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TERNARY PARTIAL SEMIRINGS

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Abstract: In this paper, the notion of ternary partial semiring is introduced and several examples of ternary partial semirings are given. Also, congruence relations and quotients of ternary partial semirings are studied.

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1. Introduction

Partially defined infinitary operations occur in the contexts ranging from integration theory to programming language semantics. Many authors studied various types of semirings. Σ-structures were studied by Higgs in 1980. Arbib and Manes [3], [11] introduced partial monoids and sum ordered partial monoids in 1980. Partial semirings were studied by Streenstrup [12] in 1985. A partial semiring is a quadruple $(R, \Sigma, \cdot, 1)$, where (R, Σ) is a partial monoid, $(R, \cdot, 1)$ is an ordinary monoid with multiplicative binary operation '$\cdot$' and unit 1, and the additive and multiplicative structures obey the both distributive laws.

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The begining of ternary algebra was started before 1932. In 1932, D. H. Lehmer [8] investigated certain ternary algebraic systems called triplexes which turned out to be ternary groups. In 1971, Lister [9] introduced the notion of ternary rings and studied some of their properties. In 2003, Dutta and Kar [5] introduced the notion of ternary semiring which is a generalization of ternary ring.

In this paper, the notion of ternary partial semiring is introduced and the concept of quotient structure on ternary partial semiring is studied. A necessary and sufficient condition for S/θ to be ternary partial semiring is obtained, where S is a ternary partial semiring and θ is a congruence relation on S.

2. Preliminaries

Let M be a nonempty set and let I be a set. An *I-indexed family* in M is a function $x : I \to M$, denoted by $(x_i : i \in I)$ where $x_i = ix$ for each i in I. Two families $(x_i : i \in I)$ and $(y_j : j \in J)$ are *isomorphic* if there is a bijection $\sigma : I \to J$ with $y_{i\sigma} = x_i$ for each i in I. A *subfamily* of $(x_i : i \in I)$ is a family $(x_k : k \in K)$ such that $K \subseteq I$. The *empty family* is indexed by $\emptyset$. A family $(x_i : i \in I)$ in M *infinite (finite)* if the cardinality of the index set I is infinite (finite).

Now we consider an infinitary operation Σ which takes (finite or infinite) families in M to elements of M, but which may not be defined for all families in M. As $\Sigma(x_i : i \in I)$ need not be defined for an arbitrary family $(x_i : i \in I)$ in M, Σ is partially defined. We say that a family $(x_i : i \in I)$ in M is *summable* if $\Sigma(x_i : i \in I)$ is defined and is in M. We use the notations $\Sigma(x_i : i \in I)$, $\Sigma_{i\in I}x_i$ and $\Sigma_i x_i$ inter-changeably.

Definition 2.1. A partial monoid is a pair (M, Σ) where M is a nonempty set and Σ is a partial addition defined on some, but not necessarily all families $(x_i : i \in I)$ in M subject to the following axioms:

(1) Unary Sum Axiom. If $(x_i : i \in I)$ is a one element family in M and $I = \{j\}$, then $\Sigma(x_i : i \in I)$ is defined and equals x_j.

(2) Partition-Associativity Axiom. If $(x_i : i \in I)$ is a family in M and $(I_j : j \in J)$ is a partition of I, then $(x_i : i \in I)$ is summable if and only if $(x_i : i \in I_j)$ is summable for every j in J and $(\Sigma(x_i : i \in I_j) : j \in J)$ is summable and then $\Sigma(x_i : i \in I) = \Sigma(\Sigma(x_i : i \in I_j) : j \in J)$.

The following are consequences of the partition associativity axiom:

(i) Σ is an associative and a commutative operation.

(ii) Any two isomorphic families have the same sum.

(iii) Every sub-family of a summable family is itself summable.

(iv) There do not exist any nontrivial additive inverses.

Observation 2.2. *In a partial monoid* (M, Σ)*, the empty family is summable. Its sum, denoted by* 0*, is such that the sum of an arbitrary number of 0's is itself equal to* 0*. Furthermore '0' acts as an additive zero in* M*.*

Definition 2.3. The *support* of a family $(x_i : i \in I)$ in M is defined to be the subfamily $(x_i : i \in J)$ where $J = \{i \in I \mid x_i \neq 0\}$.

Observation 2.4. *In a partial monoid* (M, Σ)*, if* $\Sigma(x_i : i \in I)$ *is defined and equals* 0*, then* $x_i = 0$ *for all* i *in* I*.*

Example 2.5. Let D, E be sets and let the set of partial functions from D to E be denoted by $pfn(D, E)$. Then $(pfn(D, E), \Sigma)$ is a partial monoid if Σ is defined such that a family $(x_i : i \in I)$ is summable if and only if for i, j in I, and $j \neq i$, $dom(x_i) \cap dom(x_j) = \emptyset$, where $dom(f)$ is the domain of definition of the partial function f. If $(x_i : i \in I)$ is summable then for any d in D,

$$d(\Sigma_i x_i) = \begin{cases} dx_i, \ if \ d \in dom(x_i) \ for \ some \ (necessarily \ unique) \ i \in I; \\ undefined, \ otherwise. \end{cases}$$

Example 2.6. Let D, E be sets. A multifunction $x : D \to E$ maps each element in D to an arbitrary subset of E. Such multifunctions correspond bijectively to relations $R \subseteq D \times E$, where $(d, e) \in R$ if and only if $e \in dx$. The set of multifunctions from D to E, denoted by $Mfn(D, E)$, together with Σ defined such that for d in D, $d(\Sigma_i x_i) = \bigcup_i (dx_i)$, is a partial monoid in which every family is summable.

Definition 2.7. Let (M, Σ) and (M', Σ') be partial monoids. Then (M', Σ') is said to be a partial submonoid of (M, Σ) if:

(i) M' is a subset of M; and

(ii) $(x_i : i \in I)$ is a summable family in M' implies that $(x_i : i \in I)$ is a summable family in M and $\Sigma'_i x_i = \Sigma_i x_i$.

If (M, Σ) is any partial monoid, then any subset M' of M which is closed under the restriction of Σ is a partial sub-monoid of M.

Definition 2.8. Let (M, Σ) and (M', Σ') be partial monoids. Then a function $\theta : M \to M'$ is said to be an additive map of (M, Σ) into (M', Σ') if $(x_i : i \in I)$ is a summable family in M implies $(x_i\theta : i \in I)$ is a summable family in M' and $(\Sigma_i x_i)\theta = \Sigma'_i(x_i\theta)$.

Observation 2.9. *If M_1 and M_2 are partial monoids and if $\theta : M_1 \to M_2$ is an additive map, then $0\theta = 0$.*

Products 2.10. *Let $((M^i, \Sigma^i) : i \in I)$ be a family of partial monoids. Their product is the partial monoid $(\prod_{i\in I} M^i, \Sigma)$ together with the projection maps $pr_i : (\prod_{i\in I} M^i, \Sigma) \to (M^i, \Sigma^i)$, defined as follows:*

The set $\prod_{i\in I} M^i$ is the cartesian product of M^i's. Let $(x_j : j \in J)$ be a family in $\prod_{i\in I} M^i$. Then each $x_j = (x_j^i : i \in I)$, where $x_j^i \in M^i$.The family $(x_j : j \in J)$ is summable in $\prod_{i\in I} M^i$ if for each $i \in I$, $(x_j^i : j \in J)$ is summable in M^i, in which case $\Sigma(x_j : j \in J) = \Sigma((x_j^i : i \in I) : j \in J) = (\Sigma^i(x_j^i : j \in J) : i \in I)$.

Each of the projection maps $pr_i : (\prod_{i\in I} M^i, \Sigma) \to (M^i, \Sigma^i) : x \to x^i$ is an additive map.

Definition 2.11. Let M be a partial monoid, and let E be an equivalence relation on the elements of M. Then E is a partial monoid congruence on M if E is closed under the additive operation of the product partial monoid $M \times M$.

Definition 2.12. A partial semiring is a quadruple $(R, \Sigma, \cdot, 1)$ where (R, Σ) is a partial monoid, $(R, \cdot, 1)$ is an ordinary monoid with multiplicative binary operation '$\cdot$' and unit 1, and the additive and multiplicative structures obey the following distributive laws: If $\Sigma(x_i : i \in I)$ is defined in R, then for all y in R, $\Sigma_i y \cdot x_i$ and $\Sigma_i x_i \cdot y$ are defined and $y \cdot (\Sigma_i x_i) = \Sigma_i y \cdot x_i$; $(\Sigma_i x_i) \cdot y = \Sigma_i x_i \cdot y$.

Definition 2.13. A nonempty set S together with a binary operation, called addition and a ternary multiplication, denoted by juxtaposition, is said to be a ternary semiring if S is an additive commutative semigroup satisfying the following conditions:

(i) $(abc)de = a(bcd)e = ab(cde)$,

(ii) $(a + b)cd = acd + bcd$,

(iii) $a(b + c)d = abd + acd$,

(iv) $ab(c + d) = abc + abd$; for all $a, b, c, d, e \in S$.

3. Ternary Partial Semirings

We introduce the notion of Ternary partial semiring as follows:

Definition 3.1. A nonempty set S together with a partial addition Σ, defined on some (not necessarily all) families $(a_i : i \in I)$ in S and a ternary multiplication, denoted by juxtaposition, is said to be a ternary partial semiring if it satisfies the following axioms:

(i) (S, Σ) is a partial monoid,

(ii) $(abc)de = a(bcd)e = ab(cde)$,

(iii) a family $(a_i : i \in I)$ is summable in S implies $(a_i ab : i \in I)$ is summable in S and $(\Sigma_{i \in I} a_i)ab = \Sigma_{i \in I}(a_i ab)$,

(iv) a family $(a_i : i \in I)$ is summable in S implies $(aa_i b : i \in I)$ is summable in S and $a(\Sigma_{i \in I} a_i)b = \Sigma_{i \in I}(aa_i b)$,

(v) a family $(a_i : i \in I)$ is summable in S implies $(aba_i : i \in I)$ is summable in S and $ab(\Sigma_{i \in I} a_i) = \Sigma_{i \in I}(aba_i)$;

(vi) $0ab = a0b = ab0 = 0$ where $0 = \Sigma(a_i : i \in \emptyset)$ (empty family), acts as an additive zero for binary sums in S, for all $a, b, c, d, e \in S$.

Now we give some examples of ternary partial semirings.

Example 3.2. Let $S = \mathbb{Q}^- \bigcup \{0\}$ be the set of all nonpositive rational numbers. Then with Σ, defined as the usual addition over families of finite support and with usual ternary multiplication, S forms a ternary partial semiring.

Example 3.3. Let $S' = \mathbb{Z}^- \bigcup \{0\}$ be the set of all nonpositive integers. Then S' together with Σ, defined as the usual addition over families of finite support and with usual ternary multiplication, is a ternary partial semiring.

Example 3.4. The set $\mathbb{N} \bigcup \{0\}$ for all nonnegative integers is a ternary partial semiring with Σ, defined as the usual addition over families of finite support and with usual ternary multiplication.

Example 3.5. Let $\mathbb{Q}^+$ be the set of all positive rational numbers and $S = \{q\sqrt{2} : q \in \mathbb{Q}^+ \bigcup \{0\}\}$. Then with Σ, defined as the usual addition over families of finite support and with usual ternary multiplication, S becomes a ternary partial semiring.

Example 3.6. Suppose $\mathbb{M}$ is the set of all $m \times m$ square matrices over nonpositive integers. Then $\mathbb{M}$ becomes a ternary partial semiring with normal addition over families of finite support and ternary multiplication.

Example 3.7. Let D be any set. Then $Pfn(D, D)$ together with the partial summation, defined as in example 2.5 and the ternary multiplication, defined as $d(fgh) = (((df)g)h)$, for all $d \in D$ and $f, g, h \in Pfn(D, D)$, is a ternary partial semiring.

Throughout this paper, S stands for a ternary partial semiring unless otherwise stated.

Definition 3.8. A partial submonoid A of S is called a ternary partial subsemiring if $a_1a_2a_3 \in A$ for all $a_1, a_2, a_3 \in A$.

Example 3.9. Consider the ternary partial semirings S and S', given in example 3.2 and 3.3. Now S' is a ternary partial subsemiring of S.

4. Homomorphisms and Congruence Relations of Ternary Partial Semirings

Definition 4.1. Let S and S' be two ternary partial semirings. A function $\alpha : S \to S'$ is said to be a homomorphism if it satisfies the following conditions:

(i) If $(a_i : i \in I)$ is a summable family in (S, Σ), then $(\alpha(a_i) : i \in I)$ is a summable family in (S', Σ'), and $\alpha(\Sigma_i a_i) = \Sigma'_i \alpha(a_i)$. (In this case, α is called an additive map of (S, Σ) into (S', Σ')).

(ii) $\alpha(abc) = \alpha(a)\alpha(b)\alpha(c)$ for all $a, b, c \in S$.

Note that the identity map on a ternary partial semiring is a homomorphism of ternary partial semirings. We also note that if α is a homomorhism of ternary partial semirings, then $\alpha(0) = 0$.

Definition 4.2. Let S and S' be two ternary partial semirings. A homomorphism $\alpha : S \to S'$ is said to be a:

(i) monomorphism if α is one-to-one.

(ii) epimorphism if α is onto.

(iii) isomorphism if α is both one-to-one and onto.

If S and S' are ternary partial semirings and if $\alpha : S \to S'$ is a homomorphism then $\alpha(S)$ need not be closed under the partial addition of S'. Hence, $\alpha(S)$ together with the partial addition of S' is not necessarily a ternary partial subsemiring of S'.

In the following example, we give two ternary partial semirings S and S' and a homomorphism $\alpha : S \to S'$ such that $\alpha(S)$ is not closed under the partial addition of S'.

Example 4.3. Let $S = \{0, 1\}$. For any family $(a_i : i \in I)$ of elements of S, define

$$\Sigma_i x_i = \begin{cases} 0, \ if \ a_i = 0 \ for \ all \ i \in I \\ 1, \ if \ a_j = 1, \ for \ some \ j \ \& \ a_i = 0 \ for \ all \ i \in I \ with \ i \neq j \\ undefined, \ otherwise. \end{cases}$$

For any $a, b, c \in S$, define

$$abc = \begin{cases} 1, \ if \ a \neq 0, \ b \neq 0 \ and \ c \neq 0 \\ 0, \ otherwise \end{cases}$$

Then with the partial summation and ternary multiplication, S becomes a ternary partial semiring.

Consider the ternary partial semiring $S' = \mathbb{Z}^- \bigcup \{0\}$, given in example 3.3.

Define $\alpha : S \to S'$ by $\alpha(0) = 0$ and $\alpha(1) = -1$. Now α is a partial ternary semiring homomorphism of S into S'. But $\alpha(1) + \alpha(1) = -1 + (-1) = -2 \notin \{0, -1\} = \alpha(S)$. Therefore, $\alpha(S)$ is not closed under the partial summation of S' and hence, $\alpha(S)$ not a ternary partial subsemiring.

Products 4.4. *: Let (S^i, Σ^i, f^i) be a family of ternary partial semirings indexed by I, where f^i denotes a ternary multiplication on S^i, $i \in I$. Let S be the Cartesian product of S^i, $i \in I$.*

That is, $S = X_{i \in I} S^i$. Define a partial summation Σ, and a ternary multiplication f on S as follows: Let $(a_j : j \in J)$ be a family of elements in S. Then each a_j is a tuple of of the form $a_j = (a^i_j : i \in I)$, where a^i_j is in S^i. If, for each $i \in I, (a^i_j : j \in J)$ is summable in S^i, then the family $(a_j : j \in J)$ is summable in S, and $\Sigma(a^*_j : j \in J) = \Sigma((a^i_j : i \in I) : j \in J) = (\Sigma^i(a^i_j : j \in J) : i \in I)$. For any three elements $a = (a^i : i \in I), b = (b^i : i \in I)$ and $c = (c^i : i \in I)$ in S, $abc = f(a, b, c) = (f^i(a^i, b^i, c^i) : i \in I) = (a^i b^i c^i : i \in I)$.

Then (S, Σ, f) is a ternary partial semiring. This is called the product ternary partial semiring, and is denoted by $\prod_{i \in I} S_i$. Thus $S = \prod_{i \in I} S_i$.

Let (S^i, Σ^i, f^i) be a family of ternary partial semirings indexed by I. Then for each $i \in I$, the mapping $\pi_i : \prod_{i \in I} S^i \to S^i$, defined by $\Pi_i(a) = a^i$ $(i^{th}$

component of a) is a ternary partial semiring homomorphism. This π_i is called the i^{th} projection map of $(\prod_{i\in I} S^i, \Sigma, f)$ into (S^i, Σ^i, f^i) for all $i \in I$.

Definition 4.5. Let S be a ternary partial semiring and let θ be a binary relation defined on S. Then θ is called a ternary partial semiring congruence relation on S. If it satisfies the following conditions:

(1) θ is an equivalence relation on S,

(2) θ is closed under the partial summation and the ternary operation of the product partial ternary semiring $S \times S$. That is,

(i) If $(a_i : i \in I)$ and $(b_i : i \in I)$ are summable families in S such that $a_i\theta b_i$ for all $i \in I$, then $(\Sigma_i a_i)\theta(\Sigma_i b_i)$ or $\Sigma_i(a_i, b_i) \in \theta$,

(ii) If $a\theta a'$, $b\theta b'$ and $c\theta c'$ where $a,\ b,\ c,\ a',\ b',\ c' \in S$, then $(abc)\theta(a'b'c')$.

Definition 4.6. Let S be a ternary partial semiring. Then the congruence relation θ on S, defined by $s\theta s'$ if and only if $s = s'$ for all $s, s' \in S$ is the trivial congruence relation and all other congruence relations are called nontrivial.

If $s\theta s'$ for all $s, s' \in S$, then θ is called improper and all other congruence relations are called proper.

Lemma 4.7. *Let S and S' be ternary partial semirings and let $\alpha : S \to S'$ be a homomorphism. Define $a\theta b \Leftrightarrow \alpha(a) = \alpha(b)$ for all $a, b \in S$. Then θ is a ternary partial semiring congruence relation on S.*

Proof. It is clear that θ is an equivalence relation on S. Let $(a_i : i \in I)$ and $(b_i : i \in I)$ be summable families in S such that $(a_i, b_i) \in \theta$ for all $i \in I$. Then $\Sigma_i a_i$ and $\Sigma_i b_i$ exists in S and $\alpha(a_i) = \alpha(b_i)$ for all $i \in I$ and so $\Sigma'_i\alpha(a_i) = \Sigma'_i\alpha(b_i)$ and that $\alpha(\Sigma_i a_i) = \alpha(\Sigma_i b_i)$ and implies that $(\Sigma_i a_i, \Sigma_i b_i) \in \theta$ and hence $\Sigma_i(a_i, b_i) \in \theta$. Suppose that $a\theta a'$, $b\theta b'$ and $c\theta c'$ where $a,\ b,\ c,\ a',\ b',\ c' \in S$. Then $\alpha(a) = \alpha(a')$, $\alpha(b) = \alpha(b')$ and $\alpha(c) = \alpha(c')$. So, $\alpha(abc) = \alpha(a)\alpha(b)\alpha(c) = \alpha(a')\alpha(b')\alpha(c') = \alpha(a'b'c')$. This implies that $(abc)\theta(a'b'c')$. Hence, θ is a ternary partial semiring congruence relation on S. ☐

Construction 4.8. *Let S be a ternary partial semiring and let θ be a ternary partial semiring congruence relation on S.*

For $s \in S$, let θs denote the equivalence class containing s with respect to θ. We write $S/\theta = \{\theta s \mid s \in S\}$. The natural addition $\widehat{\Sigma}$ and natural ternary multiplicative operation on S/θ are defined as follows: For any family

$(\theta a_i : i \in I)$ of elements in S/θ, define

$$\widehat{\Sigma}_i \theta a_i = \begin{cases} \theta(\Sigma_i a_i), \ if \ \Sigma_i a_i, \ exists \ in \ S \\ undefined, \ otherwise. \end{cases}$$

For any θa, θb, $\theta c \in S/\theta$, define $\theta a \theta b \theta c = \theta(abc)$. However, there is no guarantee that S/θ is a ternary partial semiring with respect to the above natural addition and the natural ternary multiplicative operation.

In the following example, we show that if S is a ternary partial semiring and θ is a congruence relation on S, then S/θ is not a ternary partial semiring with respect to the natural addition and the natural ternary multiplication.

Example 4.9. Let $\mathfrak{P}(D)$ be the power set of a set D. For any family $(A_i : i \in I)$ of elements in $\mathfrak{P}(D)$, define

$$\Sigma_i A_i = \begin{cases} \bigcup A_i \ (usual \ set \ union), \ if \ A_i \bigcap A_j = \phi \ \forall i \neq j \\ undefined, \ otherwise \end{cases}$$

For any three elements $A, B, C \in \mathfrak{P}(D)$, define $ABC = A \bigcap B \bigcap C$ (Usual set intersection). Then $\mathfrak{P}(D)$ together with the above partial summation and the ternary multiplication, forms a ternary partial semiring. Take $D = \{x, y\}$ and $\theta = \{(\emptyset, \emptyset), (\{x\}, \{x\}), (\{y\}, \{y\}), (D, D), \ (\{x\}, D), (D, \{x\}), (\emptyset, \{y\}), (\{y\}, \emptyset)\}$.

Then θ is a ternary partial semiring congruence relation on $\mathfrak{P}(D)$. Now $\mathfrak{P}(D)/\theta = \{\overline{\emptyset}, \overline{\{x\}}\}$ where $\overline{\emptyset} = \{\emptyset, \{y\}\} = \overline{\{y\}}, \ \overline{\{x\}} = \{\{x\}, D\} = \overline{D}$. In this case, $\{x\} + \{y\} = D$ is defined in $\mathfrak{P}(D)$ but $\overline{\{x\}} + \overline{\{y\}}$ is not well defined in $\mathfrak{P}(D)/\theta$, as $D + \{y\}$ is not defined in $\mathfrak{P}(D)$. Hence, $\mathfrak{P}(D)/\theta$ is not a ternary partial semiring.

The following is a necessary and sufficient condition for S/θ to be a ternary partial semiring where S is a ternary partial semiring and θ is a ternary partial semiring congruence relation on S.

If $(a_i : i \in I)$ is a summable family in S and $(b_i : i \in I)$ is any family in S with $a_i \theta b_i \forall i \in I$, then $(b_i : i \in I)$ is summable in S.

Theorem 4.10. *Let S be a ternary partial semiring and θ be a ternary partial semiring congruence relation on S. Suppose that $(a_i : i \in I)$ and $(b_i : i \in I)$ are any two families in S, $a_i \theta b_i \ \forall \ i \in I$ and $(a_i : i \in I)$ is summable in S implies $(b_i : i \in I)$ is summable in S. Then S/θ is a ternary partial semiring.*

Proof. First we show that if $(a_i : i \in I)$ is a summable family in S, then $(\theta a_i : i \in I)$ is a summable family in S/θ. Let $(a_i : i \in I)$ be a summable family in S. For $i \in I$, let $b_i \in \theta a_i$. Now $(b_i : i \in I)$ is a family of elements of S such that $a_i \theta b_i\ \forall\ i \in I$. Since $(a_i : i \in I)$ is summable in S and $a_i \theta b_i\ \forall\ i \in I$, by hypothesis, $(b_i : i \in I)$ is summable in S. Therefore $(\theta a_i : i \in I)$ is summable in S/θ, since $(b_i : i \in I)$ was arbitrary.

Now we show that S/θ is a ternary partial semiring with the natural partial summation $\widehat{\Sigma}$ and the natural ternary multiplication.

(i) **Unary sum axiom:** If $(\theta a_i : i \in I)$ is a one element family in S/θ and $I = \{j\}$, then $(a_i : i \in I)$ is a one element family in S and $I = \{j\}$ and hence $\Sigma(a_i : i \in I)$ exists in S and equals a_j since S is a partial monoid and implies that $\widehat{\Sigma}(\theta a_i : i \in I)$ exists in S/θ and equals θa_j.

(ii) **Partition associativity axiom:** Let $(\theta a_i : i \in I)$ be a summable family in S/θ and $(I_j : j \in J)$ be a partition of I. Then $(a_i : i \in I)$ is a summable family in S and $(I_j : j \in J)$ is a partition of I and hence each family $(a_i : i \in I_j)$ is summable in S for $j \in J$ and the family $((\Sigma a_i : i \in I_j) : j \in J)$ is summable in S, and also $(\Sigma a_i : i \in I) = \Sigma(\Sigma(a_i : i \in I_j) : j \in J)$.

This implies that each family $(\theta a_i : i \in I_j)$ is summable in S/θ for $j \in J$ and the family $(\widehat{\Sigma}(\theta a_i : i \in I_j) : j \in J)$ is summable in S/θ, and also $\widehat{\Sigma}(\theta a_i : i \in I) = \widehat{\Sigma}(\widehat{\Sigma}(\theta a_i : i \in I_j) : j \in J)$.

Therefore, S/θ is a partial monoid. For any $\theta a, \theta b, \theta c$ in S/θ, we have $\theta a \theta b \theta c = \theta(abc)$. For any θa, θb, θc, θd, θe in S/θ $(\theta a \theta b \theta c)\theta d \theta e = \theta(abc)\theta d \theta e = \theta((abc)de) = \theta(a(bcd)e) = \theta a \theta(bcd)\theta e = \theta a(\theta b \theta c \theta d)\theta e$, and $(\theta a \theta b \theta c)\theta d \theta e = \theta a \theta b \theta(cde) = \theta a \theta b(\theta c \theta d \theta e)$. Let $(\theta a_i : i \in I)$ be a summable family in S/θ and θa, $\theta b \in S/\theta$. Then $(a_i : i \in I)$ is a summable family in S and a, $b \in S$, and hence $(a_i ab : i \in I)$ is a summable family in S and $(\Sigma(a_i : i \in I))ab = \Sigma(a_i ab : i \in I)$. This implies that $(\theta(a_i ab) : i \in I)$ is a summable family in S/θ and $\theta((\Sigma(a_i : i \in I))ab) = \theta(\Sigma(a_i ab : i \in I))$ and that $(\theta(a_i ab) : i \in I)$ is a summable family in S/θ and $(\theta(\Sigma(a_i : i \in I))\theta a \theta b) = \widehat{\Sigma}(\theta(a_i ab) : i \in I)$ and that $(\theta a_i \theta a \theta b : i \in I)$ is a summable family in S/θ and $(\widehat{\Sigma}\theta a_i : i \in I)\theta a \theta b = \widehat{\Sigma}(\theta a_i \theta a \theta b : i \in I)$. Similarly, $(\theta a \theta a_i \theta b : i \in I)$ is a summable family in S/θ and $\theta a(\widehat{\Sigma}(\theta a_i : i \in I))\theta b = \widehat{\Sigma}(\theta a \theta a_i \theta b : i \in I)$ and $(\theta a \theta b \theta a_i : i \in I)$ is a summable family in S/θ and $\theta a \theta b(\widehat{\Sigma}(\theta a_i : i \in I)) = \widehat{\Sigma}(\theta a \theta b \theta a_i : i \in I)$. Also, $\theta 0 \theta a \theta b = \theta(0ab) = \theta 0$, $\theta a \theta 0 \theta b = \theta(a0b) = \theta 0$ and $\theta a \theta b \theta 0 = \theta(ab0) = \theta 0$. Therefore $\theta 0 \theta a \theta b = \theta a \theta 0 \theta b = \theta a \theta b \theta 0 = 0$ in S/θ. Hence, S/θ is a ternary partial semiring. □

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