International Journal of Pure and Applied Mathematics
Volume 111 No. 1 2016, 141-147
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: 10.12732/ijpam.v111i1.13
ijpam.eu

ON THE WEAK CONVERGENCE OF A MAXIMUM LIKELIHOOD ESTIMATOR

Sofiane Grira
Department of Mathematics and Applied Sciences
College of Arts and Sciences
Abu Dhabi University
Al Ain Campus, P.O. Box 1790, Abu Dhabi, UNITED ARAB EMIRATES

Abstract: In this paper we study the behavior of a maximum likelihood estimator $\hat{\theta}$. Feigin (1985) had shown that if M_{t} is a square integrable martingale then $\frac{M_{\mathrm{t}}}{\sqrt{< M_{\mathrm{t}} >}}$ converges in distribution to a normal . We will prove a similar result for the maximum likelihood estimator. We will show that we could write $e^{\theta \mathrm{t}}(\hat{\theta}_{\mathrm{t}}-\theta)$ under the form $e^{\theta \mathrm{t}}\frac{M_{\mathrm{t}}}{< M_{\mathrm{t}} >}$ which converges weakly to a Cauchy distribution.

AMS Subject Classification: 60F05, 60G44
Key Words: stochastic differential equation, martingale, maximum likelihood estimator

1. Preliminaries and Notations

Let Ω be the set of continuous functions $\omega : I\!R^+ \to I\!R$. And let's define a filtration on Ω by

$$\mathcal{F}_t = \cap_{s>t}\sigma(\omega(u) : u \leq s).$$

We define $\mathcal{F}$ as the smallest σ-algebra containing $\mathcal{F}_t$. $(\Omega, \mathcal{F}, \mathcal{F}_t)$ is called the canonical space. Let $(\Omega', \mathcal{F}', \mathcal{F}'_t, P')$ be the basic probability space. Let Θ be

Received: October 26, 2016
Revised: November 11, 2016
Published: December 6, 2016

a subset of $I\!R^n$ with a non-empty interior. Suppose that for all $\theta \in \Theta$ the stochastic differential equation:

$$dY_t = \beta(\theta; t, Y_t)dt + \gamma(t, Y_t)dW_t, \quad Y_0 = x_0 \tag{1}$$

has a solution Y^θ. x_0 is a d-dimensional vector and

$$\gamma : I\!R^+ \times I\!R^d \to I\!R^d \times I\!R^m$$

$$\beta(\theta, ., .) : I\!R^+ \times I\!R^d \to I\!R^d$$

are Borel functions.

The solution Y^θ induces a probability measure P_θ on $(\Omega, \mathcal{F})$.

Let $c(t, x) = \gamma(t, x)\gamma(t, x)^T$.

Definition 1.1. *A strong solution of the stochastic differential equation (1) is a process $Y = \{Y_t; 0 \le t < \infty\}$ continuous with the following properties:*

(i) Y is adapted to the filtration $\mathcal{F}_t$,

(ii) $P(X_0 = \xi) = 1$,

(iii)$P(\int_0^t (|\beta(\theta; s, Y_s)| + \gamma^2(s, Y_s))ds < \infty) = 1$ holds for every $1 \le i \le d$,

(iv) (1) holds almost surely.

Theorem 1.2. *Suppose that for every $n \ge 1$ there exists a constant $K_n > 0$ such that for every $t \ge 0$, $\|x\| \le n$ and $\|y\| \le n$:*

$$\|\beta(t, x) - \beta(t, y)\| + \|\gamma(t, x) - \gamma(t, y)\| \le K_n(\|x\| - \|y\|).$$

Then strong uniqueness holds for equation (1) .

Proof. See [7].

We define for each $\theta \in \Theta$ a d-dimensional process $y(\theta)$ by:

$$y_t(\theta) = \beta(\theta, t, Y_t) - \beta(\theta_0, t, Y_t)$$

where $\theta_0 \in \Theta$ is fixed, and another process α_t defined by

$$\alpha_t(\theta) = c_t^{-1} y_t(\theta)$$

where $c_t = c(t, Y_t)$. Finally let's set

$$A_t(\theta) = \int_0^t \alpha_s(\theta)^T c_s \alpha_s(\theta) ds.$$

Now we can express a Theorem which will give us the expression of the likelihood function

$$L_t(\theta) = \frac{dP_\theta}{dP_{\theta_0}}\bigg|_{\mathcal{F}_t} = \frac{dP_\theta^t}{dP_{\theta_0}^t},$$

where $dP_\theta^t = P_\theta|_{\mathcal{F}_t}$.

Theorem 1.3. *Suppose that*

$$P_\theta^t(A_t(\theta) < \infty) = P_{\theta_0}^t(A_t(\theta) < \infty) = 1$$

Then

$$P_\theta^t \sim P_{\theta^{(0)}}^t$$

and

$$\frac{dP_\theta^t}{dP_{\theta^{(0)}}^t} = \exp\{\int_0^t y_s(\theta)^T c_s^{-1} dY_s^c(\theta^{(0)}) - \frac{1}{2}\int_0^t y_s(\theta)^T c_s^{-1} y_s(\theta) ds\}$$

where

$$Y_s^c(\theta^{(0)}) = Y_t - x_0 - \int_0^t b(\theta; s, Y_s) ds.$$

Proof. See [9].

Theorem 1.4. *Let X_n be a submartingale. Suppose it exists $p > 1$ such that $\sup_n E[|X_n|^p] < \infty$, then it exists an integrable random variable X_∞ such that*

$$X_n \xrightarrow[n\to\infty]{} X_\infty \ \ a.s.$$

and

$$X_n \xrightarrow[n\to\infty]{L^1} X_\infty.$$

Proof. See [8].

2. Main Result

Theorem 2.1. *Let's suppose that the space $(\Omega, \mathcal{F}, P)$ is the classical case of $\Omega = \mathcal{C}([0,1] : I\!R)$on which we know that we can build an arbitrary number of independent Brownian movements.*

Let's consider the set of one-dimensional Gaussian diffusions solutions of the differential stochastic equation :

$$dX_t = \theta X_t dt + dW_t, \quad X_0 = 0, \tag{2}$$

where $\theta > 0$ and W is a standard, one-dimensional Brownian motion . Then it exists an integrable random variable H_∞ and a normal distribution Z independent of H_∞ such that $e^{\theta t}(\hat{\theta}_t - \theta)$ converges in distribution to Z/H_∞ which is distributed according to a Cauchy law.

Proof. We have

$$d(e^{-\theta s}X_s) = e^{-\theta s}(dX_s - \theta X_s ds) = e^{-\theta s}dW_s$$

by integrating between 0 and t we will have

$$e^{-\theta t}X_t = \int_0^t e^{-\theta s}dW_s \tag{3}$$

therefore the solution of the equation (2) is

$$X_t = \int_0^t e^{\theta(t-s)}dW_s.$$

According to Theorem 1.2 this solution is unique. This unique solution induces a probability measure P_θ (obviously unique) on $(\Omega, \mathcal{F})$ that is P_θ is the image of P' by the application $\omega' \in \Omega' \to X(\omega') \in \Omega$.

Since X_t is a continuous process (because W is continuous) therefore

$$A_t = \int_0^t X_s^2 ds < \infty \quad \forall t > 0.$$

Then by Theorem 1.3 by taking $\theta^{(0)} = 0$ the likelihood function is

$$L_t(\theta) = \frac{dP_\theta^t}{dP_{\theta^{(0)}}^t} = e^{(N_t\theta - \frac{1}{2}\theta^2 I_t)},$$

where

$$N_t = \int_0^t X_s dX_s$$

and

$$I_t = \int_0^t X_s^2 ds.$$

Let's calculate the maximum likelihood estimator $\hat{\theta}$.

$$\frac{\partial}{\partial\theta}\log(L_t(\theta)) = N_t - \theta I_t$$

$\Longrightarrow$

$$\hat{\theta}_t = I_t^{-1} N_t.$$

On the other hand, let's set

$$H_t = e^{-\theta t} X_t = \int_0^t e^{-\theta s} dW_s$$

according to (3). But

$$E[H_t^2] = \int_0^t e^{-2\theta s} ds = \frac{1 - e^{-2\theta t}}{2\theta} \leq \frac{1}{2\theta} < \infty$$

which imply according to Theorem 1.4 that it exists H_∞ such that

$$H_t = e^{-\theta t} X_t \underset{t \to \infty}{\longrightarrow} H_\infty \quad p.s. \tag{4}$$

Moreover H_∞ is distributed according to a normal distribution $N(0, \frac{1}{2\theta})$. Let's apply Itô's formula for X_t with $f(x) = \dfrac{x^2}{2}$, we shall have

$$\frac{X_t^2}{2} = N_t + \frac{t}{2}$$

which imply

$$N_t = \frac{1}{2}(e^{2\theta t} H_t^2 - t).$$

Therefore

$$\begin{aligned} e^{\theta t}(\hat{\theta}_t - \theta) = e^{\theta t}(\frac{N_t}{I_t} - \theta) &= e^{\theta t} \left(\frac{\frac{1}{2} e^{2\theta t} H_t^2 - \frac{t}{2}}{I_t} - \theta \right) \\ &= \frac{e^{\theta t}}{2 I_t} (e^{2\theta t} H_t^2 - t - 2\theta I_t). \end{aligned}$$

But

$$I_t = \int_0^t e^{2\theta s} H_s^2 ds \sim \frac{e^{2\theta t}}{2\theta} H_t^2 \sim \frac{e^{2\theta t}}{2\theta} H_\infty^2,$$

which imply

$$e^{\theta t}(\hat{\theta}_t - \theta) \sim \frac{\theta e^{-\theta t}}{H_t^2} \left(e^{2\theta t} H_t^2 - t - 2\theta \int_0^t e^{2\theta s} H_s^2 ds \right).$$

On the other hand

$$\begin{aligned}\int_0^t e^{2\theta s} H_s^2 ds &= \int_0^t e^{2\theta s}(H_s - H_t + H_t)^2 ds \\ &= \int_0^t e^{2\theta s}(H_s - H_t)^2 ds + H_t^2 \int_0^t e^{2\theta s} ds \\ &\quad +2H_t \int_0^t e^{2\theta s}(H_s - H_t) ds.\end{aligned}$$

But

$$\int_0^t e^{2\theta s} ds = \frac{e^{2\theta t} - 1}{2\theta} \sim e^{2\theta t}/(2\theta)$$

and

$$E\left\{\int_0^t e^{2\theta s}(H_s - H_t)^2 ds\right\} \sim \frac{t}{2\theta}.$$

Hence

$$e^{\theta t}(\hat{\theta}_t - \theta) \sim \frac{4\theta^2 e^{-\theta t}}{H_t} \int_0^t e^{2\theta s}(H_t - H_s) ds$$

as $H_t - H_s$ is distributed according to a normal distribution, then

$$4\theta^2 e^{-\theta t} \int_0^t e^{2\theta s}(H_t - H_s) ds$$

converges in distribution to Z, distributed according to a normal distribution with zero mean and variance $2\theta^2$, because

$$\begin{aligned}\sigma_t^2 &= \mathrm{Var}\left(4\theta^2 e^{-\theta t} \int_0^t e^{2\theta s}(H_t - H_s) ds\right) \\ &= 16\theta^4 e^{-2\theta t} \int_0^t \int_0^t e^{2\theta s} e^{2\theta s'} \int_{\max(s,s')} e^{-2\theta u} du ds ds' \\ &= 8\theta^3 e^{-2\theta t} \int_0^t \int_0^t e^{2\theta(s+s')} \left(e^{-2\theta \max(s,s')} - e^{-2\theta t}\right) ds ds' \\ &= 16\theta^3 e^{-2\theta t} \int_0^t \int_0^s e^{2\theta s'} \left(1 - e^{-2\theta(t-s)}\right) ds' ds \\ &= 8\theta^2 \int_0^t \left(1 - e^{-2\theta s}\right)\left(e^{-2\theta s} - e^{-2\theta t}\right) ds \\ &\to 2\theta.\end{aligned}$$

Finally,

$$\begin{aligned}\operatorname{Cov}\left(H_t, 4\theta^2 e^{-\theta t}\int_0^t e^{2\theta s}(H_t - H_s)ds\right) &= 4\theta^2 e^{-\theta t}\int_0^t e^{2\theta s}\int_s^t e^{-2\theta u}duds \\ &= 2\theta e^{-\theta t}\int_0^t \left(1 - e^{-2\theta s}\right) ds \\ &\to 0.\end{aligned}$$

As H_t converges almost surely to H_∞ and Z is independent of H_∞, then $e^{\theta t}(\hat{\theta}_t - \theta)$ converges in distribution to Z/H_∞ which is distributed according to Cauchy distribution with density function

$$f(x) = \frac{1}{\pi}\frac{2\theta}{1 + 4\theta^2 x^2}.$$

References

[1] B.M. Brown, Martingale central limit theorems, *Ann. Math. Statist*, **42** (1971), 59-66, **doi:** 10.1214/aoms/1177693494.

[2] D.J. Daley, D. Vere-Jones, *An introduction to the theory of point process,* Springer-Verlag, (1988), **doi:** 10.1137/1032025.

[3] P.D Feigin, Stable convergence of semimartingales, *Stoch. Proc. App*, **19** No 1 (1985), 125-134, **doi:** 10.1016/0304-4149(85)90044-4.

[4] P. Hall, C.C. Heyde, *Martingale limit theory and its application,* Academic Press, Inc., Harcourt Brace Jovanovich, Publishers, New York-London (1980). MR 624435

[5] N. Ikeda, S. Watanabe, *Stochastic differential equations and diffusion processes,* North-Holland Publishing Company, (1981), **doi:** 10.1002/bimj.4710280425.

[6] J. Jacod, A.N. Shiryaev, *Limit theorems for stochastics process,* Springer-Verlag, (1987),**doi:** 10.1007/978-3-662-05265-5.

[7] I.Karatzas, E.S. Shreve, *Brownian motion and Stochastic calculus,* Springer-Verlag, (1988), **doi:** 10.1007/978-1-4684-0302-2.

[8] A.N. Shiryayev, *Probability,* Springer-Verlag, (1984), **doi:** 10.1007/978-1-4899-0018-0.

[9] M. Sorensen, Likelihood methods for diffusions with jumps, *Prob. Pure. Applied*, **6** (1991), 67-105.