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SPECTRAL THEORY FOR INTEGRATED SEMIGROUPS

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Abstract: In this paper, we investigate the transfer of some spectral properties from the integrated semigroup to its generator.

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1. Introduction and Preliminaries

Throughout this work, X denotes a complex Banach space and $\mathcal{B}(X)$ denotes the Banach algebra of all bounded linear operators on X. Let T a closed operator with domain $D(T)$, we denote by T^*, $R(T)$, $N(T)$, $R^\infty(T) = \bigcap_{n\geq 0} R(T^n)$, $\rho(T)$, $\sigma(T)$, $\sigma_p(T)$, $\sigma_r(T)$,$\sigma_k(T)$, $\sigma_{es}(T)$ respectively the adjoint, the range, the kernel, the hyper-range, the resolvent set, the spectrum, the point spectrum, the residual spectrum, the Kato spectrum and the essential kato spectrum of T.

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The ascent of T is defined by $a(T) = \min\{p \in \mathbb{N} : N(T^p) = N(T^{p+1})\}$; if such p does not exist, we let $a(T) = \infty$. Similarly, the descent of T is $d(T) = \min\{q \in \mathbb{N} : R(T^q) = R(T^{q+1})\}$, if such q does not exist we let $d(T) = \infty$ [3]. It is well known that if both $a(T)$ and $d(T)$ are finite then $a(T) = d(T)$ and therefore we have the decomposition $X = R(T^p) \oplus N(T^p)$ where $p = a(T) = d(T)$.

The descend and ascent spectrum defined by:

$$\sigma_{desc}(T) = \{\lambda \in \mathbb{C} : d(\lambda - T) = \infty\}$$

$$\sigma_{asc}(T) = \{\lambda \in \mathbb{C} : a(\lambda - T) = \infty\}$$

The essential ascent and descend of T are defined respectively by:

$$d_e(T) = \min\{n \in \mathbb{N} : \dim R(T^n)/_{R(T^{n+1})} < \infty\}$$

$$a_e(T) = \min\{n \in \mathbb{N} : \dim N(T^{n+1})/_{N(T^n)} < \infty\}$$

The essential ascent and descend spectrum are defined respectively by:

$$\sigma^e_{asc}(T) = \{\lambda \in \mathbb{C} : a_e(\lambda - T) = \infty\}$$

$$\sigma^e_{des}(T) = \{\lambda \in \mathbb{C} : d_e(\lambda - T) = \infty\}$$

Recall that T is a Drazin invertible if $d(T) < \infty$ and $a(T) < \infty$. The Drazin spectrum is $\sigma_D(T) = \{\lambda \in \mathbb{C} : d(\lambda - T) = \infty \text{ or } a(\lambda - T) = \infty\}$.

The set of all upper semi-Fredholm operators is defined by:

$$\Phi_+(X) = \{T \in B(X) : \dim N(\lambda - T) < \infty \text{ and } R(\lambda - T) \text{ is closed }\}$$

The upper semi-Fredholm spectrum defined by:

$$\sigma_{\Phi_+}(T) = \{\lambda \in \mathbb{C} : \lambda - T \notin \Phi_+(X)\}$$

while the set of all lower semi-Fredholm operators is defined by:

$$\Phi_-(X) = \{T \in B(X) : \text{codim} R(\lambda - T) < \infty\}$$

The lower semi-Fredholm spectrum defined by:

$$\sigma_{\Phi_-}(T) = \{\lambda \in \mathbb{C} : \lambda - T \notin \Phi_-(X)\}$$

The set of all semi-Fredholm operators is defined by:

$$\Phi_\pm(X) = \Phi_+(X) \cup \Phi_-(X)$$

The semi-Fredholm spectrum defined by:

$$\sigma_{\Phi_{\pm}}(T) = \{\lambda \in \mathbb{C} : \lambda - T \notin \Phi_{\pm}(X)\}$$

The class $\Phi(X)$ of all Fredholm operators is defined by:

$$\Phi(X) = \Phi_{+}(X) \cap \Phi_{-}(X)$$

The semi-Fredholm spectrum defined by:

$$\sigma_{\Phi}(T) = \sigma_{\Phi_{+}}(T) \cup \sigma_{\Phi_{-}}(T)$$

The class of all upper semi-Browder operators on a Banach space X that is defined by:

$$\mathcal{B}_{+}(X) = \{T \in \Phi_{+}(X) : a(T) < \infty\}$$

The upper semi-Browder spectrum defined by:

$$\sigma_{\mathcal{B}_{+}}(T) = \{\lambda \in \mathbb{C} : \lambda - T \notin \mathcal{B}_{+}(X)\}$$

The class of all lower semi-Browder operators that is defined by:

$$\mathcal{B}_{-}(X) = \{T \in \Phi_{-}(X) : d(T) < \infty\}$$

The lower semi-Browder spectrum defined by:

$$\sigma_{\mathcal{B}_{-}}(T) = \{\lambda \in \mathbb{C} : \lambda - T \notin \mathcal{B}_{-}(X)\}$$

The class of all Browder operators is defined by:

$$\mathcal{B}(X) = \mathcal{B}_{+}(X) \cap \mathcal{B}_{-}(X) = \{T \in \Phi(X) : a(T) < \infty \text{ and } d(T) < \infty\}$$

The Browder spectrum defined by:

$$\sigma_{\mathcal{B}}(T) = \{\lambda \in \mathbb{C} : \lambda - T \notin \mathcal{B}(X)\}$$

Recall that T is said to be Kato operator or semi-regular if $R(T)$ is closed and $N(T) \subseteq R^{\infty}(T)$. The set $\rho_{\gamma}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is Kato }\}$ denotes the Kato resolvent and $\sigma_{\gamma}(T) = \mathbb{C}\backslash\rho_{\gamma}(T)$ the Kato spectrum of T. It is well known that $\rho_{\gamma}(T)$ is an open subset of $\mathbb{C}$.

T is essential Kato if $R(T)$ is closed and there exists a subspace L in X with $\dim L < \infty$ such that $N(T) \subseteq R^{\infty}(T) + L$. The essential Kato spectrum of T is defined by $\sigma_{\gamma}^{e}(T) = \{\lambda \in \mathbb{C} : \lambda - T \text{ is not a essential Kato operator }\}$.

Integrated semigroups were first defined by Arendt [1] for integer-valued α. Arendt showed that certain natural classes of operators, such as adjoint semigroups of C_0 semigroups on non-reflexive Banach spaces, give rise to integrated semigroups which are not integrals of C_0 semigroups.

A family of bounded linear operators $(S(t))_{t\geq 0}$, on a Banach space X is called an integrated semigroup iff:

i) $S(0) = 0$.

ii) $S(t)$ is strongly continuous in $t \geq 0$.

iii) $S(r)S(t) = \int_0^r (S(\tau + t) - S(\tau))d\tau = S(t)S(r)$.

The differentation spaces C^n, $n \geq 0$, are defined by $C^0 = X$ and

$$C^n = \{x \in X : S(.)x \in C^n(\mathbb{R}^+; X)\}$$

Using this notion iii) can equivalently be formulated by $S(t)x \in C^1$ and $S'(r)S(t) = S(r+t) - S(r)$.

The set $N = \{x \in X; S(t)x = 0, \forall t \geq 0\}$ is called the degeneration space of the integrated semigroup $(S(t))_{t\geq 0}$. $(S(t))_{t\geq 0}$ is called non-degenerate if $N = \{0\}$ and degenerate otherwise.

The generator $A : D(A) \subseteq X \to X$ of a non-degenerate integrated semigroup $(S(t))_{t\geq 0}$ is defined as follows: $x \in D(A)$ and $Ax = y$ iff $x \in C^1$ and $S'(t)x - x = S(t)y$ for $t \geq 0$.

$C^2 \subseteq D(A) \subseteq C^1$ and $Ax = S''(0)x$ for $X \in C^2$. Moreover $AC^2 \subseteq C^1$.

A is a closed linear operator. $S(t) : C^1 \to C^2 \subseteq D(A)$ and $AS(t)x = S''(0)S(t)x = S'(t)x - x$.

Further $AS(t)x = S(t)Ax$ for $x \in D(A)$.

$\int_0^t S(r)dr$ maps X into $D(A)$ and $A\int_0^t S(r)xdr = S(t)x - tx$.

A non-degenerate integrated semigroup is uniquely determined by its generator.

Let $u : [0, T) \to X$ be continuous such that

$$\int_0^t u(s)ds \in D(A) \text{ and } A(\int_0^t u(s)ds) = u(t),$$

for $0 \leq t \leq T$. Then $u = 0$ in $[0, T)$.

Arendt [1] showed that if A generates S_t as an n-times integrated semigroup, then the Abstract Cauchy Problem $u'(t) = Au(t), u(0) = x$ has a classical solution for all $x \in D(A^{n+1})$.

In the theory of strongly continuous semigroups, some authors motivated by the relationships between the strongly continuou semigroup and its generator, like the spectral inclusion [5, 2]. In the next section, we investigate the transfer of some spectral properties from the integrated semigroup to its generator. After, we will prove the spectral inclusion for some spectra.

2. Main Results

In the sequel, we will prove that the following lemma holds.

Lemma 2.1. *Let A be the generator of an integrated semigroup $(S(t))_{t\geq 0}$. Then, for all $\lambda \in \mathbb{C}$, $t \geq 0$, and $n \in \mathbb{N}$,*

1. $(\int_0^t e^{\lambda s}ds - S(t))x = (\lambda - A)D_\lambda(t)x, \forall x \in X.$
2. $(\int_0^t e^{\lambda s}ds - S(t))x = D_\lambda(t)(\lambda - A)x, \forall x \in D(A).$
3. $R(\int_0^t e^{\lambda s}ds - S(t))^n \subseteq R(\lambda - A)^n.$
4. $N((\lambda - A)^n) \subseteq N(\int_0^t e^{\lambda s}ds - S(t))^n.$

Proof. 1. For all r, $t \in [0, +\infty[$ and $x \in X$ we have

$$\begin{aligned} S(r)D_\lambda(t)x &= S(r)\int_0^t e^{\lambda(t-s)}S(s)xds \\ &= \int_0^t e^{\lambda(t-s)}S(r)S(s)xds \\ &= \int_0^t \int_0^r e^{\lambda(t-s)}[S(\tau+s) - S(\tau)]xd\tau ds \\ &= \int_0^r \int_0^t e^{\lambda(t-s)}[S(\tau+s) - S(\tau)]xdsd\tau \end{aligned}$$

Then, for all $x \in X$, $D_\lambda(t)x \in C^1$ and

$$\begin{aligned} \frac{d}{dr}S(r)D_\lambda(t)x &= \int_0^t e^{\lambda(t-s)}[S(r+s) - S(r)]xds \\ &= \int_0^t e^{\lambda(t-s)}[S(r+s) - S(s)]xds + \int_0^t e^{\lambda(t-s)}S(s)xds \\ &\quad - \int_0^t e^{\lambda(t-s)}S(r)xds \end{aligned}$$

$$\begin{aligned}
&= \int_0^t e^{\lambda(t-s)} \frac{d}{ds}[S(s)S(r)]xds - S(r)\int_0^t e^{\lambda s}xds + D_\lambda(t)x \\
&= S(r)S(t)x + \lambda S(r)D_\lambda(t)x - S(r)\int_0^t e^{\lambda s}xds + D_\lambda(t)x \\
&= S(r)[S(t) + \lambda D_\lambda(t)x - \int_0^t e^{\lambda s}xds] + D_\lambda(t)x
\end{aligned}$$

Therefore $D_\lambda(t)x \in D(A)$ and $AD_\lambda(t)x = S(t) + \lambda D_\lambda(t)x - \int_0^t e^{\lambda s}xds$. Thus

$$(\int_0^t e^{\lambda s}ds - S(t))x = (\lambda - A)D_\lambda(t)x.$$

2. For all $x \in D(A)$, we have

$$\begin{aligned}
D_\lambda(t)Ax &= \int_0^t e^{\lambda(t-s)}S(s)Axds \\
&= \int_0^t e^{\lambda(t-s)}(S'(s)x - x)ds \\
&= \int_0^t e^{\lambda(t-s)}S'(s)xds - \int_0^t e^{\lambda s}xds \\
&= S(s) + \lambda D_\lambda(t)x - \int_0^t e^{\lambda s}xds
\end{aligned}$$

Hence

$$(\int_0^t e^{\lambda s}ds - S(t))x = D_\lambda(t)(\lambda - A)x. \qquad \square$$

Theorem 1. *Let $(S(t))_{t\geq 0}$ be an integrated semigroup and A be its generator. Then*

$$\int_0^t e^{s\sigma(A)}ds \subseteq \sigma(S(t)),\ \int_0^t e^{s\sigma_p(A)}ds \subseteq \sigma_p(S(t))$$

Proof. 1. Let $\int_0^t e^{s\lambda}ds \in \rho(S(t))$ and let $L = (\int_0^t e^{\lambda s}ds - S(t))^{-1}$. The operator $D_\lambda(t)$ and L commute. From lemma2.1, we deduce

$$(\lambda - A)D_\lambda(t)Lx = x \text{ for every } x \in X$$

and

$$L(\lambda - A)D_\lambda(t)x = x \text{ for every } x \in D(A)$$

Since $D_\lambda(t)$ and L commute we also have

$$(\lambda - A)LD_\lambda(t)x = x \text{ for every } x \in D(A)$$

Therefore, $\lambda \in \rho(A)$ and $\int_0^t e^{s\sigma(A)}ds \subseteq \sigma(S(t))$

2. If $\lambda \in \sigma_p(A)$ then there is an $x_0 \in D(A)$, $x_0 \neq 0$, such that $(A-\lambda I)x_0 = 0$. From lemma2.1 it then follows that $(\int_0^t e^{\lambda s}ds - S(t))x_0 = 0$ and therefore $\int_0^t e^{s\lambda}ds \in \sigma_p(S(t))$.

□

Lemma 2.2. *Let $(S(t))_{t\geq 0}$ an integrated semigroup on X with generator A. For $\lambda \in \mathbb{C}$ and $t \geq 0$, let $L_\lambda(t)x = \int_0^t e^{-\lambda s}D_\lambda(s)xds$, then*

1. *$L_\lambda(t)$ is a bounded linear operator on X .*
2. *$\forall x \in X$, $L_\lambda(t)x \in D(A)$ and $(\lambda - A)L_\lambda(t) + G_\lambda(t)D_\lambda(t) = \phi_\lambda(t)I$ with $G_\lambda(t) = e^{-\lambda t}I$ and $\phi_\lambda(t) = \int_0^t \int_0^\tau e^{-\lambda\sigma}xd\sigma d\tau$.*
3. *The operators $L_\lambda(t)$, $G_\lambda(t)$, $D_\lambda(t)$ and $(\lambda - A)$ are pairwise commute.*

Proof. 1. Obvious.

2. For all $r \geq 0$ we have:

$$\begin{aligned}
S(r)L_\lambda(t)x &= \int_0^t e^{-\lambda\tau}S(r)D_\lambda(\tau)xd\tau \\
&= \int_0^t \int_0^\tau e^{-\lambda\sigma} \int_0^r [S(u+\sigma) - S(u)]xdud\sigma d\tau \\
&= \int_0^t \int_0^\tau \int_0^r e^{-\lambda\sigma}[S(u+\sigma) - S(u)]xdud\sigma d\tau \\
&= \int_0^r \int_0^t \int_0^\tau e^{-\lambda\sigma}[S(u+\sigma) - S(u)]xd\sigma d\tau du
\end{aligned}$$

Therefore, for all $x \in X$, $L_\lambda(t)x \in C^1$ and

$$\begin{aligned}
\frac{d}{dr}S(r)L_\lambda(t)x &= \int_0^t \int_0^\tau e^{-\lambda\sigma}[S(r+\sigma) - S(r)]xd\sigma d\tau \\
&= \int_0^t \int_0^\tau e^{-\lambda\sigma}[S(r+\sigma) - S(\sigma)]xd\sigma d\tau + L_\lambda(t)x \\
&\quad -S(r)\phi_\lambda(t)
\end{aligned}$$

$$\begin{aligned} &= \int_0^t \int_0^\tau e^{-\lambda\sigma} \frac{d}{d\sigma} S(\sigma)S(r)x d\sigma d\tau + L_\lambda(t)x - S(r)\phi_\lambda(t) \\ &= S(r)[e^{-\lambda t} D_\lambda(t)x + \lambda L_\lambda(t)x - \phi_\lambda(t)x] + L_\lambda(t)x \end{aligned}$$

Therefore, $AL_\lambda(t)x = e^{-\lambda t} D_\lambda(t)x + \lambda L_\lambda(t)x - \phi_\lambda(t)x$.
$(\lambda - A)L_\lambda(t) + G_\lambda(t)D_\lambda(t) = \phi_\lambda(t)I$ with $G_\lambda(t) = e^{-\lambda t} I$.

3. For all $t \geq 0$, $L_\lambda(t)$ and $D_\lambda(t)$ commuting.
Indeed, for $t, s \geq 0$ we have:

$$\begin{aligned} D_\lambda(t)D_\lambda(s)x &= \int_0^t e^{\lambda(t-u)} S(u) D_\lambda(s) x du \\ &= \int_0^t e^{\lambda(t-u)} S(u) \int_0^s e^{\lambda(s-v)} S(v) x dv du \\ &= \int_0^t \int_0^s e^{\lambda(t-u)} e^{\lambda(s-v)} S(u)S(v) x dv du \\ &= \int_0^s e^{\lambda(s-v)} S(v) \int_0^t e^{\lambda(t-u)} S(u) x du dv \\ &= D_\lambda(s) D_\lambda(t) x \end{aligned}$$

Therefore:

$$\begin{aligned} L_\lambda(t)D_\lambda(t)x &= \int_0^t e^{-\lambda u} D_\lambda(u) D_\lambda(t) x du \\ &= \int_0^t e^{-\lambda u} D_\lambda(t) D_\lambda(u) x du \\ &= D_\lambda(t) \int_0^t e^{-\lambda u} D_\lambda(u) x du \\ &= D_\lambda(t) L_\lambda(t) x \end{aligned}$$

For all $x \in D(A)$ we have

$$\begin{aligned} L_\lambda(t)(\lambda - A)x &= \int_0^t e^{-\lambda s} D_\lambda(s)(\lambda - A) x ds \\ &= \int_0^t e^{-\lambda s} (e^{\lambda s} - S(s)) x ds \\ &= \phi_\lambda(t)x - \int_0^t e^{-\lambda s} S(s) x ds \\ &= \phi_\lambda(t)x - G_\lambda(t) D_\lambda(t) x \end{aligned}$$

$$= (\lambda - A)L_\lambda(t)x$$

For all $x \in D(A)$ $(\lambda - A)G_\lambda(t)x = G_\lambda(t)(\lambda - A)x$ obvious.

For all $x \in D(A)$ $(\lambda - A)D_\lambda(t)x = D_\lambda(t)(\lambda - A)x$ see lemma 2.1

□

Lemma 2.3. *Let $(S(t))_{t\geq 0}$ an integrated semigroup on X with generator A. If $R(\int_0^t e^{\lambda s}ds - S(t))$ is a closed, then $R(\lambda - A)$ is closed.*

Proof. Let $y_n = (\lambda - A)x_n$ be a convergent sequence with limit $y \in X$, according to (1) we have $\phi_\lambda(t)x_n = (\lambda - A)L_\lambda(t)x_n + G_\lambda(t)D_\lambda(t)x_n$ and $\phi_\lambda(t)y_n = (\lambda - A)L_\lambda(t)y_n + G_\lambda(t)(\int_0^t e^{\lambda s}ds - S(t))x_n$. Then:

$(\int_0^t e^{\lambda s}ds - S(t))G_\lambda(t)x_n = y_n - (\lambda - A)L_\lambda(t)y_n$ tends to $\phi_\lambda(t)y - (\lambda - A)L_\lambda(t)y \in R(\int_0^t e^{\lambda s}ds - S(t))$ since $(\lambda - A)L_\lambda(t)$ is a linear bounded operator and $R(\int_0^t e^{\lambda s}ds - S(t))$ is closed. Then there exists $z \in X$ such that $\phi_\lambda(t)y - (\lambda - A)L_\lambda(t)y = (\int_0^t e^{\lambda s}ds - S(t))z$ then $y = (\lambda - A)[L_\lambda(t)y + D_\lambda(t)z]$, hence $y \in R(\lambda - A)$. □

Theorem 2. *Let $(S(t))_{t\geq 0}$ an integrated semigroup on X with generator A. Then for all $t > 0$ we have*

1. *If $\int_0^t e^{\lambda s}ds - S(t)$ is a Kato operator, then $\lambda - A$ is a Kato operator.*
2. *If $\int_0^t e^{\lambda s}ds - S(t)$ is a essential Kato operator, then $\lambda - A$ is a essential Kato operator.*

Proof.

1. Suppose that $\int_0^t e^{\lambda s}ds - S(t)$ is a Kato operator, then for all $n \in \mathbb{N}$, we have $N(\int_0^t e^{\lambda s}ds - S(t)) \subseteq R(\int_0^t e^{\lambda s}ds - S(t))^n$ and $R(\int_0^t e^{\lambda s}ds - S(t))$ is closed. According to lemma2.1 and lemma2.3 we have $N(\lambda - A) \subseteq N(\int_0^t e^{\lambda s}ds - S(t)) \subseteq R(\int_0^t e^{\lambda s}ds - S(t))^n \subseteq R(\lambda - A)^n$ and $R(\lambda - A)$ is closed.
2. Suppose that $\int_0^t e^{\lambda s}ds - S(t)$ is a essential Kato operator, then there exists a subspace L in X such that $\dim L < \infty$, $N(\int_0^t e^{\lambda s}ds - S(t)) \subseteq R^\infty(\int_0^t e^{\lambda s}ds - S(t)) + L$ and $R(\int_0^t e^{\lambda s}ds - S(t))$ is closed. According to lemma2.1 and lemma2.3 we have $N(\lambda - A) \subseteq N(\int_0^t e^{\lambda s}ds - S(t)) \subseteq R^\infty(\int_0^t e^{\lambda s}ds - S(t)) + L \subseteq R^\infty(\lambda - A) + L$ and $R(\lambda - A)$ is closed , hence $\lambda - A$ is essential Kato operator

□

Corollary 1. *Let $S(t)_{t\geq 0}$ be an integrated semigroup on X with generator A. Then for all $t > 0$:*

$$\int_0^t e^{s\sigma_\gamma(A)}ds \subseteq \sigma_\gamma(S(t)) \text{ and } \int_0^t e^{s\sigma_\gamma^e(A)}ds \subseteq \sigma_\gamma^e(S(t))$$

Theorem 3. *Let $(S(t))_{t\geq 0}$ an integrated semigroup on X with generator A. Then for all $t > 0$ we have*

1. *If $\int_0^t e^{\lambda s}ds - S(t)$ is a upper semi-Fredholm operator, then $\lambda - A$ is a upper semi-Fredholm operator.*
2. *If $\int_0^t e^{\lambda s}ds - S(t)$ is a lower semi-Fredholm operator, then $\lambda - A$ is a lower semi-Fredholm operator.*
3. *If $\int_0^t e^{\lambda s}ds - S(t)$ is a semi-Fredholm operator, then $\lambda - A$ is a semi-Fredholm operator.*
4. *If $\int_0^t e^{\lambda s}ds - S(t)$ is a Fredholm operator, then $\lambda - A$ is a Fredholm operator.*

Proof. 1. If $\int_0^t e^{\lambda s}ds - S(t)$ is a upper semi-Fredholm operator, then $\dim N(\int_0^t e^{\lambda s}ds - S(t)) < \infty$ and $R(\int_0^t e^{\lambda s}ds - S(t))$ is closed. According to lemma2.1 and lemma 2.3 we have $\dim(\lambda - A) < \infty$ and $R(\lambda - A)$ is closed, hence $\lambda - A$ is a upper semi-Fredholm operator.

2. If $\int_0^t e^{\lambda s}ds - S(t)$ is a lower semi-Fredholm operator, then $\text{codim}R(\int_0^t e^{\lambda s}ds - S(t)) < \infty$. According to lemma2.1 we have codim $R(\lambda - A) < \infty$, hence $\lambda - A$ is a lower semi-Fredholm operator.

3. Obvious.

4. Obvious. □

Corollary 2. *Let $S(t)_{t\geq 0}$ be an integrated semigroup on X with generator A. Then for all $t > 0$:*

$$\int_0^t e^{s\sigma_{\Phi_+}(A)}ds \subseteq \sigma_{\Phi_+}(S(t)),\ \int_0^t e^{s\sigma_{\Phi_-}(A)}ds \subseteq \sigma_{\Phi_-}(S(t))$$

$$\int_0^t e^{s\sigma_{\Phi_\pm}(A)}ds \subseteq \sigma_{\Phi_\pm}(S(t)),\ \int_0^t e^{s\sigma_{\Phi}(A)}ds \subseteq \sigma_{\Phi}(S(t))$$

Theorem 4. *Let $(S(t))_{t\geq 0}$ an integrated semigroup on X with generator A. Then for all $t > 0$ we have*

1. *If $d(\int_0^t e^{\lambda s}ds - S(t)) < \infty$, then $d(\lambda - A) < \infty$.*

2. *If* $a(\int_0^t e^{\lambda s}ds - S(t)) < \infty$*, then* $a(\lambda - A) < \infty$*.*

3. *If* $d_e(\int_0^t e^{\lambda s}ds - S(t)) < \infty$*, then* $d_e(\lambda - A) < \infty$*.*

4. *If* $a_e(\int_0^t e^{\lambda s}ds - S(t)) < \infty$*, then* $a_e(\lambda - A) < \infty$*.*

5. *If* $\int_0^t e^{\lambda s}ds - S(t)$ *is a Drazin invertible, then* $\lambda - A$ *is a Drazin invertible.*

Proof. 1. If $d(\int_0^t e^{\lambda s}ds - S(t)) < \infty$, then there exists $n \in \mathbb{N}$ such that

$$R(\int_0^t e^{\lambda s}ds - S(t))^n = R(\int_0^t e^{\lambda s}ds - S(t))^{n+1}$$

There exists two operators $L_n(t)$ and $G_n(t)$ such that:

$$(\lambda - A)^n L_n(t) + G_n(t)B_\lambda^n(t) = I \tag{1}$$

and $L_n(t)$, $H_n(t)$, $D_\lambda(t)$ and $(\lambda - A)$ are pairwise commute.
Let $y \in R(\lambda - A)^n$ and $x \in D(A^n)$ such that $y = (\lambda - A)^n x$. According to (1) we have:

$$\begin{aligned}(\lambda - A)^n x &= (\lambda - A)^n L_n(t)(\lambda - A)^n x + G_n(t)D_\lambda^n(t)(\lambda - A)^n x \\ &= (\lambda - A)^{n+1} L_n(t)(\lambda - A)^{n-1} x + G_n(t)(\int_0^t e^{\lambda s}ds - S(t))^n x\end{aligned}$$

Let $z \in X$ such that $(\int_0^t e^{\lambda s}ds - S(t))^n x = (\int_0^t e^{\lambda s}ds - S(t))^{n+1} z$, then

$$(\lambda - A)^n x = (\lambda - A)^{n+1}[(\lambda - A)^{n-1}L_n(t)x + G_n(t)D_\lambda^{n+1}(t)z]$$

Therefore $R(\lambda - A)^n = R(\lambda - A)^{n+1}$, hence $d(\lambda - A) < \infty$.

2. If $a(\int_0^t e^{\lambda s}ds - S(t)) < \infty$, there exists $n \in \mathbb{N}$ such that

$$N(\int_0^t e^{\lambda s}ds - S(t))^n = N(\int_0^t e^{\lambda s}ds - S(t))^{n+1}$$

Let $x \in N(\lambda - A)^{n+1}$ then $(\lambda - A)^{n+1}x = 0$

$$\begin{aligned}(\lambda - A)^{n+1}x = 0 &\Rightarrow (\int_0^t e^{\lambda s}ds - S(t))^{n+1}x = 0 \\ &\Rightarrow (\int_0^t e^{\lambda s}ds - S(t))^n x = 0 \\ (\lambda - A)^n x &= (\lambda - A)^n L_n(t)(\lambda - A)^n x\end{aligned}$$

$$
\begin{aligned}
& +G_n(t)(\int_0^t e^{\lambda s}ds - S(t))^n x \\
= \; & (\lambda - A)^{n-1} L_n(t)(\lambda - A)^{n+1} x \\
& +G_n(t)(\int_0^t e^{\lambda s}ds - S(t))^n x \\
= \; & 0
\end{aligned}
$$

Therefore $N(\lambda - A)^n = N(\lambda - A)^{n+1}$, hence $a(\lambda - A) < \infty$.

3. If $d_e(\int_0^t e^{\lambda s}ds - S(t)) < \infty$, there exists $n \in \mathbb{N}$ such that

$$\dim R(\int_0^t e^{\lambda s}ds - S(t))^n / R(\int_0^t e^{\lambda s}ds - S(t))^{n+1} < \infty$$

Consider the application:

$$
\begin{aligned}
\psi : R(\lambda - A)^n & \rightarrow R(\int_0^t e^{\lambda s}ds - S(t))^n / R(\int_0^t e^{\lambda s}ds - S(t))^{n+1} \\
(\lambda - A)^n x & \mapsto (\int_0^t e^{\lambda s}ds - S(t))^n x + R(\int_0^t e^{\lambda s}ds - S(t))^{n+1}
\end{aligned}
$$

ψ is well defined, linear, surjective. According to (1) we have

$$N(\psi) \subseteq R(\lambda - A)^{n+1} \subseteq R(\lambda - A)^n$$

According to the theorem of isomorphism we have

$$\dim R(\lambda - A)^n / N(\psi) = \dim R(\int_0^t e^{\lambda s}ds - S(t))^n / R(\int_0^t e^{\lambda s}ds - S(t))^{n+1} < \infty$$

Since

$$R(\lambda - A)^{n+1} / N(\psi) \subseteq R(\lambda - A)^n / N(\psi)$$

Then

$$\dim R(\lambda - A)^{n+1} / N(\psi) < \infty$$

According to the theorem of isomorphism we have

$$\dim R(\lambda - A)^n / R(\lambda - A)^{n+1} < \infty$$

Hence $d_e(\lambda - A) < \infty$.

4. If $a_e(\int_0^t e^{\lambda s}ds - S(t)) < \infty$, there exist $n \in \mathbb{N}$ such that

$$\dim N(\textstyle\int_0^t e^{\lambda s}ds - S(t))^{n+1}/N(\int_0^t e^{\lambda s}ds - S(t))^n < \infty$$

We have $\dim N(\lambda - A)^{n+1}/N(\lambda - A)^n < \infty$.
Indeed consider the application:

$$\begin{aligned} \varphi : N(\lambda - A)^{n+1} &\rightarrow N(\int_0^t e^{\lambda s}ds - S(t))^{n+1}/N(\textstyle\int_0^t e^{\lambda s}ds - S(t))^n \\ x &\mapsto x + N(\int_0^t e^{\lambda s}ds - S(t))^n \end{aligned}$$

φ is well defined, linear, and $N(\varphi) \subseteq N(\lambda - A)^n \subseteq N(\lambda - A)^{n+1}$.
According to the theorem of isomorphism we have

$$\begin{aligned} \dim N(\lambda - A)^{n+1}/N(\varphi) &= \dim Im(\varphi) \\ &\leq \dim N(\int_0^t e^{\lambda s}ds - S(t))^{n+1}/N(\textstyle\int_0^t e^{\lambda s}ds - S(t))^n \\ &< \infty \end{aligned}$$

Then $\dim N(\lambda - A)^n/N(\varphi) < \infty$. According to the theorem of isomorphism we have $\dim N(\lambda - A)^{n+1}/N(\lambda - A)^n < \infty$ hence $a_e(\lambda - A) < \infty$.

5. If $\int_0^t e^{\lambda s}ds - S(t)$ is Drazin invertible, then $d(\int_0^t e^{\lambda s}ds - S(t)) < \infty$ and $a(\int_0^t e^{\lambda s}ds - S(t)) < \infty$ then $d(\lambda - A)) < \infty$ and $a(\lambda - A) < \infty$, hence $\lambda - A$ is invertible Drazin.

□

Corollary 3. *Let $S(t)_{t\geq 0}$ be an integrated semigroup on X with generator A. Then for all $t > 0$:*

$$\int_0^t e^{s\sigma_{desc}(A)}ds \subseteq \sigma_{desc}(S(t)),\ \int_0^t e^{s\sigma_{asc}(A)}ds \subseteq \sigma_{asc}(S(t))$$

$$\int_0^t e^{s\sigma^e_{desc}(A)}ds \subseteq \sigma^e_{desc}(S(t)),\ \int_0^t e^{s\sigma^e_{asc}(A)}s \subseteq \sigma^e_{asc}(S(t))$$

$$\int_0^t e^{s\sigma_D(A)}ds \subseteq \sigma_D(S(t))$$

Theorem 5. *Let $(S(t))_{t\geq 0}$ an integrated semigroup on X with generator A. Then for all $t > 0$ we have*

1. *If* $\int_0^t e^{\lambda s}ds - S(t)$ *is a upper semi-Browder operator, then* $\lambda - A$ *is a upper semi-Browder operator.*

2. *If* $\int_0^t e^{\lambda s}ds - S(t)$ *is a lower semi-Browder operator, then*$\lambda - A$ *is a lower semi-Browder operator.*

3. *If* $\int_0^t e^{\lambda s}ds - S(t)$ *is a Browder operator, then* $\lambda - A$ *is a Browder operator.*

Proof. 1. If $\int_0^t e^{\lambda s}ds - S(t)$ is a upper semi-Browder operator, then $\int_0^t e^{\lambda s}ds - S(t)$ is a upper semi-Fredholm and $d(\int_0^t e^{\lambda s}ds - S(t)) < \infty$. According to theorem3 and theorem4 we have $\lambda - A$ is a upper semi-Fredholm and $d(\lambda - A) < \infty$, hence $\lambda - A$ is a upper semi-Browder operator.

2. If $\int_0^t e^{\lambda s}ds - S(t)$ is a lower semi-Browder operator, then $\int_0^t e^{\lambda s}ds - S(t)$ is a lower semi-Fredholm and $a(\int_0^t e^{\lambda s}ds - S(t)) < \infty$. According to theorem3 and theorem4 we have $\lambda - A$ is a lower semi-Fredholm and $d(\lambda - A) < \infty$, hence $\lambda - A$ is a lower semi-Browder operator.

3. obvious.

□

Corollary 4. *Let* $S(t)_{t\geq 0}$ *be an integrated semigroup on* X *with generator* A*. Then for all* $t > 0$*:*

$$\int_0^t e^{s\sigma_{\mathcal{B}_+}(A)}s \subseteq \sigma_{\mathcal{B}_+}(S(t)), \int_0^t e^{s\sigma_{\mathcal{B}_-}(A)}s \subseteq \sigma_{\mathcal{B}_-}(S(t))$$

$$\int_0^t e^{s\sigma_{\mathcal{B}}(A)}s \subseteq \sigma_{\mathcal{B}}(S(t))$$

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