International Journal of Pure and Applied Mathematics
Volume 109 No. 4 2016, 827-836
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: 10.12732/ijpam.v109i4.6
ijpam.eu

HOPF BIFURCATION ANALYSIS AND DESIGN OF HYBRID CONTROL FOR GROWTH MODEL WITH DELAY

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Abstract: In this paper, we investigate the problem of bifurcation control for a delayed logistic growth model. By choosing the timedelay as the bifurcation parameter, we present a Hybrid controler to control Hopf bifurcation. We show that the onset of Hopf bifurcation can be delayed or advanced via a hybeid controller by setting proper controling parameter. Under consideration model as operator Equation, apply orthogonal decomposition, compute the center manifold and normal form we determined the direction and stability of bifurcating periodic solutions. Therefore the Hopf bifurcation of the model became controllable to achieve desirable behaviors which are applicable in certain circumstances.

AMS Subject Classification: 34H20, 34D23
Key Words: Hopf bifurcation, bifurcation control, hybrid control, stability, timedelay

1. Introduction

The single-species logistic growth model governed by delay differential (and integro-differential) equations plays an important role in population dynamics and ecology that has been investigated in-depth involving the stability, persistent, oscillations and chaotic behavior of solutions [3]-[8]. Gopalsamy and Weng

Received: July 27, 2016
Revised: September 18, 2016
Published: October 6, 2016

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[4] considered the following control system:

$$\begin{cases} \dot{n}(t) = rn(t)[1 - \dfrac{an(t) + bn(t-\tau)}{k} - cv(t)], \\ \dot{v}(t) = -dv(t) + en(t-\tau), \end{cases} \tag{1.1}$$

where a, b, c, d, e, k, r $\in (0, +\infty)$ and $\tau \in [0, +\infty)$. The authors presented some sufficient conditions for the global asymptotic stability of the positive equilibrium of the system. On one hand, in Song et al. [6], the authors considered the Hopf bifurcation for a regulated logistic growth model which is a special case of (1.1) as follows:

$$\begin{cases} \dot{n}(t) = rn(t)[1 - \dfrac{n(t-\tau)}{k} - cv(t)], \\ \dot{v}(t) = -dv(t) + en(t-\tau), \end{cases} \tag{1.2}$$

The authors gave the explicit algorithm determining the direction of Hopf bifurcations and the stability of the periodic solutions, while they didnt discuss the existence of stability switches of this system. On the other hands, Gopalsamy and Weng [4] investigate the following control system:

$$\begin{cases} \dot{n}(t) = rn(t)[1 - \dfrac{n(t-\tau)}{k} - cv(t)], \\ \dot{v}(t) = -dv(t) + en(t), \end{cases} \tag{1.3}$$

the initial conditions for the system (1.3) take the form of $n(s) = \phi(s) \geq 0, \phi(0) > 0, \phi \in C([-\tau, 0], R_+)$, $v(0) = v_0$. The solutions of (1.3) are defined for all $t > 0$ and also satisfy $n(t) > 0, v(t) > 0$ for $t > 0$. And the system (1.3) has unique positive equilibrium

$$(n^{\square}, v^{\square}) = \left(\frac{dk}{d + kec}, \frac{ek}{d + kec} \right).$$

Then by the linear chain trick technique[4], system(1.3) can be transformed into the following equivalent system:

$$\begin{cases} \dot{x}(t) = -dx(t) + en^{\square}y(t), \\ \dot{y}(t) = -crx(t) - \dfrac{rn^{\square}}{k}y(t-\tau) - crx(t)y(t) - \dfrac{rn^{\square}}{k}y(t-\tau)y(t). \end{cases} \tag{1.4}$$

The author obtained when the condition (H) $\dfrac{ec}{d} > \dfrac{1}{k}$ and $d > (1 + \sqrt{2})r$ hold, the positive equilibrium $(n^{\square}, v^{\square})$ of (1.3) is linearly asymptotically stable irrespective of the size of the delay τ. XIE(2015) [5] interested in the effect of delay τ on dynamics of system (1.3) when the condition (H) is not satisfied. Taking the delay τ as a parameter, they showed that the stability and a Hopf bifurcation occurs when the delay τ passes through a critical value. We summarize these features of the solution via the existence and stability of a positive equilibrium in following:

Theorem 1. *If $\frac{ec}{d} < \frac{1}{k}$ then $(n^{\square}, v^{\square})$ is locally asymptotically stable for $0 \leq \tau < \tau_0$ and unstable for $\tau > \tau_0$ and system (1.3) undergoes Hopf bifurcation at $(n^{\square}, v^{\square})$ when $\tau = \tau_n$, n=0,1,2,...*

The organization of this paper is as follows. In Section 2, we study the stability and the Hopf bifurcation of Control system. In the next section, by the normal form method and the center manifold theory introduced by Hassard et al. [2], the direction of Hopf bifurcation and the stability of bifurcating periodic solutions are determined. In addition, the main results illustrated by examples with numerical simulations.

2. Hopf Bifurcation in Hybrid Control Delay Differential Equation

In this section,we focus on designing a controller to control the Hopf bifurcation in model based on the hybrid control strategy(2015)[7]. Apply the hybrid control to system (1.4), we get

$$\begin{aligned} \dot{x}(t) =& \alpha[-dx(t) + en^{\square}y(t)] + (\alpha-1)x(t), \\ \dot{y}(t) =& \alpha[-crx(t) - \frac{rn^{\square}}{k}y(t-\tau) - crx(t)y(t) \\ & - \frac{rn^{\square}}{k}y(t-\tau)y(t)] + (\alpha-1)y(t-\tau), \end{aligned} \tag{2.1}$$

where $0 < \alpha \leq 1$. The characteristic linear equation (2.1) is

$$\lambda^2 + (a_1 + \frac{rn^{\square}}{b_2}e^{-\lambda\tau})\lambda + \frac{a_1 rn^{\square}}{b_2}e^{-\lambda\tau} + b_1 a_2 rn^{\square} = 0, \tag{2.2}$$

so $a_1 = 1 - \alpha + \alpha d$, $b_1 = \alpha e$,$a_2 = \alpha c$ and $\frac{rn^{\square}}{b_2} = 1 - \alpha + \frac{rn^{\square}\alpha}{k}$. If $\tau > 0$. We assume $\lambda = i\omega$ is a purely imaginary root of (2.2), then we can obtained

$$\omega^2 + \frac{\omega rn^{\square}}{b_2} sin\omega\tau + \frac{a_1 rn^{\square}}{b_2} cos\omega\tau + b_1 a_2 rn^{\square}$$

$$+ i(\omega a_1 + \frac{\omega r n^{\Box}}{b_2} cos\omega\tau - \frac{a_1 r n^{\Box}}{b_2} sin\omega\tau) = 0. \quad (2.3)$$

Separating the real and imaginary parts of (2.3), we obtain

$$\begin{cases} -b_1 a_2 r n^{\Box} + \omega^2 = \dfrac{\omega r n^{\Box}}{b_2} sin\omega\tau + \dfrac{a_1 r n^{\Box}}{b_2} cos\omega\tau, \\ -\omega a_1 = \dfrac{\omega r n^{\Box}}{b_2} cos\omega\tau - \dfrac{a_1 r n^{\Box}}{b_2} sin\omega\tau. \end{cases} \quad (2.4)$$

Since $sin^2\omega\tau + cos^2\omega\tau = 1$, we have

$$\omega^4 + [{a_1}^2 - 2b_1 a_2 r n^{\Box} - (\frac{r n^{\Box}}{b_2})^2]\omega^2 + (b_1 a_2 r n^{\Box})^2 - (\frac{a_1 r n^{\Box}}{b_2})^2 = 0. \quad (2.5)$$

If the condition $\frac{b_1 a_2}{a_1} < \frac{1}{b_2}$ holds, then $(b_1 a_2 r n^{\Box})^2 - (\frac{a_1 r n^{\Box}}{b_2})^2 < 0$ is satisfied too. So, equation (2.5) has a solution $\omega_0 > 0$, since equation of (2.4) is

$$\tau_n = \frac{1}{\omega_0} arccos(\frac{-a_1 b_1 a_2 b_2}{\omega_0^2 + {a_1}^2}) + \frac{2n\pi}{\omega_0}, \quad n = 0, 1, 2, \ldots . \quad (2.6)$$

Then $\pm\omega_0$ is the purely imaginary root of (2.2).

Lemma 2. *In the equation of (2.2), If the condition* $\frac{b_1 a_2}{a_1} < \frac{1}{b_2}$ *holds, then* $Re(\frac{d\lambda}{d\tau})|_{\lambda = i\omega_0} > 0$.

Proof. We will calculate $\frac{d\lambda}{d\tau}$ using equation (2.2):

$$\frac{d\lambda}{d\tau} = \frac{(\lambda + a_1)\lambda r n^{\Box} e^{-\lambda\tau}}{(2\lambda + a_1)b_2 + r n^{\Box}(1 - \lambda\tau - a_1\tau)e^{-\lambda\tau}}.$$

The sign of the real part of $\frac{d\lambda}{d\tau}$ is obtained from

$$A = \frac{(a_1 b_2 + r n^{\Box} cos\omega_0\tau)(r n^{\Box} a_1 \omega_0 sin\omega_0\tau - r n^{\Box} {\omega_0}^2 cos\omega_0\tau)}{(r n^{\Box} a_1 \omega_0 sin\omega_0\tau - r n^{\Box} {\omega_0}^2 cos\omega_0\tau)^2 + (r n^{\Box} a_1 \omega_0 cos\omega_0\tau + r n^{\Box} {\omega_0}^2 sin\omega_0\tau)^2}$$
$$+ \frac{(2b_2\omega_0 - r n^{\Box} sin\omega_0\tau)(r n^{\Box} a_1 \omega_0 cos\omega_0\tau + r n^{\Box} {\omega_0}^2 sin\omega_0\tau)}{(r n^{\Box} a_1 \omega_0 sin\omega_0\tau - r n^{\Box} {\omega_0}^2 cos\omega_0\tau)^2 + (r n^{\Box} a_1 \omega_0 cos\omega_0\tau + r n^{\Box} {\omega_0}^2 sin\omega_0\tau)^2} \quad (2.7)$$

Therefore $A > 0$, and the proof is complete. □

Consequently, the positive equilibrium of (2.1) yields Hopf bifurcation, if $\tau = \tau_n$.

3. Direction and Stability of the Hopf Bifurcation in Discrete Control Model

In Section 2, we obtained conditions for the Hopf bifurcation to occur when $\tau = \tau_n$ under the condition $\dfrac{b_1 a_2}{a_1} < \dfrac{1}{b_2}$. In the section we study the direction of the Hopf bifurcation and the stability of the bifurcating periodic solutions, using techniques from normal form and center manifold theory for delay differential equations(2009)[1]. If we consider $c_1 = \dfrac{\alpha n^*}{k}$ then system (2.1) can be written as

$$\begin{cases} \dot{x}(t) = \tau[-a_1 x(t) + b_1 n^* y(t)], \\ \dot{y}(t) = \tau[-a_2 r x(t) - \dfrac{r n^*}{b_2} y(t-1) - a_2 r x(t) y(t) - c_1 r y(t-1) y(t)]. \end{cases} \tag{3.1}$$

Let $\mu = \tau - \tau_n$, functional differential equation (3.1) in $C = C([-1,0], R^2)$ is

$$\acute{x}(t) = L_\mu(x_t) + f(\mu, x_t), \tag{3.2}$$

where $x(t) = (x(t), y(t))^T \in R^2$ and $L_\mu : C \longrightarrow R^2$, $f : R \times C \longrightarrow R^2$are given, respectively:

$$\begin{aligned} L_\mu(\phi) =& (\tau_n + \mu) \begin{bmatrix} -a_1 & b_1 n^* \\ -a_2 r & 0 \end{bmatrix} \begin{bmatrix} \phi_1(0) \\ \phi_2(0) \end{bmatrix} \\ &+ (\tau_n + \mu) \begin{bmatrix} 0 & 0 \\ 0 & -\dfrac{r n^*}{b_2} \end{bmatrix} \begin{bmatrix} \phi_1(-1) \\ \phi_2(-1) \end{bmatrix}, \end{aligned} \tag{3.3}$$

$$f(\mu, \phi) = (\tau_n + \mu) \begin{bmatrix} 0 \\ Q \end{bmatrix}. \tag{3.4}$$

Here $Q = -a_2 r \phi_1(0) \phi_2(0) - c_1 r \phi_2(-1) \phi_2(0)$. By the Riesz representation theorem, there exists a function $\eta(\theta, \mu)$ of bounded variation for $\theta \in [-1, 0]$ such that

$$L_\mu(\theta) = \int_{-1}^{0} d\eta(\theta, \mu) \phi(\theta), \quad \phi \in C. \tag{3.5}$$

In fact, we can choose

$$\eta(\theta, \mu) = (\tau_n + \mu) \begin{bmatrix} -a_1 & b_1 n^* \\ -a_2 r & 0 \end{bmatrix} \delta(\theta) - (\tau_n + \mu) \begin{bmatrix} 0 & 0 \\ 0 & -\dfrac{r n^*}{b_2} \end{bmatrix} \delta(\theta + 1). \tag{3.6}$$

Then

$$\delta = \begin{cases} 0, & \text{if } \theta \neq 0, \\ 1, & \text{if } \theta = 0. \end{cases}$$

For $\phi \in C([-1,0], R^2)$, define:

$$A(\mu)\phi = \begin{cases} \dfrac{d\phi(\theta)}{d\theta}, & \text{if } \theta \in [-1,0), \\ \displaystyle\int_{-1}^{0} d\eta(\theta,\mu)\phi(\theta), & \text{if } \theta = 0, \end{cases},$$
$$R(\mu)\phi = \begin{cases} 0, & \text{if } \theta \in [-1,0), \\ f(\mu,\phi), & \text{if } \theta = 0. \end{cases} \tag{3.7}$$

Then system (3.2) is equivalent to

$$\acute{x_t} = A(\mu)x_t + R(\mu)x_t, \tag{3.8}$$

where $x_t(\theta) = x(t+\theta)$, $\theta \in [-1,0]$. For $\Gamma \in C([0,1], (R^2)^{*})$, define

$$A^{*}(\mu)\Gamma = \begin{cases} -\dfrac{d\Gamma(\theta)}{d\theta} & \text{if } \theta \in (0,1], \\ \displaystyle\int_{-1}^{0} d\eta(\theta,\mu)\phi(\theta) & \text{if } \theta = 0, \end{cases}$$

and a bilinear inner product

$$< \Gamma, \phi >= \bar{\Gamma}^T(0)\phi(0) - \int_{-1}^{0}\int_{\xi=0}^{\theta} \bar{\Gamma}^T(\xi-\theta)d\eta(\theta)\phi(\xi)d\xi.$$

Here $\eta(\theta) = \eta(\theta, 0)$. As $< \Gamma, A(0)\phi >=< A^{*}\Gamma, \phi >$, obviously $A(0)$ and $A^{*}(0)$ are adjoint operators and $\pm i\omega_0$ are eigenvalues of $A(0)$ and $A^{*}(0)$. We first need to compute the eigenvector of $A(0)$ and $A^{*}(0)$ corresponding to $i\omega_0$ and $-i\omega_0$ respectively. Suppose that $q(\theta) = (1, q_1)^T e^{i\omega_0\theta}$ is the eigenvector of $A(0)$ and $q^{*}(s) = D(1, {q^{*}}_1)e^{-i\omega_0 s}$ is the eigenvector of $A^{*}(0)$ corresponding to $i\omega_0$, $-i\omega_0$ respectively. In order to assur $< q^{*}, q >= 1$, we need to determine the value of the D. From inner product we can obtain

$$D = \frac{1}{1 + \bar{q}_1^{*} q_1 - \dfrac{rn^{*}e^{i\omega_0}}{b_2}}.$$

To compute the coordinates describing center manifold C_0 at $\mu = 0$. Define $z(t) =< q^{*}, x_t >, W(t,\theta) = x_t - 2Rez(t)q(\theta)$. On the center manifold C_0 we have $W(t,\theta) = W_{20}(\theta)\frac{z^2}{2} + W_{11}(\theta)z\bar{z} + W_{02}(\theta)\frac{\bar{z}^2}{2} + ...$, For the solution x_t of (3.8) since $\mu = 0$, we have

$$\acute{z}(t) = i\omega_0 z + \bar{q}^{\square}(0)f(0,z,\bar{z}) = g_{20}\frac{z^2}{2} + g_{11}z\bar{z} + g_{02}\frac{\bar{z}^2}{2} + g_{21}\frac{z^2\bar{z}}{2}...,$$

to calculate the coefficients, we have

$$g_{20} = -2[\frac{r}{b_2}e^{-i\omega_0} + a_2rq_1]\bar{D}, g_{11} = -[\frac{r}{b_2}(e^{-i\omega_0} + e^{i\omega_0}) + a_2r(\bar{q}_1 + q_1)]\bar{D}, g_{02} = -2[\frac{r}{b_2}e^{i\omega_0} + a_2r\bar{q}_1]\bar{D}$$

and

$$g_{21} = [W_{20}{}^{(1)}(0)(-\frac{r}{b_2}e^{i\omega_0} - a_2r\bar{q}_1) + W_{11}{}^{(1)}(0)(-\frac{2r}{b_2}e^{-i\omega_0} - 2a_2rq_1) - \frac{2r}{b_2}W_{11}{}^{(1)}(-1) - \frac{r}{b_2}W_{20}{}^{(1)}(-1) - 2a_2rW_{11}{}^{(2)}(0) - a_2rW_{20}{}^{(2)}(0)]\bar{D}.$$

So $W_{11} = (W_{11}{}^{(1)}, W_{11}^{(2)})$, $W_{20} = (W_{20}{}^{(1)}, W_{20}^{(2)})$.

Let

$$\Delta = \frac{i}{2\omega_0}\{g_{11}g_{20} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3}\} + \frac{g_{21}}{2}.$$

We define

$$\begin{cases} \mu_2 = -\dfrac{Re\{\Delta\}}{Re\dot{\lambda}(\tau_n)} \\ T_2 = -\dfrac{Im\{\Delta\} + \mu_2 Im\dot{\lambda}(\tau_n)}{\omega_0} \\ \beta_2 = 2Re\{\Delta\} \end{cases}$$

According to the case described above, we can summarize the results in the following theorem.

Theorem 3. *For the controlled system (3.1), the Hopf bifurcationis determined by the parameters μ_2, T_2 and β_2, the conclusions are summarized asfollows:*

(I) Parameter μ_2 determines the direction of the Hopf bifurcation. if $\mu_2 > 0$, the Hopf bifurcation is supercritical,the bifurcating periodic solutions exist for $\tau > \tau_n$, if $\mu_2 < 0$ the Hopf bifurcation is subcritical, the bifurcating periodic solutions exist for $\tau < \tau_n$.

(II) Parameter β_2 determines the stability of the bifurcating periodic solutions. If $\beta_2 < 0$, the bifurcating periodic solutions is stable; if $\beta_2 > 0$. The bifurcating periodic solutions is unstable.

(III) Parameter T_2 determines the period of the bifurcating periodic solution. If $T_2 > 0$, the period increases; If $T_2 < 0$, the period decreases.

In (1.4): Let $r = 1$, $k = 1$, $c = \frac{1}{2}$, $d = e = 2$, see Figures 1 and 2.

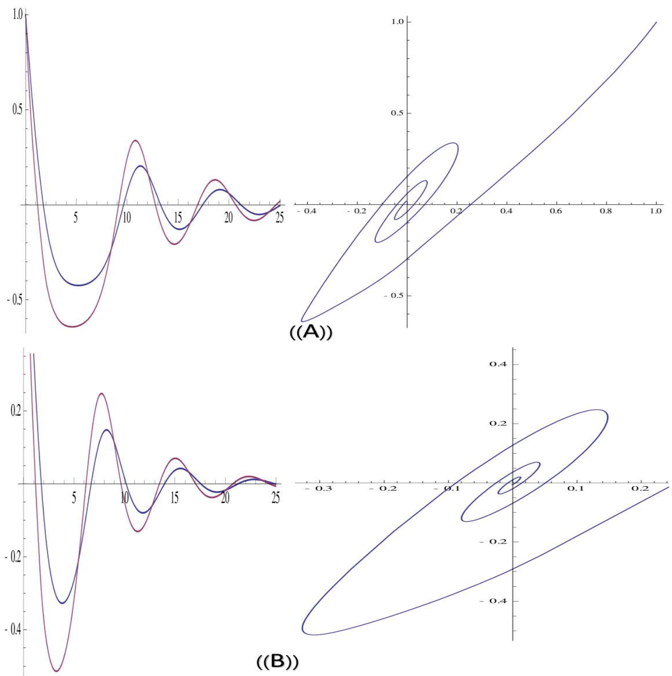

Figure 1: the numerical solution of uncontrol modle with corresponding to ((A)): $\tau = 1.7561$ and ((B)): $\tau = 1.6$.

4. Conclusion

In this paper, the problem of Hopf bifurcation control for an logistic growth model with time delay was studied. In order to control the Hopf bifurcation, a Hybrid controller is applied to the model. This Hybrid controller can successfully delay or advance the onset of an inherent bifurcation. The end theoren

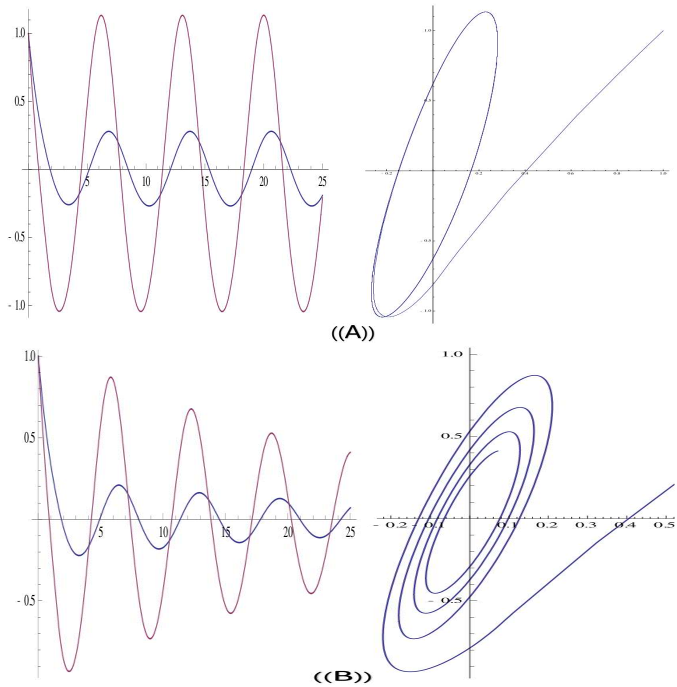

Figure 2: the numerical solution of control modle with corresponding to ((A)): $\alpha = 0.3$, $\tau = 1.7561$ and ((B)): $\alpha = 0.3$, $\tau = 1.6$.

helped to improve model.

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