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ORBIT OF TUPLE OF OPERATORS TENDING TO INFINITY

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Abstract: In this paper we prove that there is a dense set of vectors in X whose orbits under the tuple $\mathcal{T} = (T_1, T_2, ..., T_n)$ of commutative bounded linear operators on a infinite dimensional (real, complex) Banach space X tend to infinity.

Key Words: tuple of operators, orbit, spectral radius, spectrum, point spectrum, approximate point spectrum, bounded below operator

1. Introduction

By an n-tuple of operators we mean a finite sequence of length n of commuting bounded linear operators on a Banach space X.

Throughout, X denotes a infinite dimensional Banach space and $B(X)$ denotes the Banach algebra of all bounded linear operators on X. The orbit of a points $x \in X$ under an operator $T \in B(X)$ is the sequence

$$\{T^n x \ : \ n = 0, 1, ...\}.$$

For $T \in B(X)$, with r(T), $\sigma(T)$, $\sigma_p(T)$ and $\sigma_{ap}(T)$ will denote the spectral radius, the spectrum, the point spectrum and the approximate point spectrum of T,respectively. Recall that $\sigma_p(T)$ is the set of all eigenvalues of T while $\sigma_{ap}(T)$ is the set of $\lambda \in \sigma(T)$ for there which there is a sequence of unit vectors

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$(x_n)_{n\geq 1}$ such that $\|Tx_n - \lambda x_n\| \to 0$, as $n \to \infty$ any such sequence is called a sequence of almost eigenvectors for λ.

We are interested in the operators $T \in B(X)$ for which there is $x \in X$ whose $Orb(T, x)$ tends strongly to infinity, i.e.

$$\|T^n x\| \longrightarrow \infty, \text{ as } n \longrightarrow \infty.$$

Obviously, if $\sigma_p(T)$ contains a point λ with $|\lambda| > 1$, then for every corresponding nonzero vector x in the eigenspace $Ker(T - \lambda)$, the orbit will tends strongly to infinity:

$$\|T^n x\| = |\lambda^n| \, \|x\| \longrightarrow \infty, \text{ as } n \longrightarrow \infty.$$

In general, $Ker(T - \lambda)$ is not dense in X(relative to the norm topology). In order to produce a dense set of vectors in X whose orbits under T tend strongly to infinity we have to look at the points in the approximate point spectrum which are not eigenvalues.

In [5] S. Mancevska gaves a complete proof of the following theorem (originally stated by B. Beauzamy [9, Theorem 2.A.5]): If $T \in B(X)$ and the circle $\{\lambda \in \mathbb{C}/\, |\lambda| = r(T)\}$ contains a point in $\sigma(T)$ which is not an eigenvalue for T, then for every positive sequence $(\alpha_n)_{n\geq 1}$ with $\sum_{n\geq 1} \alpha_n < +\infty$, in every open ball in X with radius strictly larger then $\sum_{n\geq 1} \alpha_n$, there is $x \in X$ satisfying $\|T^n x\| \geq \alpha_n \frac{r(T)^n}{2}$ for all $n \geq 1$. Note that, if $r(T) > 1$, then the space will contain a dense set of vectors $x \in X$ with orbits under T tending strongly to infinity.

If $r(T)$ is replaced with $|\lambda|$, for any $\lambda \in \sigma_{ap}(T) \backslash \sigma_p(T)$. Thus we have.

Theorem 1.2. (see [1], Corollary 3.2.) *Let X be an infinite dimensional reflexive Banach space and $T \in B(X)$ and $S \in B(X)$.*

If the sets $\sigma_{ap}(T)\backslash\sigma_p(T)$ and $\sigma_{ap}(S)\backslash\sigma_p(S)$ both have a non-empty intersection with the domain $\{\lambda \in \mathbb{C}/\, |\lambda| > 1\}$ then, there is a dense set of vectors $x \in X$ such that both the orbits $Orb(T; x)$ and Orb(S;x) tend strongly to infinity.

In [2] S. Mancevska and M. Orovrance considered some conditions under which, given a sequence of bounded linear operators $(T_i)_{i\geq 1}$ on an infinite-dimensional complex reflexive Banach space X, and they show that there is a dense set of vectors in X whose orbits under each T_i tend strongly to infinity.

Theorem 1.3. (see [2], Corollary 10) *Let X an infinite dimensional reflexive Banach space. If $(T_i)_{i\geq 1}$ is a sequence in $B(X)$ for wich there is $\beta > 0$ such that $r(T_i) > 1 + \beta$ for all $i \geq 1$ then, there is a dense set D in X such that $Orb(T_i; x)$ tend strongly to infinity for every $x \in D$ and $i \geq 1$.*

If $(T_i)_{i\geq 1}$ is a sequence in $B(X)$ satisfying the following, weaker condition then the one in Theorem 0.3.

$$\sigma_{ap}(T_i)\backslash\sigma_a(T_i) \cap \{\lambda \in \mathbb{C}/\, |\lambda| \succ 1\} \neq \emptyset, \quad \text{for all } i \geq 1. \qquad (*)$$

the space may still contain a dense set of vectors with orbits under each T_i, $i \geq 1$ tending strongly to infinity.

2. Main Results

Definition 1.1. Let $\mathcal{T} = (T_1, T_2, ..., T_n)$ be an n-tuple of operators acting on an infinite dimensional Banach space X.

Let
$$\mathcal{F} = \left\{ T_1^{k_1} T_2^{k_2} ... T_n^{k_n} : k_i \geq 0, i = 1, ..., n \right\}.$$
be the semi-group generated by $\mathcal{T}$. For $x \in X$, the orbit of x under the tuple $\mathcal{T}$ is the set
$$Orb(\mathcal{T}, x) = \{Sx : S \in \mathcal{F}\}.$$

Definition 1.2. The orbit of x under the tuple $\mathcal{T}$ tending to infinity if:
$$\left\| T_1^{k_1} T_2^{k_2} ... T_n^{k_n} x \right\| \to \infty$$
as $k_i \to \infty$ with $k_i \geq 0$, for all $i = 1, ..., n$.

In this paper are considered some conditions under which, The orbit of x under the tuple $\mathcal{T}$ tending to infinity. For simplicity we state and prove our results for a pair that is a tuple with n= 2, and the general case follows by a similar method.

Definition 1.3. An operator T is bounded from below if and only there exists a constant $C > 0$ such that:
$$\|Tx\| \geq C\,\|x\|, \text{ for all } x \in X.$$

Theorem 1.4. *Let X an infinite dimensional reflexive Banach space and $T = (T_1, T_2)$ be the pair of operators T_1 and T_2.*

Suppose that the flowing conditions hold true:

1) T_1 and T_2 are bounded from below.

2) The sets $\sigma_{ap}(T)\backslash\sigma_p(T)$ and $\sigma_{ap}(S)\backslash\sigma_p(S)$ both have a non-empty intersection with the domain $\{\lambda \in \mathbb{C}/\ |\lambda| > 1\}$.

Then there exists $x \in X$ such that the orbit of x under the pair T tends strongly to infinity.

Proof. Using condition 2 we may apply Theorem 1.2. Then, there exists a dense set of vectors $x \in X$ such that both the orbits $Orb(T_1; x)$ and $Orb(T_2; x)$ tend strongly to infinity.

Hence T_1 and T_2 are bounded from below or,

$$\begin{aligned}\left\|T_1^{k_1}T_2^{k_2}x\right\| &= \left\|T_1^{k_1}(T_2^{k_2}x)\right\| \\ &\geq C_1\left\|T_2^{k_2}x\right\| \to \infty, \text{ as } k_2 \to \infty,\end{aligned}$$

and

$$\begin{aligned}\left\|T_1^{k_1}T_2^{k_2}x\right\| &= \left\|T_2^{k_2}T_1^{k_1}x\right\| \\ &= \left\|T_2^{k_2}(T_1^{k_1}x)\right\| \\ &\geq C_2\left\|T_1^{k_1}x\right\| \to \infty, \text{ as } k_1 \to \infty.\end{aligned}$$

Then

$$\left\|T_1^{k_1}T_2^{k_2}x\right\| \to \infty \text{ as } k_1 \to \infty, \text{ and } k_1 \to \infty,$$

i.e. $Orb(T, x)$ tend strongly to infinity. □

Corollary 1.1. *Let X an infinite dimensional reflexive Banach space and $\mathcal{T} = (T_1, T_2, ..., T_n)$ be the n-tuple of operators in $B(X)$bounded below for all $i \geq 1$.*

If there is $x \in X$ such that the orbit of x under T_i for all $i \geq 1$ tend strongly to infinity then the orbit of x under the tuple $\mathcal{T}$ tend strongly to infinity.

Remark 1.1. The converse is also true, ie If there is $x \in X$ such that the orbit of x under the tuple $\mathcal{T}$ tend strongly to infinity then the orbit of x under T_i for all $i \geq 1$ tend strongly to infinity.

Proof. T_i are the commuting bounded linear operators then,

$$\left\|T_1^{k_1}T_2^{k_2}...T_n^{k_n}x\right\| \leq \prod_{i=1, i\boxminus j}^{n} \left\|T_i^{k_i}\right\| \left\|T_j^{k_j}x\right\|,$$

or

$$\left\|T_1^{k_1}T_2^{k_2}...T_n^{k_n}x\right\| \to \infty \text{ as } k_j \to \infty \text{ for all } j \geq 1.$$

Therefore, $Orb(T_j, x) \to \infty$ for all $j \geq 1$. □

Example 1.1. Let S be the unilateral forward shift on $\ell^2(\mathbb{N})$:

$$Se_n = e_{n+1}, \quad i \geq 1,$$

where $\{e_n : n \in \mathbb{N}\}$ is the standard orthonormal basis for $\ell^2(\mathbb{N})$.

Given a sequence of positive numbers $(a_i)_{i \geq 1}$ so that $a_i \succ 1$ for all $i \geq 1$ and $a_i \to 1$ as $i \to \infty$ and let

$$T_i = a_i S, \quad i = 1, ..., n.$$

T_i is unilateral injective forward weighted shift and hence (see [10], Theorem 6):

$$\sigma_p(T_i) = \emptyset \text{ and } \sigma_{ap}(T_i) = \{\lambda \in \mathbb{C} / |\lambda| = a_i\}.$$

Then the set $\sigma_{ap}(T_i) \backslash \sigma_p(T_i)$ have a non-empty intersection with the domain $\{\lambda \in \mathbb{C} / |\lambda| \succ 1\}$.

Obviously, $(T_i)_{i \geq 1}$ satisfies the weaker condition (*) and there exists a dense set of vectors in $\ell^2(\mathbb{N})$ with orbits in each T_i tending strongly to infinity.

Actually

$$\|T_i^n x\| = \|(a_i S)^n x\| = a_i^n \|x\| \to \infty, \; n \to \infty, \text{ for all } x \neq 0 \text{ and } i \geq 1.$$

Moreover, T_i is bounded from below. Indeed:

$$\|T_i x\| = \|a_i S x\| \geq \|Sx\| = \|x\|.$$

Therefore, by the use of Corollary 1.1, the orbit of x in the tuple $(T_1, T_2, ..., T_3)$ tends strongly to infinity.

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