International Journal of Pure and Applied Mathematics
Volume 111 No. 1 2016, 9-15
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: 10.12732/ijpam.v111i1.2

ijpam.eu

COMPLEMENTARY TREE PAIRED DOMINATION VERTEX CRITICAL GRAPHS

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Abstract: A dominating set $S \subseteq V$ is a paired dominating set if the induced sub graph $\langle S \rangle$ has a perfect matching. The minimum cardinality of a paired dominating set in G is called the paired domination number of G and is denoted by $\gamma_{pr}(G)$. The graph G is paired domination vertex critical if for every vertex v of G that is not adjacent to a vertex of degree one, $\gamma_{pr}(G - v) < \gamma_{pr}(G)$. If G is γ_{pr}-vertex critical and $\gamma_{pr}(G) = k$, then we say that G is $k - \gamma_{pr}$-vertex critical. In this paper we introduce the concept complementary tree paired domination vertex critical graph and also we present some upper and lower bounds of it. Furthermore, we construct the complementary tree paired domination vertex critical graph.

Key Words: graph, paired domination, complementary tree, vertex critical graph

1. Introduction

Let $G(p, q)$ be a simple undirected graph with p vertices and q edges. The set

Received: August 30, 3015
Revised: November 23, 2016
Published: November 30, 2016

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of vertices is denoted by $V(G)$; the set of edges by $E(G)$. Order of G is denoted by $|G|$ as a symbol for the cardinality of $V(G)$. The degree, neighborhood and closed neighborhood of a vertex v in the graph G are denoted by $d(v)$, $N(v)$ and $N[v] = N(v) \cup \{v\}$ respectively. For a subset S of $V(G)$, $N(S)$ denotes the set of all vertices adjacent to some vertex of S in G and $N[S] = N(S) \cup S$. Let $\langle S \rangle$ and $G - S$ denote the sub graph of G induced by S and $V(G) - S$ respectively. The minimum degree and maximum degree of the graph G are denoted by $\delta(G)$ and $\Delta(G)$ respectively. A vertex of degree one is called a leaf or end vertex and its neighbor is a support vertex. The set of support vertices in G is denoted by $S(G)$. Two vertices at maximum distance apart in G are called diametrical vertices of G.

We generally follow graph terminology and notation from [8]. The concept of complement of a graph was defined by Bondy and Murty [1]. Many of the authors introduced the concept related with complements. In particular, Complementary perfect dominating set was introduced by Paulraj Joseph. J and Mahadevan. G [7]. The concept Complementary nil domination number of a graph was introduced by T. Tamizhchelvam and Robin Chelladurai [6]. A paired dominating set $S \subseteq V(G)$ is said to be a complementary tree paired dominating set if the induced sub graph $V - S$ is a tree. The minimum cardinality of a complementary tree paired dominating set (CTPD) of G is called the complementary tree paired domination number of G and is denoted by $\gamma_{ctpd}(G)$ and the corresponding set is γ_{ctpd} set of G. This concept was introduced by A. Meenakshi and J. Baskar Babujee [4]

A set $S \subseteq V$ is a dominating set if for every vertex v in $V - S$, there exists a vertex u in S such that v is adjacent to u. The minimum cardinality of a dominating set in G is called the domination number of G and is denoted by $\gamma(G)$. A matching M in a graph is a set of independent edges in G. The number of edges in a maximum matching of G is called the matching number of G which denote by $\alpha'(G)$. A vertex of G incident with an edge of the matching M is said to be matched by M or M-matched. A dominating set $S \subseteq V$ is a paired dominating set if the induced sub graph $\langle S \rangle$ has a perfect matching. The minimum cardinality of a paired dominating set in G is called the paired domination number of G and is denoted by $\gamma_p(G)$. This concept was introduced by Haynes and Slater [8]. Two vertices joined by an edge of M are said to be paired and are called partners in S.

The graph G is a paired domination vertex critical if for every vertex v of G that is not adjacent to a vertex of degree one, $\gamma_{pr}(G) > \gamma_{pr}(G - v)$. If G is γ_{pr}-vertex critical and $\gamma_{pr}(G) = k$, then we say that G is k-γ_{pr}-vertex critical. This concept was first studied by Edwards [3]. Likewise in this paper we have introduced the concept complementary tree paired domination vertex critical graph and it is denoted by γ_{ctpd}-vertex critical. If G is γ_{ctpd}-vertex critical and $\gamma_{ctpd}(G) = k$, then we say that G is k-γ_{ctpd}-vertex critical. We present the characterization of complementary tree paired domination (CTPD) vertex critical graph and also we construct the CTPD vertex critical graph from two smaller ones.

2. Complementary Tree Paired Domination Vertex Critical Graphs

Definition 1. The graph G is said to be a complementary tree paired domination vertex critical if for every vertex v of G that is not a cut vertex, $\gamma_{ctpd}(G - v) < \gamma_{ctpd}(G)$. If G is γ_{ctpd}-vertex critical and $\gamma_{ctpd}(G) = k$ then we say that G is $k - \gamma_{ctpd}$-vertex critical.

For example

(i) K_5 is 4-γ_{ctpd}-vertex critical, Since $\gamma_{ctpd}(G) = 4$ and $\gamma_{ctpd}(G - v) = 2$ for every non cut vertex $v \in V(G)$. Therefore $\gamma_{ctpd}(G) > \gamma_{ctpd}(G - v)$. Hence K_5 is 4-γ_{ctpd}-vertex critical.

(ii) C_7 is 6-γ_{ctpd}-vertex critical, Since $\gamma_{ctpd}(G) = 6$ and $\gamma_{ctpd}(G - v) = 4$ for every non cut vertex $v \in V(G)$. Therefore $\gamma_{ctpd}(G) > \gamma_{ctpd}(G - v)$. Hence C_7 is 6-γ_{ctpd}-vertex critical.

We made some observations

Observation 2. *(i) The removal of a vertex can decrease the CTPD number by at most two.*

(ii) If G is complete graph of order $n \equiv 1 \pmod 2$ and $N[u_i] = N[v_j]$ for every i and j then G is γ_{ctpd}-vertex critical.

In this section we present some characterization of CTPD vertex critical graph.

Theorem 3. *The graph G is γ_{ctpd}-vertex critical if and only if for every non cut vertex $v \in V(G)$ there exists a set $D \subseteq V$, $|D| \leq \gamma_{ctpd}(G) - 2$ such that D is precisely complementary tree paired dominates $G - v$.*

Proof. Suppose G is γ_{ctpd}-vertex critical. Then we have $\gamma_{ctpd}(G - v) < \gamma_{ctpd}(G) - 2$ for every non cut vertex $v \in V(G)$. Therefore for every non cut vertex $v \in V(G)$ there exists a set D such that $|D| \leq \gamma_{ctpd}(G) - 2$, D precisely complementary tree paired dominates $G - v$.

Conversely suppose that for every non cut vertex $v \in V(G)$ there exists a set D such that $|D| \leq \gamma_{ctpd}(G) - 2$, D precisely complementary tree paired dominates $G - v$. Therefore $\gamma_{ctpd}(G - v) \leq |D| < \gamma_{ctpd}(G)$. This implies that $\gamma_{ctpd}(G - v) < \gamma_{ctpd}(G)$. Hence G is γ_{ctpd}-vertex critical. □

Theorem 4. *If G is a γ_{ctpd}-vertex critical then $\gamma_{ctpd}(G - v) = \gamma_{ctpd}(G) - 2$ for every non cut vertex $v \in V(G)$.*

Proof. Let G be a γ_{ctpd} critical graph. Then by theorem 3, for every non cut vertex $v \in V(G)$ there exists a set D with $|D| \leq \gamma_{ctpd}(G) - 2$ such that D precisely complementary tree paired dominates $G - v$. Hence $\gamma_{ctpd}(G - v) + 2 \geq \gamma_{ctpd}(G)$. The deletion of a vertex decreases the complementary tree paired domination number by at most 2, which implies that $\gamma_{ctpd}(G - v) \leq \gamma_{ctpd}(G) - 2$. Hence $\gamma_{ctpd}(G - v) = \gamma_{ctpd}(G) - 2$ for every non cut vertex $v \in V(G)$. □

Theorem 5. *If G is a γ_{ctpd}-vertex critical then for every non cut vertex $v \in V(G)$ there exists a γ_{ctpd} set $D \subseteq V(G)$ such that $v \in D$*

Proof. Let G be a γ_{ctpd}-vertex critical graph. By theorem 3, for every non cut vertex $v \in V(G)$ there exists set $D' \subseteq V(G)$ with $|D'| = \gamma_{ctpd}(G) - 2$ such that D' precisely complementary tree paired dominates $G - v$. Then the set $D = D' \cup \{v, v'\}$ is a CTPD set of G that contains v. □

Theorem 6. *If a connected graph G has any two vertices u and v such that $N(u) \subseteq N(v)$ then G is not γ_{ctpd}-vertex critical.*

Proof. If v is a cut vertex then $\gamma_{ctpd}(G)$ does not exists. Hence G is not γ_{ctpd} vertex critical. If v is not a cut vertex then any γ_{ctpd} set of $G - v$ contains a vertex in $N(u)$ and which is a CTPD set of G. Hence G is not γ_{ctpd} vertex critical. □

3. Construction of Complementary Tree Paired Domination Vertex Critical Graphs

We present the similar construction from Brigham et al [2] that makes it possible to build γ_{ctpd} vertex critical graph. Suppose F and H are nonempty graphs. Let u and w be non cut vertices of F and H respectively. Then $(F.H)(u, w : v)$ the coalescence of F and H via u and w.

Theorem 7. *Let F and H be any two connected graphs with $\delta(F) \geq 2$ and $\delta(H) \geq 2$, u and w be any two non cut vertices of F and H respectively and let $G = (F \circ H)(u, w : v)$. Then $\gamma_{ctpd}(F) + \gamma_{ctpd}(H) - 2 \leq \gamma_{ctpd}(G) \leq \gamma_{ctpd}(F) + \gamma_{ctpd}(H)$.*

Proof. Let D_F and D_H be γ_{ctpd}-sets of F and H respectively. If $u \notin D_F$ and $w \notin D_H$ then $D = D_F \cup D_H$ is a CTPD set of G with cardinality $\gamma_{ctpd}(F) + \gamma_{ctpd}(H)$. If $u \notin D_F$ and $w \in D_H$ or $u \in D_F$ and $w \notin D_H$ then $D = (D_F - \{u\}) \cup (D_H - \{w\}) \cup \{v\}$ is a CTPD set of G with cardinality $\gamma_{ctpd}(F) + \gamma_{ctpd}(H)$. If $u \in D_F$ and $w \in D_H$, let $D^v = (D_F - \{u\}) \cup (D_H - \{w\})$. Let M^v be a maximum matching of $G[D^v]$. Then there exists exactly two vertices $u' \in D^v \cap N_F(u)$ and $w' \in D^v \cap N_H(w)$ such that u' and w' are not saturated by M^v. Define $D = D^v \cap \{u'', w''\}$. Since M^v does not have perfect matching there exists two vertices say u'' and w'' such that $u'' \notin D^v$ and $w'' \notin D^v$ but $u'' \in N_F(u')$ and $w'' \in N_H(w')$. Since $\langle V(F) - D_F \rangle$ is a tree and $\langle V(H) - D_H \rangle$ is a tree, $\langle V - D \rangle$ is a tree. Hence D is a CTPD set of G with $|D| \leq \gamma_{ctpd}(F) + \gamma_{ctpd}(H)$. Hence the upper bound is attained.

Let D be a γ_{ctpd}-set of G and let $D_F = D \cap V(F)$ and $D_H = D \cap V(H)$, If $v \notin D$ then v is complementary tree paired by D_F or D_H. If v is complementary paired dominated by both of D_F and D_H, then D_F and D_H are CTPD set of F and H respectively. Hence $\gamma_{ctpd}(G) = |D| \geq \gamma_{ctpd}(F) + \gamma_{ctpd}(H)$. Suppose v is complementary tree paired dominated uniquely by D_F, then D_F is a CTPD set of F and $D_H \cup \{w, w'\}$ is a CTPD set of H, where w' is the partner of w. So $|D| + 2 \geq \gamma_{ctpd}(F) + \gamma_{ctpd}(H)$, hence $\gamma_{ctpd}(G) + 2 \geq \gamma_{ctpd}(F) + \gamma_{ctpd}(H)$. If $v \in D$, then $D_F = D \cap V(F) \cup \{u\}$ is a CTPD set of F. If $N_H(w) \subseteq D$ in H, then $D_H = D \cap V(H)$ is a CTPD set of H. Hence $|D| = |D_F| + |D_H| \geq \gamma_{ctpd}(F) + \gamma_{ctpd}(H)$. If there is a vertex $w' \in N_H(w)$ such that $w' \notin D$, then $D_H = D \cap V(H) \cup \{w, w'\}$ is a CTPD set of H. Hence $\gamma_{ctpd}(G) + 2 \geq \gamma_{ctpd}(F) + \gamma_{ctpd}(H)$, this is the lower bound of G. □

Theorem 8. *[5] Let F and H be a $j - \gamma_{pr}$ vertex critical and $k - \gamma_{pr}$ vertex critical respectively, with minimum degrees at least two, and let G be a graph formed by identifying a vertex of F with a vertex of H. If $\gamma_{pr}(G) = j + k - 2$, then G is γ_{pr}-vertex critical.*

Theorem 9. *[9] Let F and H be connected γ_{pr}-vertex critical graphs and let $G = F \circ H$. Then G is a γ_{pr}-vertex critical with $\gamma_{pr}(G) = \gamma_{pr}(F) + \gamma_{pr}(H) - 2$ and $diam(G) = diam(F) + diam(H)$.*

Theorem 10. *Let F be a i-γ_{ctpd}-vertex critical and H be a j-γ_{ctpd}-vertex critical with $\delta \geq 2$ and let G be a graph with $\delta(G) \geq 2$ formed by identifying a* *vertex of F with a vertex of H. If $\gamma_{ctpd}(G) = i + j - 2$ then G is γ_{ctpd}-vertex critical.*

Proof. Let $u \in V(G)$ be any non cut vertex and label the identified vertex v. Without loss of generality, $u \in V(F)$. Since F is i-γ_{ctpd}-vertex critical, $\gamma_{ctpd}(F - u) = i - 2$. If $u \neq v$, then every $\gamma_{ctpd}(F - u)$-set can be extended to a CTPD set of $G - u$ by adding to it $\gamma_{ctpd}(H)$. Hence $\gamma_{ctpd}(G - u) \leq i - 2 + j = \gamma_{ctpd}(G) - 2 < \gamma_{ctpd}(G)$. If $u = v$ then v is a cut vertex, so $\gamma_{ctpd}(G)$ does not exist. □

Theorem 11. *[9] Let F and H be two graphs with no isolated vertices and let $G = (F \circ H)(u, w : v)$. If $u \notin S(F)$ and $w \notin S(H)$, then G is γ_{pr}-vertex critical if and only if both F and H are γ_{pr}-vertex critical. Furthermore, if G is γ_{pr} critical, then $\gamma_{pr}(G) = \gamma_{pr}(F) + \gamma_{pr}(H) - 2$.*

Theorem 12. *Let F and H be two connected graphs with $\delta \geq 2$ and let $G = (F.H)(u, w : v)$. If $u \in V(F)$ and $w \in V(H)$ are not cut vertices then G is γ_{ctpd}-vertex critical if and only if both F and H are γ_{ctpd}-vertex critical.*

Proof. Assume that G is a γ_{ctpd}-vertex critical. To prove that F and H are γ_{ctpd}-vertex critical. Suppose that F is not γ_{ctpd}-vertex critical, then there exists a non cut vertex $x \in V(F)$ such that $\gamma_{ctpd}(F - x) \geq \gamma_{ctpd}(F)$. Let D_x be a γ_{ctpd}-set of $G - x$. Then

$|D_x| = \gamma_{ctpd}(G) - 2 = \gamma_{ctpd}(F) + \gamma_{ctpd}(H) - 2$. Let $D_F = D_x \cap V(F)$ and $D_H = D_x \cap V(H)$. If $x = u$ then $\gamma_{ctpdf}(G - x) = \gamma_{ctpd}(F - u) + \gamma_{ctpd}(H - w) \geq \gamma_{ctpd}(F) + \gamma_{ctpd}(H) - 2$ this is a contradiction. If $x \neq u$ and $v \notin D_x$ then D_H is a CTPD set of $H - w$. Hence $|D_H| \geq \gamma_{ctpd}(H) - 2$. If v is paired dominated by D_F then D_F is a CTPD set of $F - x$. So, $|D_F| \geq \gamma_{ctpd}(F - x) \geq \gamma_{ctpd}(F)$. Since $|D_F| + |D_H| = \gamma_{ctpd}(F) + \gamma_{ctpd}(H) - 2$, $|D_H| \leq \gamma_{ctpd}(H) - 2$ which is a contradiction. Hence v is paired dominated by D_H and D_H is a CTPD set of H. So $|D_H| \geq \gamma_{ctpd}(H)$ and $|D_F| = |D_x| - |D_H| \leq \gamma_{ctpd}(F) - 2$. Hence $D_F \cup \{u, u'\}$ is a CTPD set of $F - x$ with cardinality at most $\gamma_{ctpd}(F) - 2$ which is a contradiction to $\gamma_{ctpd}(F - x) \geq \gamma_{ctpd}(F)$. If $x \neq u$ and $v \in D_x$ then $\gamma_{ctpd}(G - x) = \gamma_{ctpd}(F - u) + \gamma_{ctpd}(H - w)\gamma_{ctpd}(F) + \gamma_{ctpd}(H) - 2$, which is a contradiction. If v is paired dominated by D_F, then D_F is a CTPD set of $F - x$. This is impossible since $\gamma_{ctpd}(F - x) \geq \gamma_{ctpd}(F)$. Thus v is not paired dominated by D_F. Let u' be a neighbor of u in F. Then $D_F \cup \{u, u'\}$ is a CTPD set of $F - x$ with cardinality at most $\gamma_{ctpd}(F) - 2$ which is a contradiction to $\gamma_{ctpd}(F - x) \geq \gamma_{ctpd}(F)$. Hence F is γ_{ctpd}-vertex critical. Similar way we can prove that H is also γ_{ctpd}-vertex critical.

Suppose that F and H are γ_{ctpd}-vertex critical. We prove that $G = (F.H)(u, w : v)$ is γ_{ctpd}-vertex critical. Let x be any non cut vertex of $V(G)$. If $x \neq v$, say $x \in V(F) - \{u\}$ and x is also a non cut vertex of F. Let D_F and D_H be γ_{ctpd}-sets of $F - x$ and $H - w$ respectively. Then $D_F \cup D_H$ is a CTPD set of $G - x$. Hence $\gamma_{ctpd}(G - x) \leq |D_F| + |D_H| \leq \gamma_{ctpd}(F) + \gamma_{ctpd}(H) - 2 = \gamma_{ctpd}(G) - 2$. If $x = v$ then v is a cut vertex, $\gamma_{ctpd}(G)$ does not exist. □

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