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r-DYNAMIC CHROMATIC NUMBER OF CYCLES, COMPLETE GRAPHS AND FORESTS

Hermie V. Inoferio[1], Michael P. Baldado Jr.[2 §]

[1]College of Education
Jose Rizal Memorial State University
Katipunan, Zamboanga del Norte, PHILIPPINES

[2]Mathematics Department
Negros Oriental State University
Dumaguete City, Philippines

Abstract: Let $G = (V, E)$ be a graph. An r-dynamic k-coloring of G is a function f from V to a set C of colors such that (1) f is a proper coloring, and (2) for all vertices v in V, $|f(N(v))| \geq min\{r, deg_G(v)\}$. The r-dynamic chromatic number of a G, denoted by $\chi_r(G)$, is the smallest k such that f is an r-dynamic k-coloring of G.

This study gave the r-dynamic chromatic number of paths, cycles, complete graphs, empty graphs, the vertex gluing of graphs and the union of graphs. Some characterizations are also given.

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Key Words: r-dynamic chromatic number, r-dynamic k-coloring, cycles, vertex gluing

1. Introduction

As mentioned in [11], the earliest results about graph coloring is on planar graphs motivated by a problem regarding the coloring of maps. While coloring a map of the counties of England, Francis Guthrie presented the *four color conjecture* which stated that four colors were sufficient to color the map so that no regions sharing a common border received the same color. This idea may have motivated the concept proper vertex coloring. Using graph theoretic

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§Correspondence author

parlance, a proper vertex coloring of a graph $G = (V, E)$ is a function from V to a finite set C, whose elements are called *colors*, such that no two adjacent vertex are assigned the same color.

A generalization of the proper vertex coloring is the dynamic coloring of graphs. The dynamic coloring of a graph G is a proper coloring such that every vertex of G with degree at least two has atleast two neighbors that are colored differently. This concept was introduced by Montgomery in [3]. Since then, the concept was studied by many authors, see for example [1], [2], [4], [5], [6] and [7].

A generalization of the dynamic coloring was also introduced by Montgomery in [3]. The generalized concept is now called the r-dynamic k-coloring. The proper vertex coloring and the dynamic coloring are special cases of the concept r-dynamic k-coloring, that is, the 1-dynamic k-coloring is the proper vertex coloring concept, while the 2-dynamic k-coloring is the dynamic coloring concept.

The general r-dynamic k-coloring of planar grids is studied well in [8] and [9].

A vertex coloring of a graph $G = (V, E)$ is a function f from V to a finite set C, whose elements are called color. A *k-coloring* is a vertex coloring with at most k-colors. In this case, we always assume that $C = \{1, 2, \ldots, k\}$. A *k-coloring* may also be viewed as a vertex partition $V = V_1 \cup V_2 \cup \cdots \cup V_k$ or $(V_1, V_2, \ldots, V_k)$, where $V_i = f^{-1}(i)$ are called the *color classes*. A graph is *k-colorable* if it admits a proper vertex coloring with at most k-colors. The *chromatic number* of a graph G, denoted by $\chi(G)$, is the smallest k such that G is k-colorable.

An r-dynamic k-coloring of G is a proper coloring f such that for all vertices v in V, $|f(N(v))| \geq min\{r, deg_G(v)\}$. The r-dynamic chromatic number of a G, denoted by $\chi_r(G)$, is the smallest k such that f is an r-dynamic k-coloring of G.

Let $G_1, G_2, \ldots, G_t$ be disjoint graphs, each containing a complete subgraph K_r $(r \geq 1)$. Let G be a graph obtained from the union of t graphs G_i by identifying the complete graphs K_r (from each G_i) in an arbitrary way. We call G a K_r-gluing of $G_1, G_2, \ldots, G_t$. In particular, when $r = 1$ we say that G is a *vertex-gluing.*

Hereafter, please refer to [12] for concepts that were used but were not discussed in this paper.

2. r-Dynamic Chromatic Number of Paths and Cycles

In this section we verified the r-dynamic chromatic number of paths and cycles. The next Lemma, Lemma 2.1, characterizes r-dynamic k-coloring in paths.

Lemma 2.1. *Let $P_n = (v_1, v_2, \ldots, v_n)$ be a path of order $n \geq 3$. Then $f : P_n \rightarrow \{1, 2, \ldots, k\}$ is an r-dynamic $(r > 1)$ k-coloring of P_n if and only if for all i with $deg(v_i) = 2$, $f(v_{i-1})$, $f(v_i)$, $f(v_{i+1})$ are distinct.*

Proof. Assume that f is an r-dynamic k-coloring of P_n and there exist i such that at least two of $f(v_{i-1})$, $f(v_i)$, $f(v_{i+1})$ are not distinct. Consider the following cases: (Case 1. $f(v_{i-1}) = f(v_i)$) If $f(v_{i-1}) = f(v_i)$, then f is not proper. This is a contradiction. (Case 2. $f(v_{i-1}) = f(v_{i+1})$) If $f(v_{i-1}) = f(v_{i+1})$, then $f(N(v_i)) = f(\{v_{i-1}, v_{i+1}\}) = \{f(v_{i-1})\}$, that is, $|f(N(v_i))| = |\{f(v_{i-1})\}| = 1 < 2 = min\{2, r\}$. This is a contradiction.

Conversely, assume that for all i with $deg(v_i) = 2$, $f(v_{i-1})$, $f(v_i)$, $f(v_{i+1})$ are distinct. Let $v \in V(P_n)$ and consider the following cases: (Case 1. $v = v_1$ or $v = v_n$) If $v = v_1$, then $deg(v_1) = 1$. Hence, $|f(N(v))| = |f(N(v_1))| = |\{f(v_1)\}| = 1 = min\{1, r\}$. (Case 2. $v \neq v_1$ and $v \neq v_n$) Without loss of generality, let $v = v_2$. Then $deg(v) = deg(v_2) = 2$. Thus, $|f(N(v))| = |f(N(v_2))| = |\{f(v_1), f(v_3)\}| = 2 = min\{2, r\}$. This shows that f is an r-dynamic k-coloring of P_n. □

Remark 2.2. Lemma 2.1 says that if f is an r-dynamic k-coloring in P_n, then $k \geq 3$.

An observation in [9] stated that $\chi_r \geq min\{\Delta(G, r)\} + 1$ with equality holding when G is a tree. Thus, when this observation is applied for paths of order $n \geq 3$, we have the following remark.

Remark 2.3. Let $P_n = (v_1, v_2, \ldots, v_n)$ be a path of order $n \geq 3$. If $r > 1$, then $\chi_r(P_n) = 3$.

The ideas presented in Lemma 2.1 and Remark 2.2 may be used to verify Remark 2.3 as follows. Define $f : V(P_n) \rightarrow \{1, 2, 3\}$ as follows

$$f(v_i) = \begin{cases} 1, & \text{if } i \equiv 1\,(mod\,3) \\ 2, & \text{if } i \equiv 2\,(mod\,3) \\ 3, & \text{if } i \equiv 0\,(mod\,3). \end{cases}$$

Then by Lemma 2.1 f is an r-dynamic 3-coloring of P_n. Hence, $\chi_r(P_n) \leq 3$. Now by Remark 2.2 $\chi_r(P_n) = 3$.

The next Lemma 2.4 characterizes an r-dynamic k-coloring in cycles.

Lemma 2.4. *Let $C_n = [v_1, v_2, \ldots, v_n]$ be a cycle of order $n \geq 3$. Then $f : V(C_n) \rightarrow \{1, 2, \ldots, k\}$ is an r-dynamic ($r > 1$) k-coloring of C_n if and only if for all i, $f(v_{i-1})$, $f(v_i)$, $f(v_{i+1})$ are distinct.*

Proof. Assume that f is an r-dynamic k-coloring of C_n and there exist i such that at least two of $f(v_{i-1})$, $f(v_i)$, $f(v_{i+1})$ are not distinct. Consider the following cases: (Case 1. $f(v_{i-1}) = f(v_i)$) If $f(v_{i-1}) = f(v_i)$, then f is not proper. This is a contradiction. (Case 2. $f(v_{i-1}) = f(v_{i+1})$) If $f(v_{i-1}) = f(v_{i+1})$, then $f(N(v_i)) = f(\{v_{i-1}, v_{i+1}\}) = \{f(v_{i-1})\}$, that is, $|f(N(v_i))| = |\{f(v_{i-1})\}| = 1 < 2 = min\{2, r\}$. This is a contradiction.

Conversely, assume that for all i, $f(v_{i-1})$, $f(v_i)$, $f(v_{i+1})$ are distinct. Let $v \in C_n$. Then $v = v_i$ for some i and $deg(v) = deg(v_i) = 2$. Thus, $|f(N(v))| = |f(N(v_i))| = |\{f(v_{i-1}), f(v_{i+1})\}| = 2 = min\{2, r\}$. This shows that f is an r-dynamic k-coloring of C_n. □

Remark 2.5. Lemma 2.4 says that if f is an r-dynamic k-coloring in C_n, then $k \geq 3$.

Lemma 2.6. *Let $G = (V, E)$ be a graph and $S = \{v_1, v_2, \ldots, v_n\} \subseteq V$ with $\langle\{v_1, v_2, \ldots, v_n\}\rangle = (v_1, v_2, \ldots, v_n)$. Let $f : V \rightarrow \{1, 2, 3\}$ be a vertex coloring of G. Then for all $i = 2, 3, \ldots, n-1$, $f(v_{i-1})$, $f(v_i)$, $f(v_{i+1})$ are distinct if and only if*

$$f|_S(v_j) = \begin{cases} k_1, & \text{if } j \equiv 0\,(mod\, 3) \\ k_2, & \text{if } j \equiv 1\,(mod\, 3) \\ k_3, & \text{if } j \equiv 2\,(mod\, 3) \end{cases}$$

where $k_1k_2k_3$ is a permutation of 1,2 and 3.

Proof. Assume that $f(v_i)$, $f(v_{i+1})$ are distinct and without loss of generality

$$f|_S(v_j) \neq \begin{cases} 1, & \text{if } j \equiv 0\,(mod\, 3) \\ 2, & \text{if } j \equiv 1\,(mod\, 3) \\ 3, & \text{if } j \equiv 2\,(mod\, 3). \end{cases}$$

Since f must be proper, there exists i such that $f|_S(v_i) = 1$, $f|_S(v_{i+1}) = 2$, $f|_S(v_{1+2}) = 3$, and $f|_S(v_{i+3}) = 2$. This is a contradiction.

Conversely, assume that

$$f|_S(v_j) = \begin{cases} k_1, & \text{if } j \equiv 0\,(mod\, 3) \\ k_2, & \text{if } j \equiv 1\,(mod\, 3) \\ k_3, & \text{if } j \equiv 2\,(mod\, 3) \end{cases}$$

where $k_1k_2k_3$ is a permutation of 123, say without loss of generality

$$f|_S(v_j) = \begin{cases} 1, & \text{if } j \equiv 0\,(mod\,3) \\ 2, & \text{if } j \equiv 1\,(mod\,3) \\ 3, & \text{if } j \equiv 2\,(mod\,3). \end{cases}$$

Let $j \in \{2, 3, \ldots, n-1\}$, say without loss of generality $j \equiv 1\,(mod\,3)$. Then $j-1 \equiv 0\,(mod\,3)$ and $j+1 \equiv 2\,(mod\,3)$. Hence, $f(v_{j-1}) = 3$, $f(v_j) = 1$, $f(v_{j+1}) = 2$ are distinct. □

A result in [10] stated that

$$\chi_2(C_n) = \begin{cases} 3, & \text{if } n \equiv 0\,(mod\,3) \\ 5, & \text{if } n = 5 \\ 4, & \text{otherwise} \end{cases}$$

and an observation in [8] stated that if $r \geq \Delta(G)$, then $\chi_r(G) = \chi_{\Delta(G)}(G)$. By these and since $\Delta(C_n) = 2$, we have the following remark.

Remark 2.7. Let $C_n = [v_1, v_2, \ldots, v_n]$ be a cycle of order $n \geq 3$. If $r > 1$, then

$$\chi_r(C_n) = \begin{cases} 3, & \text{if } n \equiv 0\,(mod\,3) \\ 5, & \text{if } n = 5 \\ 4, & \text{otherwise} \end{cases}$$

The ideas presented in Lemma 2.4, Remark 2.5 and Lemma 2.6 may be used to verify Remark 2.7 as follows. Let $C_n = [v_1, v_2, \ldots, v_n]$ be a cycle of order $n \geq 3$. Consider the following cases: (Case 1. $n \equiv 0\,(mod\,3)$) If $n \equiv 0\,(mod\,3)$, then we define $f : V(C_n) \to \{1, 2, 3\}$ as follows

$$f(v_i) = \begin{cases} 1, & \text{if } i \equiv 1\,(mod\,3) \\ 2, & \text{if } i \equiv 2\,(mod\,3) \\ 3, & \text{if } i \equiv 0\,(mod\,3). \end{cases}$$

Then by Lemma 2.4 f is an r-dynamic 3-coloring of C_n. Thus, by Remark 2.5 $\chi_r(C_n) = 3$.

(Case 2. $n \equiv 1\,(mod\,3)$) If $n \equiv 1\,(mod\,3)$, then we define $f : V(C_n) \to \{1, 2, 3, 4\}$ as follows

$$f(v_i) = \begin{cases} 1, & \text{if } i \equiv 1\,(mod\,3) \text{ and } j \neq n \\ 2, & \text{if } i \equiv 2\,(mod\,3) \\ 3, & \text{if } i \equiv 0\,(mod\,3) \\ 4, & \text{if } i = n. \end{cases}$$

Then by Lemma 2.4 f is an r-dynamic 3-coloring of C_n. Hence, $\chi_r(C_n) \leq 4$. Suppose that $\chi_r(C_n) < 4$, say without loss of generality $\chi_r(C_n) = 3$. Let $f : V(C_n) \to \{1,2,3\}$ be an r-dynamic 3-coloring of C_n. Then by Lemma 2.4 $f(v_{i-1})$, $f(v_i)$, $f(v_{i+1})$ are distinct for all $i = 1,2,\ldots,n$. Let $S = \{1,2,\ldots, n-1\}$. Then $\langle S\rangle = (1,2,\ldots,n-1)$. By Lemma 2.6, without loss of generality

$$f(v_j) = \begin{cases} 1, & \text{if } j \equiv 0\,(mod\,3) \\ 2, & \text{if } j \equiv 1\,(mod\,3) \\ 3, & \text{if } j \equiv 2\,(mod\,3). \end{cases}$$

for all $i = 1,2,\ldots,n-1$. Since there are only 3 colors, $f(v_n)$ is either 1, 2 or 3. Since f must be a proper vertex coloring, $f(v_n) = 2$. This implies that $f(v_{n-2})$, $f(v_{n-1})$, $f(v_n)$ are not distinct. This is a contradiction. Therefore, $\chi_r(C_n) = 4$.

(Case 3. $n \equiv 2\,(mod\,3)$ with $n \neq 5$) If $n \equiv 1\,(mod\,3)$ with $n \neq 5$, then we define $f : V(C_n) \to \{1,2,3,4\}$ as follows

$$f(v_i) = \begin{cases} 1, & \text{if } i \equiv 1\,(mod\,4) \\ 2, & \text{if } i \equiv 2\,(mod\,4) \\ 3, & \text{if } i \equiv 3\,(mod\,4) \\ 4, & \text{if } i \equiv 0\,(mod\,4). \end{cases}$$

Then clearly f is an r-dynamic 4-coloring of C_n. Hence, $\chi_r(C_n) \leq 4$. Suppose that $\chi_r(C_n) < 4$, say $\chi_r(C_n) = 3$. Using the same arguments in the proof of Case 2, it can be shown also that $\chi_r(C_n)$ can not be less than 4. Therefore, $\chi_r(C_n) = 4$.

(Case 4. $n = 5$) If $n = 5$, then we define $f : V(C_5) \to \{1,2,3,4,5\}$ as follows $f(v_i) = i$. Then clearly f is an r-dynamic 5-coloring of C_5. Hence, $\chi_r(C_5) \leq 5$. Suppose that $\chi_r(C_5) < 5$, say $\chi_r(C_5) = 4$. Then by Lemma 2.4 (without loss of generality) $f(v_i) = i$ for $i = 1,2,3,4$. Since there are only 4 colors $f(v_5)$ is either 1 or 2 or 3 or 4. This is a contradiction by Lemma 2.4. Therefore, $\chi_r(C_5) = 5$.

3. r-Dynamic Chromatic Number of Complete Graphs and Empty Graphs

This section presents the r-dynamic chromatic number of complete graphs and empty graphs. Theorem 3.1 characterizes graphs with r-dynamic chromatic number equal to its order when r is less than the minimum degree δ.

Theorem 3.1. *Let $G = (V, E)$ be graph of order $n \geq 3$, and $r \in \mathbb{N}$ with $r < \delta$. Then $\chi_r(G) = n$ if and only if $G \cong K_n$.*

Proof. Assume that $\chi_r(G) = n$ and $G \ncong K_n$. If $G \ncong K_n$, then there exist $u, v \in V$ such that $uv \notin E$. Assume that $V = \{v_1, v_2, \ldots, v_{n-2}, v_{n-1} = u, v_n = v\}$. Define $f : V(K_n) \to \{1, 2, \ldots, n-1\}$ as follows

$$f(v_i) = \begin{cases} i, & \text{if } i \neq n \\ n-1, & \text{if } i = n. \end{cases}$$

Let $w \in V(G)$ and consider the following cases: (Case 1. $w = u$ or $w = v$) Without loss of generality, if $w = u$, then $|f(N(w))| = min\{r, deg(u)\}$. (Case 2. $w \neq u$ and $w \neq v$) If $w \neq u$ and $w \neq v$, then $|f(N(w))| \geq r = min\{r, deg(u)\}$ since $r < \delta$. Thus, f is an r-dynamic k-coloring of G, that is, $\chi_r(G) \leq n-1$. This is a contradiction.

Conversely, assume that $G = K_n$. Then $n = \chi(G) \leq \chi_r(G) \leq n$. Hence, $\chi_r(G) = n$. □

Theorem 3.2 characterizes graphs with r-dynamic chromatic number equal to 1.

Theorem 3.2. *Let $G = (V, E)$ be graph of order n, and $r \in \mathbb{N}$. Then $\chi_r(G) = 1$ if and only if $G \cong \overline{K}_n$.*

Proof. Assume that $\chi_r(G) = 1$ and $G \ncong \overline{K}_n$. If $G \ncong K_n$, then there exist $u, v \in V$ such that $uv \in E$. Let $f : V(\overline{K}_n) \to \{1\}$ be an r-dynamic 1-coloring of G. Then $f(u) = 1 = f(v)$. This is a contradiction since f must be proper.

Conversely, assume that $G = \overline{K}_n$. Define $f : V \to \{1\}$ by $f(v) = 1$ for all $v \in V$. Let $u \in V$. Then $|f(N(u))| = |f(\varnothing)| = 0 = min\{r, 0\}$. Clearly, f is a proper coloring. Hence, f is an r-dynamic 1-coloring of G. Accordingly, $\chi_r(G) = 1$. □

4. 2-Dynamic Chromatic Number of the Vertex Gluing of Graphs

In this section we present the r-dynamic chromatic number of the vertex gluing of graphs. Lemma 4.1 characterizes an r-dynamic k-coloring in a graph.

Lemma 4.1. *Let $G = (V, E)$ be graph and $f = \{V_1, V_2, \ldots, V_k\}$ be a k-coloring of G. Then f is an r-dynamic k-coloring of G if and only if*

1. $\{u, v\} \nsubseteq V_i$ for all $i = 1, 2, \ldots, k$ whenever $uv \in E$, and

2. *there exist* $\left\{q_1, q_2, \dots, q_{min\{r,deg(v)\}}\right\} \subseteq \{1, 2, \dots, k\} \setminus \{i\}$ *such that* $N(v) \cap V_{q_j} \neq \varnothing$ *for all* $j = 1, 2, \dots, min\{r, deg(v)\}$ *whenever* $v \in V_i$ *for some* $i \in \{1, 2, \dots, k\}$.

Proof. Assume that $f = \{V_1, V_2, \dots, V_k\}$ is an r-dynamic k-coloring of G. Let $uv \in E$. Since f must be proper $u \in V_s$ and $v \in V_t$ for some $s, t \in \{1, 2, \dots, k\}$ with $s \neq t$. Since $\{V_1, V_2, \dots, V_k\}$ is a pairwise disjoint family, we must have $\{u, v\} \nsubseteq V_i$ for all $i = 1, 2, \dots, k$. Next, let $v \in V_i$ for some $i \in \{1, 2, \dots, k\}$. Let $N(v) = \{v_1, v_2, \dots, v_m\}$ where $m \geq min\{r, deg(v)\}$. Since f is an r-dynamic k-coloring,

$$|f(N(v))| = |f(\{v_1, v_2, \dots, v_m\})| = |\{f(v_1), f(v_2), \dots, f(v_m)\}| \geq min\{r, deg(v)\}.$$

This implies that there exist $\left\{q_1, q_2, \dots, q_{min\{r,deg(v)\}}\right\} \subseteq \{1, 2, \dots, k\} \setminus \{i\}$ such that $v_j \in V_{q_j}$ for all $j = 1, 2, \dots, min\{r, deg(v)\}$. Hence, there exist

$$\left\{q_1, q_2, \dots, q_{min\{r,deg(v)\}}\right\} \subseteq \{1, 2, \dots, k\} \setminus \{i\}$$

such that $N(v) \cap V_{q_j} \neq \varnothing$ for all $j = 1, 2, \dots, min\{r, deg(v)\}$.

Conversely, assume that conditions (1) and (2) hold. Let $uv \in E$. Then by (1), $\{u, v\} \nsubseteq V_i$ for all $i = 1, 2, \dots, k$. Since $\{V_1, V_2, \dots, V_k\}$ is a partition of V, $u \in V_s$ and $v \in V_t$ for some $s, t \in \{1, 2, \dots, k\}$ with $s \neq t$, that is, $f(u) \neq f(v)$. This shows that f is proper. Next, let $v \in V$. Then $v \in V_i$ for some $i \in \{1, 2, \dots, k\}$. By (2), there exist $\left\{q_1, q_2, \dots, q_{min\{r,deg(v)\}}\right\} \subseteq \{1, 2, \dots, k\} \setminus \{i\}$ such that $N(v) \cap V_{q_j} \neq \varnothing$ for all $j = 1, 2, \dots, min\{r, deg(v)\}$. Thus, there exist $\left\{q_1, q_2, \dots, q_{min\{r,deg(v)\}}\right\} \subseteq \{1, 2, \dots, k\} \setminus \{i\}$ such that $v_j \in V_{q_j}$ for all $j = 1, 2, \dots, min\{r, deg(v)\}$. This implies that if $N(v) = \{v_1, v_2, \dots, v_m\}$ where $m \geq min\{r, deg(v)\}$, then $|f(N(v))| = |f(\{v_1, v_2, \dots, v_m\})| = |\{f(v_1), f(v_2), \dots, f(v_m)\}| \geq min\{r, deg(v)\}$. ☐

Lemma 4.1, says that the colors of the color classes may be interchanged and the new coloring will still be an r-dynamic k-coloring.

Corollary 4.2. *Let* $G = (V, E)$ *be a graph and* $(V_1, V_2, \dots, V_k)$ *be an* r*-dynamic* k*-coloring of* G*. If* $\{i_1, i_2, \dots, i_k\}$ *is a permutation of* $\{1, 2, \dots, k\}$*, then* $(V_{i_1}, V_{i_2}, \dots, V_{i_k})$ *is also an* r*-dynamic* k*-coloring of* G*.*

Next, we have Lemma 4.3. This Lemma says that if f is a proper coloring of G, then the restriction of f to any subgraphs of G is also a proper coloring.

Lemma 4.3. *Let* $G = (V, E)$ *be graph and* H *be a subgraph of* G*. If* f *is a proper coloring of* G*, then* $f|_{V(H)}$ *is a proper coloring of* H*.*

Proof. Let $f: V \to C$ be a proper coloring of G. If $f: V \to C$ be a proper coloring of G, then $f(u) \neq f(v)$ for all $uv \in E$. Let $ab \in E(H)$ and consider $f|_{V(H)}$. If $ab \in E(H)$, then $ab \in E$. Hence, $f(a) \neq f(b)$, that is, $f|_{V(H)}(a) \neq f|_{V(H)}(b)$. This shows that $f|_{V(H)}$ is a proper coloring of H. □

Next, we have Lemma 4.3. The Lemma says that if f is an r-dynamic k-coloring of G, then the restriction of f to the vertex set of any subgraph H of G is an r-dynamic k-coloring of H.

Lemma 4.4. *Let $G = (V, E)$ be graph and H be a subgraph of G. If f is an r-dynamic k-coloring of G, then $f|_{V(H)}$ is an r-dynamic k-coloring of H.*

Proof. Let $f: V \to C$ be an r-dynamic k-coloring. If $f: V \to C$ is an r-dynamic k-coloring of G, then f is a proper coloring, and for all vertices v in V, $|f(N(v))| \geq min\{r, deg_G(v)\}$. By Lemma 4.3 $f|_{V(H)}$ is a proper vertex coloring of H. Now, let $v \in V(H)$. Then

$$\begin{array}{rrcl} & |f(N_G(v))| - |f(N_H(v))| & \leq & |N_G(v)| - |N_H(v)| \\ \Rightarrow & |f(N_G(v))| - |f(N_H(v))| & \leq & \deg_G(v) - \deg_H(v) \\ \Rightarrow & |f(N_G(v))| - \deg_G(v) & \leq & |f(N_H(v))| - \deg_H(v) \\ \Rightarrow & 0 & \leq & |f(N_H(v))| - \deg_H(v) \\ \Rightarrow & \deg_H(v) & \leq & |f(N_H(v))| \\ \Rightarrow & min\{r, \deg_H(v)\} & \leq & |f(N_H(v))|. \end{array}$$

This shows that $f|_{V(H)}$ is an r-dynamic k-coloring of H. □

We call Corollary 4.5 the *monotonicity* property.

Corollary 4.5. *Let $G = (V, E)$ be graph and H be a subgraph of G. Then $\chi_r(G) \geq \chi_r(H)$.*

Proof. Let $\mathscr{F} = \{f : f$ is an r-dynamic k-coloring of $G\}$ and $\mathscr{G} = \{g : g$ is an r-dynamic k-coloring of $H\}$. Then by Lemma 4.4, $\mathscr{I} = \left\{f|_{V(H)} : f \in \mathscr{F}\right\}$ is a subset of $\mathscr{G}$. Next, let $\mathscr{A} = \{\left|f|_{V(H)}(V(H))\right| : f|_{V(H)} \in \mathscr{I}\}$ and $\mathscr{B} = \{|g(V(H))| : g \in \mathscr{G}\}$. Then, $\mathscr{A} \subseteq \mathscr{B}$. Thus, $\chi_r(G) \geq \chi_r(H)$ since $|f(V(H))| \leq |f(V)|$ for all $f \in \mathscr{F}$. □

Theorem 4.6. *Let G_1 and G_2 be connected graphs with $\chi_r(G_1) \geq 3$. Let G be the graph obtained by identifying a vertex in G_1 and a vertex in G_2, that is G is a vertex gluing of G_1 and G_2. If $r = 2$, then $\chi_r(G) = max\{\chi_r(G_1), \chi_r(G_2)\}$.*

Proof. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be a graphs. Let G be the graph obtained by identifying $u \in V_1$ and $v \in V_2$, that is G is a vertex gluing of G_1 and G_2. Let $f_1 = \left(V_1^{(1)}, V_2^{(1)}, \ldots, V_{k_1}^{(1)}\right)$ and $f_2 = \left(V_1^{(2)}, V_2^{(2)}, \ldots, V_{k_2}^{(2)}\right)$ be r-dynamic k_1-coloring of G_1 and r-dynamic k_2-coloring of G_2, respectively, such that $k_1 = \chi_r(G_1)$ and $k_2 = \chi_r(G_2)$. Consider the following cases: (Case 1. $deg_{G_1}(u) = 1$ or $deg_{G_2}(v) = 1$) Without loss of generality assume that $deg_{G_1}(u) = 1$. Let w be a neighborhood of u in G_1 and z be a neighborhood of v in G_2. Without loss of generality, assume that $u \in V_1^{(1)}$, $w \in V_2^{(1)}$, $v \in V_s^{(2)}$ and $z \in V_t^{(2)}$. Then by Corollary 4.2, $f_3 = \left(V_{i_1}^{(2)}, V_{i_2}^{(2)}, \ldots, V_{i_{k_2}}^{(2)}\right)$ with

$$i_m = \begin{cases} 1 & \text{if } m = s \\ s & \text{if } m = 1 \\ 3 & \text{if } m = t \\ t & \text{if } m = 3 \\ m & \text{otherwise} \end{cases}$$

is also an r-dynamic k-coloring of G_2. Now, let $f : V(G) \to \{1, 2, \ldots, max\{\chi_2(G_1), \chi_2(G_2)\}\}$ be given by

$$f(w) = \begin{cases} f_1(w) & \text{if } w \in V_1 \\ f_3(w) & \text{if } w \in V_2, \end{cases}$$

that is,

$$\begin{aligned} f &= \left(V_{j_1}, V_{j_2}, V_{j_3}, \ldots, V_{j_{k_1}}\right) \\ &= (V_1^{(1)} \cup V_{i_s}^{(2)}, V_2^{(1)} \cup V_{i_2}^{(2)}, V_3^{(1)} \cup V_{i_t}^{(2)}, \ldots, V_{k_2}^{(1)} \cup V_{i_{k_2}}^{(2)}, V_{k_2+1}^{(1)}, \\ &\quad \ldots, V_{k_1}^{(1)}) \end{aligned}$$

(without loss of generality we assume here that $k_1 \geq k_2$). Let $xy \in E(G)$. Then either $xy \in E_1$ or $xy \in E_2$. Without loss of generality we assume that $xy \in E_1$. Since $f_1 = \left(V_1^{(1)}, V_2^{(1)}, \ldots, V_{k_1}^{(1)}\right)$ is an r-dynamic k-coloring, by Lemma 4.1, $\{x, y\} \nsubseteq V_i^{(1)}$ for all $i = 1, 2, \ldots, k_1$. Since $xy \in E_1$, $x, y \in V_1$. Hence, there exists $s, t \in \{1, 2, \ldots, k_1\}$ with $s \neq t$ such that $x \in V_s$ and $y \in V_t$, that is, $x \in V_{j_s}$ and $y \in V_{j_t}$. Since is $\left\{V_{j_1}, V_{j_2}, V_{j_3}, \ldots, V_{j_{k_1}}\right\}$ a pairwise disjoint family, we have $\{x, y\} \nsubseteq V_{j_i}$ for all $i = 1, 2, \ldots, k_1$.

Next, let $y \in V(G)$. Consider the following subcases: (Subcase 1. either $y \in V_1 \backslash V_2$ or $y \in V_2 \backslash V_1$). Without loss of generality, assume that $y \in V_1 \backslash V_2$, say $y \in V_{j_r}$. Then since f_1 is an r-dynamic k-coloring, by Lemma 4.1 there exist $s, t \in \{1, 2, \ldots, k_1\} \setminus \{r\}$ such that $V_{j_s}^{(1)} \cap N_{G_1}(y) \neq \varnothing$ and $V_{j_t}^{(1)} \cap N_{G_1}(y) \neq \varnothing$. This implies that there exist $s, t \in \{1, 2, \ldots, k_1\} \setminus \{r\}$ such that $V_{j_s} \cap N_G(y) \neq \varnothing$ and $V_{j_t} \cap N_G(y) \neq \varnothing$.

(Subcase 2. $y \in V_1 \cap V_2$) If $y \in V_1 \cap V_2$, then $y = v = u$. Since $w \in V_2^{(1)} \subseteq V_{j_2}$ and $z \in V_3^{(2)}$, we have $V_{j_2} \cap N_G(y) \neq \varnothing$ and $V_{j_3} \cap N_G(y) \neq \varnothing$.

(Case 2. $deg_{G_1}(u) > 1$ or $deg_{G_2}(v) > 1$) Without loss of generality, assume that $u \in V_1^{(1)}$ and $v \in V_2^{(2)}$. Then by Corollary 4.2, $f_3 = \left(V_{i_1}^{(2)}, V_{i_2}^{(2)}, \ldots, V_{i_{k_2}}^{(2)}\right)$ with

$$i_m = \begin{cases} 1 & \text{if } m = 2 \\ 2 & \text{if } m = 1 \\ m & \text{otherwise} \end{cases}$$

is also an 2-dynamic k-coloring of G_2. Now, let $f : V(G) \rightarrow \{1, 2, \ldots, max\{\chi_2(G_1), \chi_2(G_2)\}\}$ be given by

$$f(w) = \begin{cases} f_1(w) & \text{if } w \in V_1 \\ f_3(w) & \text{if } w \in V_2, \end{cases}$$

that is,

$$\begin{aligned} f &= \left(V_{j_1}, V_{j_2}, V_{j_3}, \ldots, V_{j_{k_1}}\right) \\ &= \left(V_1^{(1)} \cup V_{i_1}^{(2)}, V_2^{(1)} \cup V_{i_2}^{(2)}, \ldots, V_{k_2}^{(1)} \cup V_{i_{k_2}}^{(2)}, V_{k_2+1}^{(1)}, \ldots, V_{k_1}^{(1)}\right) \end{aligned}$$

(without loss of generality we assume here that $k_1 \geq k_2$). Let $xy \in E(G)$. Then either $xy \in E_1$ or $xy \in E_2$. Without loss of generality we assume that $xy \in E_1$. Since $f_1 = \left(V_1^{(1)}, V_2^{(1)}, \ldots, V_{k_1}^{(1)}\right)$ is an r-dynamic k-coloring, by Lemma 4.1, $\{x, y\} \nsubseteq V_i^{(1)}$ for all $i = 1, 2, \ldots, k_1$. Since $xy \in E_1$, $x, y \in V_1$. Hence, there exists $s, t \in \{1, 2, \ldots, k_1\}$ with $s \neq t$ such that $x \in V_s$ and $y \in V_t$, that is, $x \in V_{j_s}$ and $y \in V_{j_t}$. Since is $\left\{V_{j_1}, V_{j_2}, V_{j_3}, \ldots, V_{j_{k_1}}\right\}$ a pairwise disjoint family, we have $\{x, y\} \nsubseteq V_{j_i}$ for all $i = 1, 2, \ldots, k_1$.

Next, let $y \in V(G)$. Without loss of generality, assume that $y \in V_1$, say $y \in V_r^{(1)}$, that is $y \in V_{j_r}$. Then since f_1 is an r-dynamic k-coloring, by Lemma 4.1 there exist $s, t \in \{1, 2, \ldots, k_1\} \setminus \{r\}$ such that $V_s^{(1)} \cap N_{G_1}(y) \neq \varnothing$ and $V_t^{(1)} \cap N_{G_1}(y) \neq \varnothing$. This implies that there exist $s, t \in \{1, 2, \ldots, k_1\} \setminus \{r\}$ such that $V_{j_s} \cap N_G(y) \neq \varnothing$ and $V_{j_t} \cap N_G(y) \neq \varnothing$.

Accordingly, by Lemma 4.1 f is a 2-dynamic k-coloring. Therefore, $\chi_2(G) \leq k_1 = max\{\chi_r(G_1), \chi_r(G_2)\}$.

Suppose that $\chi_2(G) < k_1 = max\{\chi_r(G_1), \chi_r(G_2)\}$. Note that G_1 is a subgraph of G. Thus by Corollary 4.5, $\chi_2(G_1) \leq \chi_2(G)$. This is a contradiction. □

Theorem 4.6 may be extended to the vertex gluing of a finite number of graphs.

Corollary 4.7. *Let $n \in \mathbb{N}$ and $G_1, G_2, \ldots, G_n$ be graphs. Let G be the vertex gluing of $G_1, G_2, \ldots, G_n$. If $r = 2$, then*

$$\chi_2(G) = max\{\chi_r(G_1), \chi_r(G_2), \ldots, \chi_r(G_n)\}$$

.

Proof. This follows from Theorem 4.6 by induction. □

5. r-Dynamic Chromatic Number of Forest

This section presents the r-dynamic chromatic number of forest. We note that a forest is a finite union of trees, and a tree is a finite vertex gluing of paths. The next remark gives the 2-dynamic chromatic number of trees. An observation in [8] states that $\chi_r(G) \geq min\{\Delta(G), r\} + 1$ where equality holds when G is a tree. The next remark verified this idea for $r = 2$.

Remark 5.1. Let T be a tree of order $n \geq 3$. If $r = 2$, then $\chi_r(T) = 3$.

Proof. Assume that T is the vertex gluing of n paths $P_1, P_2, \ldots, P_n$. Then by Corollary 4.7,

$$\chi_2(T) = max\{\chi_r(P_1), \chi_r(P_2), \ldots, \chi_r(P_n)\} = 3.$$

□

The next result gives the r-dynamic chromatic number of the union of two graphs.

Theorem 5.2. *Let G and H be graphs. Then*

$$\chi_r(G \cup H) = max\{\chi_r(G), \chi_r(H)\}.$$

Proof. Let

$$f_1 : V(G) \to \{1, 2, \ldots, \chi_r(G)\} \text{ and } f_1 : V(G) \to \{1, 2, \ldots, \chi_r(H)\}$$

be r-dynamic k-colorings of G and H, respectively. Define $f : V(G \cup H) \to \{1, 2, \ldots, max\{\chi_r(G), \chi_r(H)\}\}$ by

$$f(v) = \begin{cases} f_1(v) & \text{if } v \in V(G) \\ f_2(v) & \text{if } v \in V(H). \end{cases}$$

(Claim 1. f is a proper vertex coloring) Let $uv \in E(G \cup H)$. Then either $uv \in E(G)$ or $uv \in E(H)$. Without loss of generality, assume that $uv \in E(G)$. If $uv \in E(G)$, then since f_1 is proper, $f_1(u) \neq f_1(v)$, that is, $f(u) \neq f(v)$. This shows the claim.

(Claim 2. $|f(N(v))| \geq min\{r, deg_{G\cup H}(v)\}$ for all $v \in V(G \cup H)$) Let $v \in V(G \cup H)$. Then either $v \in V(G)$ or $v \in V(H)$. Without loss of generality, assume that $v \in V(G)$. If $v \in V(G)$, then since f_1 is an r-dynamic k-coloring, $|f_1(N(v))| \geq min\{r, deg_G(v)\}$, that is, $|f(N(v))| \geq min\{r, deg_{G\cup H}(v)\}$. This shows the claim.

By Claims 1 and 2, $\chi_r(G \cup H) \leq max\{\chi_r(G), \chi_r(H)\}$. And by Corollary 4.5, $\chi_r(G \cup H) = max\{\chi_r(G), \chi_r(H)\}$. □

Theorem 5.2 may be extended to finite union of graphs.

Corollary 5.3. *Let $n \in \mathbb{N}$ and $G_1, G_2, \ldots, G_n$ be graphs. Then*

$$\chi_r(G_1 \cup G_2 \cup \cdots \cup G_n) = max\{\chi_r(G_1), \chi_r(G_2), \ldots, \chi_r(G_n)\}.$$

We revisit an observation in [8] which states that $\chi_r(G) \geq min\{\Delta(G), r\} + 1$ where equality holds when G is a tree. Since a forest is a finite union of trees, the next result gives the r-dynamic chromatic number of forest.

Corollary 5.4. *Let F be a forest with a component of order $n \geq 3$. Then $\chi_r(F) = min\{\Delta(G), r\} + 1$.*

Proof. Assume that F is the union of n trees $T_1, T_2, \ldots, T_n$. Then by Corollary 5.3,

$$\chi_r(F) = max\{\chi_r(T_1), \chi_r(T_2), \ldots, \chi_r(T_n)\} = min\{\Delta(G), r\} + 1.$$

□

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