International Journal of Pure and Applied Mathematics
Volume 110 No. 3 2016, 427-432
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: 10.12732/ijpam.v110i3.4

ijpam.eu

IDEAL SEMIRINGS AND THE IDEAL EXTENSION PROPERTY FOR SEMIRINGS

Ahamd Khaksari

Department of Mathematics
Payame Noor University
P.O. Box 19395-3697, Tehran, IRAN

Abstract: Suppose that R is a semiring. In this paper, we define the ideal extension property, ideal semiring, the congruence extension property for semiring R and prove that R has the ideal extension property if and only if every subsemiring of R has the ideal extension property. We prove that if R has the ideal extension property then the homomorphic image of R has the ideal extension property. Also, we show that if R is an ideal semiring then the homomorphic image of R is an ideal semiring and it is proved that each ideal semiring with the congruence extension property has the ideal extension property.

AMS Subject Classification: 16Y60
Key Words: semiring, ideal semiring, ideal extension property, congruence extension property

1. Introduction

Definition 1.1. Let $R \neq \emptyset$ be a set with $+$ and $.$ as binary operations on R, named addition and multiplication, respectively. Then $(R, +, .)$ is called a semiring if the following conditions are satisfied:

Received: December 23, 2015
Revised: August 23, 2016
Published: November 5, 2016

1 $(R,+)$ is a commutative semigroup;

2 $(R,.)$ is a semigroup;

3 Both operations are connected by the distributive lows $a.(b+c) = a.b + a.c$ and $(a+b).c = a.c + b.c$ for all $a,b,c \in R$.

Definition 1.2. A subset H of a semiring R is called a subsemiring provided that H is a semiring under both binary operations on R.

Definition 1.3. Let R be a semiring. An equivalence relation ρ on the semiring R is called congruence if

$$(\forall s,t,a \in R)(s,t) \in \rho \Longrightarrow (a+s, a+t) \in \rho$$

and

$$(\forall s,t,a \in R)(s,t) \in \rho \Longrightarrow (a.s, a.t) \in \rho$$

and $(s.a, t.a) \in \rho$.

Definition 1.4. For a semiring $(R,+,.)$ and a non-empty subset A of R, we define the following subset of R:

$$\langle A \rangle = \left\{ \sum_{v=1}^{n} a_v : n \in I\!N, a_v \in A \right\}$$

The subset $\langle A \rangle$ is called the subset of R generated by A.

Definition 1.5. A non-empty subset 1 of a semiring R will be called an ideal if $a,b \in I$ and $r \in R$ imply that $a+b \in I$ and $ra \in I$ and $ar \in I$.

2. Ideal Semirings and Ideal Extension Property for Semirings

Definition 2.1. Let R be a semiring. R has the ideal extension property provided that for each subsemiring H of R and each ideal J of H there exists an ideal K of R such that $K \cap H = J$.

Theorem 2.2. *Let R be a semiring. Then R has the ideal extension property if and only if every subsemiring of R has the ideal extension property.*

Proof. Let every subsemiring of R has the ideal extension property. Since R is a subsemiring of itself, R has the ideal extension property. Now, Suppose that R has the ideal extension property. Let H be a subsemiring of R and L be a subsemiring of H and J be an ideal of L. Since L is also a subsemiring of R and R has the ideal extension property, there exists an ideal K of R such that $K \cap L = J, K \cap H$ is an ideal of H and we have $K \cap HL = K \cap L = J$,it follows that H has the ideal extension property.

Example 2.3. Let $R = \{a, b, c\}$. Then R with addition and multiplication defined by the following Cayley tables is a semiring.

$+$	a	b	c
a	a	b	c
b	b	b	c
c	c	c	c

$.$	a	b	c
a	a	b	c
b	a	b	c
c	c	c	c

Let $R_1 = \{a\}$, $R_2 = \{b\}$, $R_3 = \{c\}$, $R_4 = \{a, b\}$, $R_5 = \{a, c\}$, $R_6 = \{b, c\}$ and $R_7 = \{a, b, c\}$. Then $R_1, R_2, ..., R_7$ are subsemirings of R. Let $I_1 = \{a\}$, $I_2 = \{b\}$, $I_3 = \{c\}$, $I_5 = \{c\}$, $I_6 = \{c\}$ and $I_7 = \{a, b, c\}$. Then I_1 is the only ideal of R_1, I_2 is the only ideal of R_2, I_3 is the only ideal of R_3, R_4 has not any ideal, I_5 is the only ideal of R_5, I_6 is the only ideal of R_6 and I_7 is the only ideal of R_7. Let $J_1 = \{c\}$, $J_2 = \{a, b, c\}$. Then J_1 , J_2 and J_3 are ideals of R and we have $J_2 \cap R_1 = I_1, J_2 \cap R_2 = I_2, J_2 \cap R_3 = I_3, J_1 \cap R_5 = I_5, J_1 \cap R_6 = I_6, J_2 \cap R_7 = I_7$. It follows that R has the ideal extension property.

Example 2.4. Let $R = (\mathbb{Z}, +, .)$ be the semiring of all integer number under usual addition and multiplication. Let $H = \{0, 1, 2, 3, ...\}$ and $J = \{0, 4, 5, 8, 9, 10, 12, 13, 14, ...\} = H - \{1, 2, 3, 6, 7, 11\}$. Then H is a subsemiring of R and J is an ideal of H. Suppose that there exists an ideal K of R such that $K \cap H = J$. Then there exists $m \geq 0$ in $\mathbb{Z}$ such that $K = m\mathbb{Z}$. It follows that $K \cap H = mH$, but $J \neq mH$. This contradiction establishes that R does not have the ideal extension property.

Corollary 2.5. *In view of 2.2 and 2.4, the rational numbers* $\mathbb{Q}$ *does not have the ideal extension property.*

Theorem 2.6. *Let* R *be a semiring if* R *has the ideal extension property then any homomorphic image of* R *has the ideal extension property.*

Proof. Let R be a semiring with the ideal extension property. Let $\Psi : R \longrightarrow R^*$ be a homomorphism of semiring R onto a semiring R^*. Suppose that H^* be a subsemiring of R^* and J^* be an ideal of H^*. Let $H = \Psi^{-1}(H^*)$ and let $J = \Psi^{-1}(J^*)$. Then H is a subsemiring of R and J s an ideal of H. By hypothesis, there exists an ideal K of R such that $K \cap H = J$. Let $K^* = \psi(K)$. Then since Ψ is onto, we have that K^* is an ideal of R^*.

Now, we show that $K^* \cap H^* = J^*$. Let $x \in K^* \cap H^*$. Then $\Psi^{-1}(x) \subset K$ and $\Psi^{-1}(x) \subset H$. Therefore, $\Psi^{-1}(x) \subset K \cap H = J$. Then $x \in \Psi(J) = J^*$.

To establish the other inclusion let $a \in J^*$. Then $\Psi^{-1}(a) \subset J = K \cap H$. Therefore, $a \in \Psi(K \cap H) \subset \Psi K \cap \Psi H = K^* \cap H^*$. It follow that $K^* \cap H^* = J^*$. We conclude that R^* has the ideal extension property.

Definition 2.7. Let R be a semiring and let ρ be a congruence on R. Then ρ is called an ideal congruence provided that there exists an ideal J of R such that $\rho = (J \times J) \cup \Delta_R$ (where Δ_R denotes the diagonal relation on R).

Definition 2.8. A semiring R is said to be an ideal semiring provided that each congruence on R is an ideal congruence.

Theorem 2.9. *Let* R *be a semiring. If* R *is an ideal semiring then the homomorphic image of* R *is an ideal semiring.*

Proof. Suppose that R is an ideal semiring. Let $\Psi : R \longrightarrow S$ be a homomorphism of R onto a semiring S and let ρ be a congruence on S. Define $\alpha = \{(a, b) \in (R \times R) : (\Psi(a), \Psi(b)) \in \rho\}$. Then α is a congruence on R. By assumption, R is an ideal semiring. Consequently there exists an ideal J of R such that $\alpha = (J \times J) \cup \Delta_R$. Let $K = \Psi(J)$. Then K is an ideal of S.

We claim that $\rho = (K \times K) \cup \Delta_S$. Assume that $(\Psi(a), \Psi(b)) \in \rho$. Therefore $(a, b) \in \alpha$. If $a \neq b$, then $(a, b) \in (J \times J)$ and $(\Psi(a), \Psi(b)) \in (K \times K)$. If $a = b$ then $(\Psi(a), \Psi(b)) \in \Delta_S$. Thus $\rho \subset (K \times K) \cup \Delta_S$.

Let $(s, t) \in (K \times K)$. Then there exists $(a, b) \in (J \times J)$ such that $s = \Psi(a)$ and $t = \Psi(b)$. Then $(a, b) \in \alpha$ and we have $(s, t) \in \rho$. It follows that $\rho = (K \times K) \cup \Delta_S$ and S is an ideal semiring.

3. Congruence Extension Property for Semirings

Definition 3.1. Let R be a semiring. R has the congruence extension property provided that for each subsemiring H of R and each congruence ρ on H there exists a congruence α on R such that $\alpha \cap (H \times H) = \rho$. The congruence α is called an extension of ρ.

Theorem 3.2. *Let H be a subsemiring of a semiring R and ρ be a congruence on H and $\langle\rho\rangle$ be a congruence on R generated by ρ. Then ρ has an extension to R if and only if $\langle\rho\rangle$ is an extension.*

Proof. If $\langle\rho\rangle$ is an extension of congruence ρ to R, it is clear that ρ has an extension to R. Let α be an extension of congruence ρ to R. We must establish that $\rho = \langle\rho\rangle \cap (H \times H)$. Since $\rho \subset \langle\rho\rangle$ and $\rho \subset (H \times H)$ it follows that $\rho \subset \langle\rho\rangle \cap (H \times H)$. Since α is an extension of congruence ρ to R, it follows that $\alpha \cap (H \times H) = \rho \subset \langle\rho\rangle \cap (H \times H)$.

We clime that $\langle\rho\rangle \subset \alpha$. Let there exists $x \in (R \times R$ such that $x \in \langle\rho\rangle$ and $x \notin \alpha$. Then we have $x = \sum_{v=1}^{n} a_v$ for some $n \in I\!N$, where $a_v \in \rho, 1 \le v \le n$. Since $\alpha \cap (H \times H) = \rho, a_v \in \alpha, 1 \le v \le n$ and so $x = \sum_{v=1}^{n} a_v \in \alpha$. This contradiction implies that $\langle\rho\rangle \subset \alpha$ and it follows that $\langle\rho\rangle \cap (H \times H) \subset \alpha \cap (H \times H) = \rho$. We conclude that $\rho = \langle\rho\rangle \cap (H \times H)$. Therefore, to establish the existence of an extension it suffices to show that $\langle\rho\rangle \cap (H \times H) \subset \rho$.

Theorem 3.3. *Let R be an ideal semiring. If R has the congruence extension property, then R has the ideal extension property.*

Proof. Let R be an ideal semiring which has the congruence extension property, let H be a subsemiring of R and J be an ideal of H. Then $\rho = (J \times J) \cup \Delta_H$ is a congruence on H. Since R has the congruence extension property, there exists an extension $\langle\rho\rangle$ of ρ to R. Since R is an ideal semiring there exists an ideal K of R such that $\langle\rho\rangle = (K \times K) \cup \Delta_R$. Since $\rho = \langle\rho\rangle \cap (H \times H)$, we conclude that $J = K \cap H$. Therefore R has the ideal extension property.

References

[1] F.W. Anderson and K.R. fuller, *Rings and Categories of Modules*, Springe-Verlag, New York, 1974.

[2] A.H. Clifford and G.B. Preston, *The Algebraic Theory of Semigroups*, Volumes I and II, Amer. Math. Soc. Survey, No. 7, 1961.

[3] J.I. Garcia, *The Congruence Extension Property for Algebraic Semigroup*, Ph.D. Thesis, 1990.

[4] K.R. Goodearl, *Ring Theory*, Marcel Dekker, New York, 1967.

[5] U. Hebisch and H. J. Weinert, *Semirings: Algebraic Theory and Applications in Computer Science*, B.G. Teubner, Stuttgart, 1993.

[6] J.M. Howie, *An introduction to Semigroup Theory*, Academic Press, 1967.

[7] Y.S. Park, J.P. Kim and M.G. Sohn, Semiprime ideals in semigroups, *Math. Japonica*, **33** (1988), 267-273.