to as the rectangle inequality (of G). This notion was introduced by Mustafa and Sims [3] in 2006.

From the definition of G-metric space, it immediately follows that

$$G(x,y,y) \leq 2G(x,x,y) \text{ for all } x,y \in X. \tag{1}$$

We use the following notions, developed in [3]:

Definition 1.1. Let (X,G) be a G-metric space. A G-ball in X is defined by

$$B_G(x,r) = \big\{y \in X : G(x,y,y) < r\big\}.$$

It is easy to see that the family of all G-balls forms a base topology, called the G-metric topology $\tau(G)$ on X.

Also

$$\rho_G(x,y) = G(x,y,y) + G(x,x,y) \text{ for all } x,y \in X. \tag{2}$$

induces a metric on X, and the G-metric topology coincides with the metric topology induced by the metric ρ_G. This allows us to readily transform many concepts from metric space into the setting of G-metric space.

Definition 1.2. A sequence $\langle x_n\rangle_{n=1}^{\infty}$ in a G-metric space (X,G) is said to be *G-convergent* with limit $p \in X$ if it converges to p in the G-metric topology $\tau(G)$.

Lemma 1.1. *The following statements are equivalent in a G-metric space (X,G):*

(a) $\langle x_n\rangle_{n=1}^{\infty} \subset X$ *is G-convergent with limit* $p \in X$,

(b) $\lim_{n\to\infty} G(x_n,x_n,p) = 0$,

(c) $\lim_{n\to\infty} G(x_n,p,p) = 0$.

Definition 1.3. A sequence $\langle x_n\rangle_{n=1}^{\infty}$ in a G-metric space (X,G) is said to be G-Cauchy if $\lim_{n,m\to\infty} G(x_n,x_m,x_m) = 0$.

Definition 1.4. A G-metric space (X, G) is said to be G-complete if every G-Cauchy sequence in X converges in it.

Definition 1.5. Let (X, G) be a G-metric space. A set $S \subset X$ is said to be G-bounded or simply bounded if there exists a positive number M such that $G(x, y, z) < M$ for all $x, y, z \in S$.

Definition 1.6. Let (X, G) be a G-metric space. We define the diameter of $S \subset X$ by diam $S = \delta(S) = \sup\{G(x, y, z) : x, y, z \in S\}$. The set S is G-bounded if and only if $\delta(S) < \infty$.

As a part of an extensive research in G-metric spaces, we refer to a couple of interesting results from [2]. The first of them is:

Theorem 1.1. *Let (X, G) be a complete G-metric space and $f : X \to X$ satisfying one of the following conditions:*

$$G(fx, fy, fy) \leq k \max\{G(x, fy, fy), G(y, fx, fx), G(y, fy, fy)\} \tag{3}$$

or

$$G(fx, fy, fy) \leq k \max\{G(x, x, fy), G(y, y, fx), G(y, y, fy)\} \tag{4}$$

for all $x, y, z \in X$, where $0 \leq k < 1$. Then f has a unique fixed point p and f is G-continuous at p.

In the proof of Theorem 1.1, the authors used the following notations:

$$\Gamma_n = \max\{G(x_i, x_j, x_j) : \ i, j \in \{0, 1, ..., n+1\}\}, \ n \in \mathbb{N};$$
$$\Gamma = \max\{\Gamma_k : \ k = n, ..., m-1\} \text{ for } m > n,$$

where $x_n = f^n x_0$ for each $x_0 \in X$. The authors used the induction to show that

$$G(x_n, x_{n+1}, x_{n+1}) \leq k^n \Gamma_n\cdot \tag{5}$$

Then with $\Gamma = \max\{\Gamma_k : \ k = n, ..., m-1\}$ for $m > n$, and the rectangle inequality, they established

$$G(x_n, x_m, x_m) \leq \left(\frac{k^n}{1-k}\right)\Gamma\cdot \tag{6}$$

Further, the authors employed the limit as $n \to \infty$ in (6) to see that

$$G(x_n, x_m, x_m) \to 0. \tag{7}$$

That is $\langle x_n \rangle_{n=1}^{\infty}$ is a Cauchy sequence.

We have two observations in these arguments. Firstly, the sets

$$A_n = \{G(x_i, x_j, x_j) : \ i, j \in \{0, 1, ..., n+1\}\}, \ n = 1, 2, 3, ... \tag{8}$$

constitute an expanding sequence of sets of nonnegative real numbers. Hence

$$\max A_n \leq \max A_{n+1} \text{ or } \Gamma_n \leq \Gamma_{n+1} \text{ for all } n. \tag{9}$$

Therefore, $\Gamma = \max\{\Gamma_n, \Gamma_{n+1}, ..., \Gamma_{m-1}\} = \Gamma_{m-1}$.

Then for $m > n$, from the repeated application of the rectangle inequality and (5), it follows that

$$\begin{aligned} &G(x_n, x_m, x_m) \\ &\quad \leq \underbrace{G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_{n+2}, x_{n+2}) + ... + G(x_{m-1}, x_m, x_m)}_{m-n \text{ terms}} \\ &\quad \leq k^n \left(1 + k + \cdots + k^{m-n-1}\right) \Gamma_{m-1} \leq \left(\tfrac{k^n}{1-k}\right) \Gamma_{m-1}, \end{aligned}$$

which gives (6).

Definition 1.7. Given $x_0 \in X$, the orbit at x_0 is the the sequence of iterates

$$O_f(x_0) = \{x_0, x_1, ..., x_n, ...\}, \text{ where } x_n = f x_{n-1} \text{ for } n \geq 1. \tag{10}$$

Remark 1.1. We now claim that the above proof of (7) requires that $O_f(x_0)$ at each $x_0 \in X$ is bounded so that $\sup[O_f(x_0)] = \delta < \infty$.

If possible, suppose that $O_f(x_0)$ is unbounded. Then there exists a positive integer n such that

$$G(x_1, x_n, x_n) \geq \mu \max\{G(x_1, x_r, x_r) : 0 \leq r \leq n-1\}, \tag{11}$$

where

$$\mu = \max\left\{\tfrac{2k}{1-k}, 2k\right\}. \tag{12}$$

Now from the inequality (3), we have

$$G(x_1, x_n, x_n) \leq k \max\{G(x_0, x_n, x_n), G(x_{n-1}, x_1, x_1), G(x_{n-1}, x_n, x_n)\}. \tag{13}$$

Here $M = \max\{G(x_0, x_n, x_n), G(x_{n-1}, x_1, x_1), G(x_{n-1}, x_n, x_n)\}$.

Case(a): Suppose that $M = G(x_0, x_n, x_n)$. Then (13) gives

$$\begin{aligned} G(x_1, x_n, x_n) &\leq kG(x_0, x_n, x_n) \\ &\leq k[G(x_0, x_1, x_1) + G(x_1, x_n, x_n)] \\ &\leq \left(\tfrac{k}{1-k}\right) G(x_0, x_1, x_1) \\ &\leq \mu \max\{G(x_0, x_r, x_r) : 0 \leq r \leq n\} \\ &< G(x_1, x_n, x_n), \end{aligned}$$

which is a contradiction.

Case(b): Suppose that $M = G(x_{n-1}, x_1, x_1)$. Then (13) gives

$$\begin{aligned} G(x_1, x_n, x_n) &\leq kG(x_{n-1}, x_1, x_1) \leq 2kG(x_1, x_{n-1}, x_{n-1}) \\ &\leq \mu \max\{G(x_0, x_r, x_r) : 0 \leq r \leq n\} < G(x_1, x_n, x_n) \end{aligned}$$

which is again a contradiction.

Case(c): Finally, suppose that $M = G(x_{n-1}, x_n, x_n)$. Then (13) gives

$$\begin{aligned} G(x_1, x_n, x_n) &\leq kG(x_{n-1}, x_n, x_n) \\ &\leq k[G(x_{n-1}, x_1, x_1) + G(x_1, x_n, x_n) \\ &\leq \left(\tfrac{2k}{1-k}\right) G(x_1, x_{n-1}, x_{n-1}) \\ &\leq \mu \max\{G(x_0, x_r, x_r) : 0 \leq r \leq n\} \\ &< G(x_1, x_n, x_n) \end{aligned}$$

which is also a contradiction.

These three contradictions prove that $O_f(x_0)$ is bounded and $\sup[O_f(x_0)] = \delta < \infty$.

Then from (6), it follows that

$$G(x_n, x_m, x_m) \leq \frac{k^n \delta}{1-k}. \tag{14}$$

Applying the limit as $n \to \infty$ in (14), we get (7). This proves that $\langle x_n \rangle_{n=1}^{\infty}$ is a Cauchy sequence.

The second result of [2] is:

Theorem 1.2. *Let (X, G) be a complete G-metric space and $f : X \to X$ satisfying one of the following conditions:*

$$G(fx, fy, fz) \leq k \max\{G(x, fy, fy), G(x, fz, fz), G(y, fx, fx) \\ G(y, fz, fz), G(z, fx, fx), G(z, fy, fy)\} \quad (15)$$

or

$$G(fx, fy, fz) \leq k \max\{G(x, x, fy), G(x, x, fz), G(y, y, fx) \\ G(y, y, fz), G(z, z, fx), G(z, z, fy)\} \quad (16)$$

for all $x, y, z \in X$, where $0 \leq k < 1$. Then f has a unique fixed point p and f is G-continuous at p.

Remark 1.2. It was claimed in [2] that Theorem 1.2 is a Corollary to Theorem 1.1, which is a misconception. In fact, the conditions (3) and (4) follow as particular cases of conditions (15) and (16) with $y = z$ respectively. Therefore, it is appropriate to assert that Theorem 1.1 is a Corollary to Theorem 1.2. The proof for Theorem 1.2 is just similar to the above proof and is omitted here.

Remark 1.3. Also if $k = 0$, writing $z = y = fx$ in (15) or (16), we see that $G(fx, f^2x, f^2x) = 0$ so that $f^2x = fx$ for each $x \in X$. That is, every fx is a fixed point of f Theorem 1.2. In other words, the fixed point is not unique in Theorem 1.2. Therefore, Theorem 1.2 in its revised form is stated as follows:

Theorem 1.3. *Let (X, G) be a G-metric space and $f : X \to X$ satisfying either* (15) *or* (16), *where $0 < k < 1$. If X is G-complete, then f has a unique fixed point p.*

Omitting the terms $G(x, fz, fz)$, $G(y, fx, fx)$ and $G(z, fy, fy)$ in (15), and restricting k to $(0, 1/3)$, say $0 \leq \gamma < 1/3$, we get

Corollary 1.1. *Let (X, G) be a complete G-metric space and $f : X \to X$ satisfying one of the following conditions:*

$$G(fx, fy, fz) \leq \gamma \max\{G(x, fy, fy), G(y, fz, fz), G(z, fx, fx)\} \\ \text{for all } x, y, z \in X, \quad (17)$$

where $0 \leq \gamma < 1/3$. Then f has a unique fixed point.

Since the maximum of three nonnegative numbers cannot exceed their sum, (17) is weakened as

$$G(fx, fy, fz) \leq \gamma[G(x, fy, fy) + G(y, fz, fz) + G(z, fx, fx)],$$

$$\text{for all } x, y, z \in X, \tag{18}$$

where $0 < \gamma < 1/3$. This is analogous to Chatterjee's contraction [1] in metric space with the choice

$$\rho(fx, fy) \leq c[G(x, fy) + G(y, fx)] \text{ for all } x, y \in X, \tag{19}$$

where $0 < c < 1/2$. We therefore call f satisfying (18), a Chatterjee-type G-contraction.

In the next section, we shall obtain a fixed point for a Chatterjee-type G-contraction on a complete G-metric space.

2. Main Result

The notion of G-contractive fixed point was introduced by Phaneendra with Kumara Swamy in [4]. In fact

Definition 2.1. A fixed point p of f on a G-metric space (X, G) is a G-contractive fixed point of it if the orbit $O_f(x_0) = \langle x_0, fx_0, ..., f^n x_0, ...\rangle$ at each $x_0 \in X$ is G-convergent with limit p.

It was shown that the unique fixed point of the self-map f with the following choices is a G-contractive fixed point.

(a) $G(fx, fy, fz) \leq qG(x, y, z)$ for all $x, y, z \in X$, where $0 \leq q < 1$,

(b) $G(fx, fy, fz) \leq aG(x, fx, fx) + bG(y, fy, fy) + cG(z, fz, fz) + eG(x, y, z)$ for all $x, y, z \in X$, where a, b, c and e are nonnegative real numbers with $a + b + c + e < 1$.

We now prove

Theorem 2.1. *Let f be a Chatterjee-type contraction on a complete G-metric space (X, G) with the choice (18). Then f has a unique fixed point p, which will be its contractive fixed point as well.*

Proof. Let $x_0 \in X$ be arbitrary. Define $\langle x_n\rangle_{n=1}^{\infty} \subset X$ by

$$x_n = fx_{n-1} \text{ for } n \geq 1. \tag{20}$$

Writing $x = x_{n-1}$ and $y = z = x_n$ in (18) and then using (20) and (G5), we get

$$G(fx_{n-1}, fx_n, fx_n) = G(x_n, x_{n+1}, x_{n+1})$$

$$\begin{aligned}
&\leq \gamma\big[G(x_{n-1}, fx_n, fx_n) + G(x_n, fx_n, fx_n) + G(x_n, fx_{n-1}, fx_{n-1})\big]\\
&\leq \gamma\big[G(x_{n-1}, x_{n+1}, x_{n+1}) + G(x_n, x_{n+1}, x_{n+1}) + G(x_n, x_n, x_n)\big]\\
&\leq \gamma\big[G(x_{n-1}, x_n, x_n) + 2G(x_n, x_{n+1}, x_{n+1})\big]\\
&\leq kG(x_{n-1}, x_n, x_n),
\end{aligned}$$

where $k = \frac{\gamma}{1-2\gamma}\cdot$ By induction, we have

$$G(x_n, x_{n+1}, x_{n+1}) \leq k^n G(x_0, x_1, x_1) \text{ for } n \geq 1. \tag{21}$$

Now for all $n, m \in N$ with $m > n$, by $(G5)$ and (21), we obtain

$$\begin{aligned}
G(x_n, x_m, x_m) &\leq G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_{n+2}, x_{n+2})\\
&\qquad + \cdots + G(x_{m-1}, x_m, x_m) \ (m-n \text{ terms})\\
&\leq (\underbrace{k^n + k^{n+1} + k^{n+2} + \cdots + k^{n+(m-n-1)}}_{m-n \text{ terms}})G(x_0, x_1, x_1)\\
&= k^n(\underbrace{1 + k + k^2 + \cdots + k^{m-n-1}}_{m-n \text{ terms}})G(x_0, x_1, x_1)\\
&\leq k^n \cdot \tfrac{1-k^{m-n}}{1-k} \cdot G(x_0, x_1, x_1)\\
&\leq \tfrac{k^n}{1-k} \cdot G(x_0, x_1, x_1).
\end{aligned}$$

Since $k = \frac{\gamma}{1-2\gamma} < 1$, applying the limit as $n \to \infty$ in this, we find that $\lim\limits_{n,m\to\infty} G(x_n, x_m, x_m) = 0$.

Thus $\langle x_n \rangle_{n=1}^{\infty}$ is G-Cauchy sequence in X. Since X is G-Complete, there exists a point $p \in X$ such that $\langle x_n \rangle_{n=1}^{\infty}$ is G-convergent to p. That is

$$\lim_{n\to\infty} x_{n-1} = \lim_{n\to\infty} x_n = p \tag{22}$$

Now writing $x = x_{n-1}$ and $y = z = p$ in (18),

$$\begin{aligned}
G(fx_{n-1}, fp, fp) &= G(x_n, fp, fp)\\
&\leq \gamma\big[G(x_{n-1}, fp, fp) + G(p, fp, fp) + G(p, fx_{n-1}, fx_{n-1})\big]\\
&\leq \gamma\big[G(x_{n-1}, fp, fp) + G(p, fp, fp) + G(p, x_n, x_n)\big]
\end{aligned}$$

Proceeding the limit as $n \to \infty$ in this and using (22), and then simplifying, we get

$$G(p, fp, fp) \leq 2\gamma G(p, fp, fp). \tag{23}$$

If $fp \neq p$, (23) would imply that

$$0 < G(p, fp, fp) \leq 2\gamma G(p, fp, fp) < G(p, fp, fp),$$

which is a contradiction. Therefore, $fp = p$. That is p is a fixed point of f. The uniqueness of the fixed point follows easily from (18).

We finally prove that p is a G-Contractive fixed point of f. In fact, let $x_0 \in X$ be arbitrary. Writing $x = f^{n-1}x_0$ and $y = z = p$ in (18) and using (G5), we get

$$\begin{aligned} G(f^n x_0, p, p) &= G(f^n x_0, fp, fp) \\ &\leq \gamma\big[G(f^{n-1}x_0, fp, fp) + G(p, fp, fp) + G(p, f^n x_0, f^n x)\big] \\ &\leq \gamma\big[G(f^{n-1}x_0, p, p) + 2G(f^n x_0, p, p)\big] \\ &\leq \tfrac{\gamma}{1-2\gamma} G(f^{n-1}x_0, p, p). \end{aligned}$$

Since $\frac{\gamma}{1-2\gamma} < 1$, we see that $G(f^n x_0, p, p) \to 0$ as $n \to \infty$ for each $x_0 \in X$. Thus p is a G-Contractive fixed point of f. □

3. Conclusion

Let (X, d) be a G-metric space, f, a self-map on X and $x_0 \in X$. Misconceptions regarding two fixed point theorems of Mustafa et al [2] have been discussed. Then a fixed point theorem for a Chatterjee-type G-contraction on a complete G-metric space has been proved. The unique fixed point will be its contractive fixed point, to which the f-iterates $x_0, fx_0, ..., f^n x_0, ...$ converge, for each each $x_0 \in X$.

Acknowledgments

The authors are thankful for the referees for their valuable suggestions in finalizing the paper.

References

[1] S.K. Chatterjea, Fixed-point theorems, *C.R. Acad. Bulgare Sci.*, **25** (1972), 727-730, MR 48#2845.

[2] Z. Mustafa, H. Obiedat, F. Awawdeh, Some fixed point theorems for mapping on complete G-Metric Spaces, *Fixed Point Theory and Applications* (2008), Article ID 189870, 1-12, **doi:** 10.1155/2008/189870.

[3] Z. Mustafa, B. Sims, A new approach to generalized metric spaces, *Jour. Nonlinear and Convex Anal.*, **7**, No. 2 (2006), 289-297.

[4] T. Phaneendra, K. Kumara Swamy, Unique fixed point in G-metric space through greatest lower bound properties, *Novi Sad J. Math.*, **43** (2013), 107-115.