International Journal of Pure and Applied Mathematics
Volume 111 No. 1 2016, 1-8
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: 10.12732/ijpam.v111i1.1

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THE SOLUTION OF PDE MODELING OF SEMI-INFINITE STRING BY ELZAKI TRANSFORM

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Abstract: Integral transform methods can be used for solving partial differential equations, and the semi-infinite string is the typical model of wave equation. In this article, we have checked the solution of PDE modeling of semi-infinite string by Elzaki transform method.

AMS Subject Classification: 35A22, 35L05
Key Words: semi-infinite string, wave equation, Elzaki transform

1. Introduction

The wave equation is one of the most important PDE, and it models the vibrating string and the vibrating membrane. Normally, it arises in acoustics, electromagnetics and fluid dynamics, and the problem of a vibrating string was researched by D'Alembert, Euler, Bernoulli, and Lagrange. To begin with, let us take a look into preceding researches of related area. Recently(2011) Elzaki proposed Elzaki transform defined by

$$T(u) = u\int_0^\infty e^{-t/u} f(t)dt$$

for $E[f(t)] = T(u)$, and it efficiently can be used for solving problems without resorting to anew frequency domain because it preserve scales and units properties[7]. Although existing integral transforms are almost similar, these

Received: January 20, 2016
Revised: May 8, 2016
Published: November 30, 2016

kinds of researches give some considerations to establish the theory of general integral transform. There are many related papers with Elzaki transform, and the efforts to find solutions of differential equations with variable coefficients, by using integral transforms, have been pursued[1-3, 5-12, 14-15]. In this article, we have checked the solution of semi-infinite string by Elzaki transform method. The ultimate goal is the establishment of the theories of general integral transform.

2. The expression of PDE modeling of semi-infinite string

It is well-known fact that integral transform methods can be used for solving partial differential equations if one of the independent variable ranges over the positive axis. In this article, we would like to check the solution of PDEs modeling of semi-infinite string by using Elzaki transform in terms of a typical example as appears in [13].

Example 1. (semi-infinite string.) Find the displacement $w(x, t)$ of an elastic string subject to the following conditions.

1) The string is initially at rest on the x-axis from $x = 0$ to ∞.

2) For $t > 0$ the left end of the string is moved in a given fashion, namely, according to a single sine wave $w(0, t) = f(t) = \sin t$ if $0 \leq t \leq 2\pi$, and zero otherwise.

3) Furthermore, $\lim w(x, t) = 0$ as $x \to \infty$ for $t \geq 0$.

Of course there is no infinite string, but our model describes a long bar or rope(of negligible weight) with in right end fixed far out on the x-axis[13].

Physical assumptions: 1. The mass of the string per unit length is constant. The string is perfectly elastic and does not offer any resistance to bending.

2. The tension caused by stretching the string before fastening it at the ends is so large that the action of the gravitational force on the string can be neglected.

3. The string performs small transverse motions in a vertically plane; that is, every particle of the string moves strictly vertically and so that the deflection and the slope at every point of the string always remain in absolute value[13].

The representation of PDE modeling of semi-infinite string: Newton's second law states that the force is mass times acceleration, and here, the force is the resultant of all the forces acting on the given body. The model of a vibrating elastic string consists of the one-dimensional wave equation

$$\frac{\partial^2 w}{\partial t^2} = c^2 \frac{\partial^2 w}{\partial x^2}, \tag{1}$$

for positive x and t, subject to the boundary conditions

$$w(0,t) = f(t), \qquad \lim_{x\to\infty} w(x,t) = 0 \tag{2}$$

with f as given above, and the initial conditions

$$w(x,0) = 0, \quad w_t(x,0) = 0. \tag{3}$$

3. The solution of semi-infinite string by Elzaki transform

To begin with, for convenience of the reader, we would like to present a review of the Elzaki transform.

Definition 2. Elzaki transform of the functions belonging to a class A, where $A = \{f(t) | \exists M, k_1, k_2 > 0 \textit{ such that } |f(t)| < Me^{|t|/k_j},\ \textit{if } t \in (-1)^j \times [0,\infty)\}$ where $f(t)$ is denoted by $E[f(t)] = T(v)$ and defined as

$$T(v) = v^2 \int_0^\infty f(vt)e^{-t}dt, \qquad k_1, k_2 > 0,$$

or equivalently,

$$T(v) = v \int_0^\infty f(t)e^{-t/v}dt, \quad v \in (k_1, k_2)[7].$$

The following results can be obtained from the definition and simple calculations.

$$\begin{array}{ll}
1) & E[f'(t)] = T(u)/u - uf(0) \\
2) & E[f''(t)] = T(u)/u^2 - f(0) - uf'(0) \\
3) & E[tf'(t)] = u^2 \dfrac{d}{du}[T(u)/u - uf(0)] \\
- & u[T(u)/u - uf(0)] \\
4) & E[t^2 f'(t)] = u^4 \dfrac{d^2}{du^2}[T(u)/u - uf(0)] \\
5) & E[tf''(t)] = u^2 \dfrac{d}{du}[T(u)/u^2 - f(0) - uf'(0)]
\end{array}$$

$$- \quad u[T(u)/u^2 - f(0) - uf'(0)]$$

$$6) \quad E[t^2 f''(t)] = u^4 \frac{d^2}{du^2}[T(u)/u^2 - f(0) - uf'(0)].$$

for $E(f(t)) = T(u)$[7, 12].

Lemma 3. (Time shifting theorem for Elzaki transform) If $E\{f(t)\} = T(v)$, then

$$E\{f(t-a)u(t-a)\} = e^{-a/v}T(v)$$

for $u(t)$ is the unit step function[12].

The Solution of example 1. To begin with, let us check the transform of the first and second partial derivatives of $f(x,t)$. From the definition of Elzaki transform,

$$v\int_0^\infty \frac{\partial f}{\partial t} exp(-\frac{t}{v})\, dt = T(x,v).$$

By using the integration by parts, we have

$$E[\frac{\partial f(x,t)}{\partial t}] = \int_0^\infty v\,\frac{\partial f}{\partial t} exp(-\frac{t}{v})\, dt = \lim_{a\to\infty}\int_0^a v\, exp(-\frac{t}{v})\,\frac{\partial f}{\partial t}\, dt$$

$$= \lim_{a\to\infty}\{[v\, exp(-\frac{t}{v})f(x,t)]_0^a - \int_0^a exp(-\frac{t}{v})f(x,t)\, dt\}$$

$$= \frac{1}{v}T(x,v) - vf(x,0). \tag{4}$$

Similarly, putting $\partial f/\partial t = g$ and by the equation (4), we have

$$E\left[\frac{\partial^2 f(x,t)}{\partial t^2}\right] = \frac{1}{v^2}T(x,v) - f(x,0) - v\frac{\partial f(x,0)}{\partial t}. \tag{5}$$

Let us write $W(x,v) = E[w(x,t)]$. Taking Elzaki transform with respect to t on both sides of the equation (1), we have

$$E\{\frac{\partial^2 w}{\partial t^2}\} = \frac{1}{v^2}W(x,v) - w(x,0) - v\, w_t(x,0) = c^2\frac{\partial^2 W}{\partial x^2}.$$

By the equation (3), we have

$$c^2\frac{\partial^2 W}{\partial x^2} - \frac{1}{v^2}W = 0.$$

Since this is ODE, we easily obtain the general solution

$$W(x,v) = A(v)e^{x/cv} + B(v)e^{-x/cv}.$$

From the equation (2) we have

$$W(0,v) = E[w(0,t)] = E[f(t)] = T(v)$$

and

$$\lim_{x\to\infty} W(x,v) = \lim_{x\to\infty} v\int_0^\infty w(x,t)e^{-t/v}dt.$$

Since the displacements ($w(x,t)$) are an increasing Borel measurable functions and

$$\lim_{x\to\infty} w(x,t) = 0,$$

by the monotone convergence theorem[4, 9], we can interchange $\int$ and lim. Implies,

$$\lim_{x\to\infty} W(x,v) = v\int_0^\infty 0\cdot e^{-t/v}\,dt = 0.$$

Since $c > 0$, we have $A(v) = 0$ and $W(0,v) = B(v) = T(v)$. Thus,

$$W(x,v) = T(v)\,e^{-x/cv}.$$

From the lemma 3, we obtain

$$w(x,t) = f(t-\frac{x}{c})u(t-\frac{x}{c})$$

where, u is the unit step function. Implies,

$$w(x,t) = \sin(t-\frac{x}{c})$$

if $ct > x > (t-2\pi)c$ and zero otherwise. This method can be applied to another PDEs. Let us consider some examples in order to clarify this point.

Example 4. Find the solution of the equation

$$x\frac{\partial w}{\partial x} + \frac{\partial w}{\partial t} = xt, \tag{6}$$

$$\begin{aligned} w(x,0) &= 0\ (x\geq 0),\\ w(0,t) &= 0\ (t\geq 0). \end{aligned}$$

Solution. Let us put $W(x,v) = E[w(x,t)]$ and take Elzaki transform on both sides of the equation (6). By the equation (4), we have

$$xW_x + \frac{W}{v} - vw(x,0) = xv^3$$

because of $E(t) = v^3$. Then $W(0,v) = E[w(0,t)] = E(0) = 0$ and we have

$$W_x + \frac{W}{xv} = v^3.$$

Note that this is a nonhomogeneous linear ODE. Thus writing

$$h = \int \frac{1}{xv}\,dx = \frac{1}{v}\,ln\,x,$$

we have

$$\begin{aligned} W(x,v) &= exp(-\frac{1}{v}\,ln\,x)(\int exp(\frac{1}{v}\,ln\,x)\cdot v^3\,dx + C) \\ &= x^{-1/v}(\int x^{1/v}\cdot v^3\,dx + C) \\ &= \frac{v^4}{1+v}\,x + Cx^{-1/v} \end{aligned}$$

for a constant C. Since $W(x,0) = E[w(x,0)] = E(0) = 0$, we have $C = 0$. As a scan of a table of Elzaki transform (see [8]), $E(t^n) = n!v^{n+2}$ for ! is the factorial and

$$E(v^2/1 - av) = e^{at}.$$

Thus

$$W(x,v) = \frac{v^4}{1+v}\,x = (v^3 - v^2 + \frac{v^2}{v+1})x,$$

and we thus obtain the solution

$$w(x,t) = x(t - 1 + e^{-t}).$$

To check, calculate

$$xw_x + w_t = x(t - 1 + e^{-t}) + x(1 - e^{-t}) = xt.$$

This satisfies the equation (6).

$f(t)$	$E[f(t)] = T(v)$
1	v^2
t	v^3
t^n	$n!\ v^{n+2}$
e^{at}	$\frac{v^2}{1-av}$
$sinat$	$\frac{av^3}{1+a^2v^2}$
$cosat$	$\frac{v^2}{1+a^2v^2}$

Table 1: Elzaki transform of some Functions

Example 5. Find the solution of the equation

$$\frac{\partial^2 w}{\partial x^2} = 100\frac{\partial^2 w}{\partial t^2} + 100\frac{\partial w}{\partial t} + 25w, \tag{7}$$

$$\begin{aligned} w(x,0) &= 0 \ (x \geq 0), \\ w_t(x,0) &= 0 \ (t \geq 0), \\ w(0,t) &= sin\, t \ (t \geq 0). \end{aligned}$$

Solution. Let us put $W(x,v) = E[w(x,t)]$ and take Elzaki transform on (7). By the equations (4) and (5), the equation (7) becomes

$$\frac{\partial^2 W}{\partial x^2} = 100\,[\ \frac{1}{v^2}W - w(x,0) - v\frac{\partial w(x,0)}{\partial t}]$$

$$+100\,[\ \frac{1}{v}W - vw(x,0)] + 25W.$$

Organizing the equality, we have

$$\frac{\partial^2 W}{\partial x^2} = 100\,\frac{W}{v^2} + 100\,\frac{W}{v} + 25W.$$

Since this equation is ODE, we can easily obtain

$$W(x,v) = C_1(v)\,exp(\frac{5v+10}{v}x) + C_2(v)\,exp(-\frac{5v+10}{v}x).$$

$W(x,0) = E[w(x,0)] = E(0) = 0$ and so, $C_1(v) = 0$. Similarly

$$W(0,v) = E[w(0,t)] = E(sin\, t) = \frac{v^3}{1+v^2} = C_2(v).$$

Implies,

$$W(x,v) = \frac{v^3}{1+v^2}\, exp(-\frac{5v+10}{v}x)$$
$$= \frac{v^3}{1+v^2} e^{-5x} e^{-10x/v}.$$

Taking the inverse Elzaki transform and from the lemma 3, we have

$$w(x,t) = e^{-5x}\, sin(t-10x)u(t-10x)$$

for u is the unit step function.

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