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POLYNOMIAL FAMILY GRAPH OF $\mathbb{Z}_n$

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Abstract: The polynomial family graph $\Gamma_{\mathscr{P}}(R)$ of a finite ring R generated by a family of polynomials $\mathscr{P} \subseteq R[x]$ is a graph in which the elements of the ring are the vertices and two distinct elements a and b are adjacent if $p(a) = b$ for some $p(x) \in \mathscr{P}$. In this paper $\Gamma_{\mathscr{P}}(\mathbb{Z}_n)$ generated by the polynomial family $\mathscr{P} = \{ax + b \ : \ a, b \in \{1, \ -1\}\}$ is studied and bounds for some domination parameters are given.

AMS Subject Classification: 05C
Key Words: polynomial family graph, domination, chromatic, independence

1. Introduction

All graphs $\Gamma = (V, E)$ considered in this paper are connected, finite, undirected graph with neither loops nor multiple edges and $\mathbb{Z}_n$ refers to the ring of integers modulo n. For notations and terminology, see [1, 2]

The polynomial family graph $\Gamma_{\mathscr{P}}(R)$ of a finite ring R generated by a family of polynomials$\mathscr{P} \subseteq R[x]$ is a graph in which the elements of the ring are the vertices and two distinct elements a and b are adjacent if $p(a) = b$ for some $p(x) \in \mathscr{P}$. A subset S of the vertex set $V(\Gamma)$ of a nontrivial graph Γ is called a dominating set of Γ if every vertex in $V(\Gamma) - S$ is adjacent to atleast one vertex in S. The domination number $\gamma(\Gamma)$ of Γ is the minimum cardinality taken over

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all dominating sets in Γ. A subset S of the edge set $E(\Gamma)$ of a nontrivial graph Γ is called an edge dominating set of Γ if every edge in $E(\Gamma) - S$ is adjacent to atleast one edge in S. The edge domination number $\gamma'(\Gamma)$ of Γ is the minimum cardinality taken over all edge dominating sets in Γ. The minimum number of colors needed to color the vertices of Γ so that no two adjacent vertices get same color is called the chromatic number of Γ and is denoted bt $\chi(\Gamma)$. A subet S of $E(\Gamma)$ is called an edge independent set if no two edges in S are adjacent. The maximum cardinality of an edge independent set in a graph is called the edge independence number of Γ and is denoted by $\beta(\Gamma)$. In this paper some parameters such as domination and coloring for the polynomial family graph of $\mathbb{Z}_n$, $\Gamma_{\mathscr{P}}(\mathbb{Z}_n)$ generated by the polynomial family $\mathscr{P} = \{ax+b \ : \ a, b \in \{1, \ -1\}\}$ is studied in detail. Hereafter throught this paper $\mathscr{P} = \{ax + b \ : \ a, b \in \{1, \ -1\}\}$ unless otherwise specified.

2. Main Results

Consider the following subsets of the edge set $E(\Gamma_{\mathscr{P}}(\mathbb{Z}_n))$,

$$E_1 = \{(x, y) \in E(\Gamma_{\mathscr{P}}(\mathbb{Z}_n))/y = x + 1, x = 0, 1, 2, \ldots, n - 1\},$$

$$E_2 = \{(x, y) \in E(\Gamma_{\mathscr{P}}(\mathbb{Z}_n))/y = -x + 1, x = 2, \ldots, \left\lfloor \tfrac{n}{2} \right\rfloor, x \neq \tfrac{n}{2}\},$$

$$E_3 = \{(x, y) \in E(\Gamma_{\mathscr{P}}(\mathbb{Z}_n))/y = -x - 1, x = 1, 2, \ldots, \left\lfloor \tfrac{n}{2} \right\rfloor - 1, x \neq \tfrac{n}{2} - 1\}.$$

It is easy to check every edge of $\Gamma_{\mathscr{P}}(\mathbb{Z}_n)$ lies in one of E_i, so that $E_1 \cup E_2 \cup E_3 = E(\Gamma_{\mathscr{P}}(\mathbb{Z}_n))$. These sets are all mutually disjoint.

When n is odd $\left\lfloor \frac{n}{2} \right\rfloor \neq \frac{n}{2}$, so that

$$\begin{aligned}|E(\Gamma_{\mathscr{P}}(\mathbb{Z}_n))| = |E_1| + |E_2| + |E_3| &= n + (\left\lfloor \frac{n}{2} \right\rfloor - 1) + (\left\lfloor \frac{n}{2} \right\rfloor - 1) \\ &= n + (\frac{n-1}{2} - 1) + (\frac{n-1}{2} - 1) = 2n - 3.\end{aligned}$$

When n is even, $\left\lfloor \frac{n}{2} \right\rfloor = \frac{n}{2}$, so that

$$\begin{aligned}|E(\Gamma_{\mathscr{P}}(\mathbb{Z}_n))| = |E_1| + |E_2| + |E_3| &= n + (\left\lfloor \frac{n}{2} \right\rfloor - 2) + (\left\lfloor \frac{n}{2} \right\rfloor - 2) = 2n - 4 \\ &= n + (\frac{n}{2} - 2) + (\frac{n}{2} - 2) = 2n - 4.\end{aligned}$$

Summarising these results, we have:

The number of edges $|E(\Gamma_{\mathscr{P}}(\mathbb{Z}_n))|= \begin{cases} \text{2n-4} & \text{if } n \text{ is even,} \\ \text{2n-3} & \text{if } n \text{ is odd.} \end{cases}$

$\Gamma_{\mathscr{P}}(\mathbb{Z}_n)$ is Hamiltonian and so there is a dominating set of $\lceil \frac{n}{3} \rceil$ vertices. Therefore $\gamma(\Gamma_{\mathscr{P}}(Z_n)) \leq \lceil \frac{n}{3} \rceil$ holds always however the following theorem gives a better bound.

Theorem 1. *The domination number,* $\gamma(\Gamma_{\mathscr{P}}(\mathbb{Z}_n)) \leq \lfloor \frac{n}{4} \rfloor + 1$.

Proof. Consider the set

$$S_1 = \{4k + j \ : k = 0, 1, 2, \dots, \quad j = 1, 2 \text{ and } 4k + j \leq \left\lfloor \frac{n}{2} \right\rfloor\}.$$

Choose $S = \begin{cases} S_1 \cup \{\lfloor \frac{n}{2} \rfloor\} & \text{when 8 divides } n \text{ or } n-1, \\ S_1 & \text{otherwise.} \end{cases}$

Write $n = 8l + i$, where $l \in \{0, 1, 2, 3, \dots\}$ and $i \in \{0, 1, 2, \dots, 7\}$.

Let $A = \{x, n - x : 0 \leq x \leq 4l + 3\}$ be a subset of the vertex set V of $\Gamma_P(Z_n)$. For $k \leq l$, $(4k, 4k + 1)$, $(4k + 2, 4k + 3) \in E_1$ and $4k + 1$, $4k + 2 \in S$. Therefore S dominates all vertices x with $0 \leq x \leq 4l + 3$. Possibly also all $(n - 4k, 4k + 1)$, $(n - (4k + 1), 4k + 2) \in E_2$ except $(0, 1) \in E_1$ and $(n - (4k + 2), 4k + 1)$, $(n - (4k + 3), 4k + 2) \in E_3$. Therefore S dominates all vertices $n - x$ with $0 \leq x \leq 4l + 3$. Therefore S dominates A.

When $n = 8l + 7$, $V = A$ and S is a dominating set.

When $n = 8l$, $V - A = \{4l\}$ and $4l = \lfloor \frac{n}{2} \rfloor \in S$.

When $n = 8l + 1$, $V - A = \{4l, 4l + 1\}$ and $4l = \lfloor \frac{n}{2} \rfloor \in S$ and it dominates $4l$, $4l + 1$.

When $n = 8l + 2$, $V - A = \{4l, 4l + 1, 4l + 2\}$. Hence $4l + 1 = \lfloor \frac{n}{2} \rfloor \in S$ and it dominates $4l$, $4l + 2$.

When $n = 8l + 3$, $V - A = \{4l, 4l + 1, 4l + 2, 4l + 3\}$. Hence $4l + 1 = \lfloor \frac{n}{2} \rfloor \in S$ and it dominates $4l$, $4l + 2$ by edges of E_1 and dominates $4l + 3$ by the edge $(4l + 1, 4l + 3) = (4l + 1, n - (4l + 1) + 1) \in E_2$.

When $n = 8l + 4$, $V - A = \{4l + i : i = 0, 1, 2, 3 \text{ and } 4\}$. Hence $4l + 1$, $4l + 2 \in S_1$ and these dominates $4l$, $4l + 3$ by edges of E_1 and $4l + 4$ by the edge $(4l + 1, 4l + 4) = (4l + 1, n - (4l + 1) + 1) \in E_2$.

When $n = 8l + 5$, $V - A = \{4l + i : i = 0, 1, 2, 3, 4 \text{ and } 5\}$. Hence $4l + 1$, $4l + 2 \in S$ and these dominates $4l$, $4l + 3$ by edges of E_1 and dominates $4l + 4$ and $4l + 5$ by the edges $(4l + 1, 4l + 5) = (4l + 1, n - (4l + 1) + 1) \in E_2$ and $(4l + 2, 4l + 4) = (4l + 2, n - (4l + 2) + 1) \in E_2$.

When $n = 8l + 6$, $V - A = \{4l + i : i = 0, 1, 2, 3, 4, 5\text{and } 6\}$. Hence $4l + 1$, $4l + 2 \in S$ and they dominate $4l$, $4l + 3$ by edges of E_1 and dominate $4l + 5$

and $4l+6$ by the edges $(4l+1, 4l+6) = (4l+1, n-(4l+1)+1) \in E_2$ and $(4l+2, 4l+5) = (4l+2, n-(4l+2)+1) \in E_2$ and dominate $4l+4$ by the edge $(4l+1, 4l+4) = (4l+1, n-(4l+1)-1) \in E_3$. So on the whole S is a dominating set. Now $|S| = \begin{cases} 2l+1 & \text{when } n = 8l+j,\ j = 0,1,2 \text{ or } 3 \\ 2l+2 & \text{when } n = 8l+j,\ j = 4,5,6 \text{ or } 7 \end{cases}$ In all the cases it can be checked $|S| = \lfloor \frac{n}{4} \rfloor + 1$. Therefore $\gamma(\Gamma_{\mathscr{P}}(\mathbb{Z}_n)) \leq \lfloor \frac{n}{4} \rfloor + 1$ □

Theorem 2. *Edge domination number,* $\gamma'(\Gamma_{\mathscr{P}}(\mathbb{Z}_n)) \leq \lceil \frac{n}{3} \rceil$.

Proof. Let $n = 6l+i-2$, where $i \in \{0,1,2,3,4,5\}$ and l be a nonnegative integer. Consider the set

$$S_1 = \{(3k+1, 3k+2), (-3k-1, -3k-2) \in E_1 \ : k = 0,1,2,\ldots \text{ and } 3k+2 \leq \lceil \frac{n}{2} \rceil\}$$

and let $S = \begin{cases} S_1 \cup \{(\lfloor \frac{n}{2} \rfloor, \lfloor \frac{n}{2} \rfloor + 1)\} & \text{if } n = 6l+1 \text{ or } 6l+2 \\ S_1 & \text{otherwise} \end{cases}$

Every edge in $E_2 \cup E_3$ is of the form either $(x, -x+1)$ or $(x, -x-1)$. So that each edge in $E_2 \cup E_3$ has an end point of the form $3k+1$, $3k+2$, $-3k-1$ or $-3k-2$, where $3k+2 \leq \lfloor \frac{n}{2} \rfloor$. Therefore S dominates $E_2 \cup E_3$.

The edges in E_1 forms a cycle say $e_1 e_2 e_3 \ldots e_n e_{n+1}$, where $e_{n+1} = e_n$ and e_i and e_{i+1} are adjacent. The set S is constructed from these edges of E_1 in such a way that in this cycle if $e_i \in S$, then either e_{i+2} or $e_{i+3} \in S$. So the edges of S cover all the edges in E_1. So S dominates E_1. Therefore S is an edge dominating set. Therefore $\gamma'(\Gamma_{\mathscr{P}}(\mathbb{Z}_n)) \leq |S|$.

$$\begin{aligned} |S| &= \begin{cases} |S_1| + 1 & \text{if } n = 6l+1 \text{ or } 6l+2 \\ |S_1| & \text{otherwise} \end{cases} \\ &= \begin{cases} 2l+1 & \text{if } n = 6l+1, 6l+2 \text{ or } 6l+3 \\ 2l & \text{otherwise} \end{cases} \\ &= \begin{cases} \frac{n-3}{3} + 1 & \text{if } n = 6l+3 \\ \frac{n-2}{3} + 1 & \text{if } n = 6l+2 \\ \frac{n-1}{3} + 1 & \text{if } n = 6l+1 \\ \frac{n}{3} & \text{if } n = 6l \\ \frac{n+1}{3} & \text{if } n = 6l-1 \\ \frac{n+2}{3} & \text{if } n = 6l-2 \end{cases} \end{aligned}$$

$$= \left\lceil \frac{n}{3} \right\rceil .$$ □

Theorem 3. *The chromatic number,*

$$\chi(\Gamma_{\mathscr{P}}(\mathbb{Z}_n)) = \begin{cases} 2 & \text{when } n \text{ is even} \\ 3 & \text{when } n \text{ is odd} \end{cases}$$

Proof. Let $N_0 = \{0\}$, $N_1 = \{1, n-1\}$, $N_2 = \{2, n-2\} \ldots$, $N_{\lfloor \frac{n}{2} \rfloor - 1} = \{\lfloor \frac{n}{2} \rfloor - 1, n - \lfloor \frac{n}{2} \rfloor + 1\}$, $N_{\lfloor \frac{n}{2} \rfloor} = \{\lfloor \frac{n}{2} \rfloor\}$. Clearly each of $N_i's$ are independent and N_i has adjacency only with N_{i-1} and N_{i+1}. Let $A = \bigcup_{i \ even} N_i$ and $B = \bigcup_{i \ odd} N_i$. Then A and B are independent. When n is even, $\{A, B\}$ forms a bipartition of Z_n by independent sets. Therefore $\chi(\Gamma_{\mathscr{P}}(\mathbb{Z}_n)) \leq 2$. Since $\Gamma_{\mathscr{P}}(\mathbb{Z}_n)$ can not be nullgraph, $\chi(\Gamma_{\mathscr{P}}(\mathbb{Z}_n)) = 2$. When n is odd, $A \cup B = Z_n - \{\lceil \frac{n}{2} \rceil\}$. Then $\{A, B, \{\lceil \frac{n}{2} \rceil\}\}$ forms a partition of Z_n by independent set. Therefore $\chi(\Gamma_{\mathscr{P}}(\mathbb{Z}_n)) \leq 3$. When n is odd, $\{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor - 1\}$ forms a triangle. So $\chi(\Gamma_{\mathscr{P}}(\mathbb{Z}_n)) \geq 3$. Therefore $\chi(\Gamma_{\mathscr{P}}(\mathbb{Z}_n)) = 3$, when n odd. □

Theorem 4. *The edge independence number of* $\beta(\Gamma_{\mathscr{P}}(\mathbb{Z}_n)) = \lfloor \frac{n}{2} \rfloor$

Proof. As in the previous theorem, $|A| = |B| = \lfloor \frac{n}{2} \rfloor$ and they are independent. Therefore $\beta(\Gamma_{\mathscr{P}}(\mathbb{Z}_n)) \geq \lfloor \frac{n}{2} \rfloor$. $\Gamma_{\mathscr{P}}(\mathbb{Z}_n)$ has hamiltonian cycle C_n. Its known that $\beta(C_n) = \lfloor \frac{n}{2} \rfloor$. Since C_n is a hamiltonian cycle, it covers all the vertices. Addition of more edges can not increase independents number. Therefore $\beta(\Gamma_{\mathscr{P}}(\mathbb{Z}_n)) \leq \lfloor \frac{n}{2} \rfloor$. Hence $\beta(\Gamma_{\mathscr{P}}(\mathbb{Z}_n)) = \lfloor \frac{n}{2} \rfloor$ □

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