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A NEW CUBICALLY CONVERGENT ITERATIVE METHOD FOR SOLVING NONLINEAR EQUATIONS

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Abstract: In this paper, we suggest and analyze a new third order two-step iterative method for solving nonlinear equations. To illustrate the efficiency and performance of this iterative method, we have presented comparison of newly established method with Newton's method, Abbasbandy's method, Homeier's method, Chun's method and Noor's method.

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1. Introduction

In recent, various techniques have been used, such as functional approximation, sampling, decomposition, geometric and Adomian approaches, to establish higher order iterative methods for solving nonlinear equations. These iterative methods consist of multi-step iterative methods. These methods are also known as predictor-corrector methods. Abbasbandy [1], Adomain [2] and Chun [4] have established some higher order one-step and two-step iterative methods by using adomain decomposition technique. They have used higher derivatives in their methods which is a serious drawback of these methods. To over come this drawback, Noor and Noor [6] established some multi-step iterative methods by applying a different decomposition technique. Initially, we do not put any restrictions on the original function f. In fixed point method, we rewrite $f(x) = 0$ as $x = g(x)$, where:

(i) there exists $[a, b]$ such that $g(x) \in [a, b]$ for all $x \in [a, b]$,

(ii) there exists $[a, b]$ such that $|g'(x)| \leq \lambda < 1$ for all $x \in [a, b]$.

In this paper, we shall establish some algorithms using functional equation and decomposition technique given in [7]. Order of a sequence is defined as follows;

Definition 1.1. Let the sequence $\{x_n\}$ converges to α. If there is a positive integer p and real number C such that

$$\lim_{n\to\infty} \left| \frac{x_{n+1} - \alpha}{(x_n - \alpha)^p} \right| = C,$$

then p is order of convergence.

Theorem 1.2. (see [3]) *Suppose that $\varphi \in C^p[a, b]$. If $\varphi^{(k)}(x) = 0$ for $k = 0, 1, 2, ..., m - 1$ and $\varphi^{(p)}(x) \neq 0$. Then the sequence $\{x_n\}$ is of order m.*

2. Iterative Methods

Consider the nonlinear equation

$$f(x) = 0; \quad x \in \mathbb{R}. \tag{2.1}$$

We assume that α is a simple root of (2.1) and γ is an initial guess sufficiently close to α. Eqution (2.1) can be rewritten as:

$$x = g\left(x\right), \tag{2.2}$$

$$x = g\left[\gamma + (x - \gamma)\right].$$

Using Taylors series on the right hand of above, we get

$$x = g(\gamma) + (x - \gamma)\, g'(\gamma) + G(x), \tag{2.3}$$

where

$$G(x) = g(x) - g(\gamma) - (x - \gamma)\, g'(\gamma). \tag{2.4}$$

From equation (2.3) we get

$$x = \frac{g(\gamma) - \gamma g'(\gamma)}{1 - g'(\gamma)} + \frac{G(x)}{1 - g'(\gamma)}.$$

It can be written in the form

$$x = c + N(x), \tag{2.5}$$

where

$$c = \frac{g(\gamma) - \gamma g'(\gamma)}{1 - g'(\gamma)} \tag{2.6}$$

and

$$N(x) = \frac{G(x)}{1 - g'(\gamma)}. \tag{2.7}$$

Now, we establish a sequence of higher-order iteration schemes by applying the decomposition technique. The main idea of this technique consists in looking for a solution of Equation (2.7) having the infinite series of the form:

$$x = \sum_{i=0}^{\infty} x_i. \tag{2.8}$$

The nonlinear operator N can be decomposed as

$$N\left(\sum_{i=0}^{\infty} x_i\right) = N(x_0) + \sum_{i=1}^{\infty} \left\{ N\left(\sum_{j=0}^{i} x_j\right) \right\}, \tag{2.9}$$

which is mainly due to Noor et al. [7].

Thus we have from Equation (2.5), (2.8) and (2.9),

$$\sum_{i=0}^{\infty} x_i = x_0 + N(x_0) + \sum_{i=1}^{\infty} \left\{ N\left(\sum_{j=0}^{i} x_j\right) \right\}.$$

Thus we get the iteration schemes as

$$\begin{aligned}
x_0 &= c,\\
x_1 &= N(x_0),\\
x_2 &= N(x_0+x_1),\\
&\vdots,\\
x_{n+1} &= N(x_0+x_1+\cdots+x_n), \quad n=0,1,2,....
\end{aligned}$$

When

$$\begin{aligned}
x &\approx x_0\\
&= c\\
&= \frac{g(\gamma)-\gamma g'(\gamma)}{1-g'(\gamma)},
\end{aligned}$$

from above, we formulate the following algorithm:

Algorithm 2.1. *For any initial value x_0, we approximate the solution x_{n+1}, by the iterative method:*

$$x_{n+1} = \frac{g(x_n)-x_n g'(x_n)}{1-g'(x_n)},$$

which is mainly due to Kang et al. [5] and has second order convergence.

From equation (2.4), we have

$$\begin{aligned}
G(x_0) &= g(x_0)-g(\gamma)-(x_0-\gamma)g'(\gamma)\\
&= g(x_0)-g(\gamma)-\left(\frac{g(\gamma)-\gamma g'(\gamma)}{1-g'(\gamma)}-\gamma\right)g'(\gamma)\\
&= g(x_0)-g(\gamma)-\left(\frac{g(\gamma)-\gamma}{1-g'(\gamma)}\right)g'(\gamma)\\
&= g(x_0)-\left(\frac{g(\gamma)-g(\gamma)g'(\gamma)+g(\gamma)g'(\gamma)-\gamma g'(\gamma)}{1-g'(\gamma)}\right)\\
&= g(x_0)-\frac{g(\gamma)-\gamma g'(\gamma)}{1-g'(\gamma)}\\
&= g(x_0)-x_0.
\end{aligned} \tag{2.10}$$

When

$$\begin{aligned} x &\approx x_0 + x_1 \\ &= c + N(x_0) \\ &= \frac{g(\gamma) - \gamma g'(\gamma)}{1 - g'(\gamma)} + \frac{G(x_0)}{1 - g'(\gamma)} \\ &= \frac{g(\gamma) - \gamma g'(\gamma)}{1 - g'(\gamma)} + \frac{g(x_0) - x_0}{1 - g'(\gamma)}, \end{aligned}$$

from above, we formulate the algorithm as follows:

$$x_{n+1} = \frac{g(x_n) - x_n g'(x_n)}{1 - g'(x_n)} + \frac{g\left(\frac{g(x_n) - x_n g'(x_n)}{1 - g'(x_n)}\right) - \frac{g(x_n) - x_n g'(x_n)}{1 - g'(x_n)}}{1 - g'(x_n)},$$

if we take

$$y_n = \frac{g(x_n) - x_n g'(x_n)}{1 - g'(x_n)},$$

then

$$\begin{aligned} x_{n+1} &= y_n + \frac{g(y_n) - y_n}{1 - g'(x_n)} \\ &= \frac{g(y_n) - y_n g'(x_n)}{1 - g'(x_n)}. \end{aligned}$$

Algorithm 2.2. *For any initial value x_0, we approximate the solution x_{n+1}, by the iterative method:*

Predictor step:

$$y_n = \frac{g(x_n) - x_n g'(x_n)}{1 - g'(x_n)}.$$

Corrector step:

$$x_{n+1} = \frac{g(y_n) - y_n g'(x_n)}{1 - g'(x_n)}.$$

3. Convergence Analysis

Now we discuss the convergence analysis of Algorithm 2.2.

Theorem 3.1. *Let $J \subseteq \mathbb{R}$ be an open interval and $f : J \to \mathbb{R}$ be a function and consider that the nonlinear equation $f(x) = 0$ (or $x = g(x)$) has simple root $\alpha \in J$, where $g(x) : J \to \mathbb{R}$ be sufficiently smooth in the neighbourhood of the root α. If x_0 is sufficiently close to α, then convergence order of Algorithm 2.2 is at least 3.*

Proof. Let α be simple zero of $f(x) = 0$ and $x = g(x)$ be its functional equation. Let errors at nth and $(n+1)$th iterations be e_n and e_{n+1}, respectively. Then using Taylor's expansion about α, we have

$$g(x_n) = \alpha + e_n g'(\alpha) + \frac{1}{2}e_n^2 g''(\alpha) + \frac{1}{6}e_n^3 g'''(\alpha) + O(e_n^4), \tag{3.1}$$

$$g'(x_n) = \alpha g'(\alpha) + e_n g''(\alpha) + \frac{1}{2}e_n^2 g'''(\alpha) + \frac{1}{6}e_n^3 g^{(4)}(\alpha) + O(e_n^4), \tag{3.2}$$

$$\begin{aligned} g(x_n) - x_n g'(x_n) = \alpha - \alpha g'(\alpha) - \frac{1}{2}(g''(\alpha) + g'''(\alpha))e_n^2 \\ - \frac{1}{6}(2g'''(\alpha) + \alpha g^{(4)}(\alpha))e_n^3 + O(e_n^4). \end{aligned} \tag{3.3}$$

By substituting values in y_n and after simplifying, we get

$$\begin{aligned} y_n &= \frac{g(x_n) - x_n g'(x_n)}{1 - g'(x_n)} \\ &= \alpha - \frac{g''(\alpha)}{2(1-g'(\alpha))}e_n^2 - \frac{2g'''(\alpha) - 2g'''(\alpha)g'(\alpha) + 3(g'''(\alpha))^2}{6(1-g'(\alpha))^2}e_n^3 \\ &\quad + O(e_n^4). \end{aligned} \tag{3.4}$$

$$\begin{aligned} g(y_n) = \alpha &- \frac{g'(\alpha)g''(\alpha)}{2(1-g'(\alpha))}e_n^2 \\ &- \frac{g'(\alpha)(2g'''(\alpha) - 2g'''(\alpha)g'(\alpha) + 3(g'''(\alpha))^2)}{6(1-g'(\alpha))^2}e_n^3 + O(e_n^4). \end{aligned} \tag{3.5}$$

Now

$$x_{n+1} = \frac{g(y_n) - y_n g'(x_n)}{1 - g'(x_n)}.$$

By substituting values in above and simplifying, we have

$$x_{n+1} = \alpha + \frac{1}{2}\frac{(g''(\alpha))^2}{(1-g'(\alpha))^2}e_n^3 + O(e_n^4).$$

Hence Algorithm 2.2 has third order convergence. □

4. Applications

Now we present some example to illustrate the efficiency and performance of newly devolved method namely, Algorithm 2.2. We make a comparison of Newton's method (NM), Abbasbandy's method (AM), Homeier's method (HM), Chun's method (CM), Noor's method (NR) and Algorithm 2.2 (NA) devolved in this paper (see Tables 1-7). We use $\epsilon = 10^{-15}$. The following criteria are used for computer programs:

(i) $|x_n - x_{n-1}| < \epsilon$,

(ii) $|f(x_n)| < \epsilon$.

The examples are same as in Chun [4] and Noor et al. [7].

$$
\begin{aligned}
&f(x) = \sin^2 x - x^2 + 1, \quad g(x) = \sin x + \frac{1}{x + \sin x},\\
&f(x) = x^2 - e^x - 3x + 2, \quad g(x) = \frac{e^x - 2}{x - 3},\\
&f(x) = \cos x - x, \quad g(x) = \cos x,\\
&f(x) = (x-1)^3 - 1, \quad g(x) = 1 + \sqrt{\frac{1}{x-1}},\\
&f(x) = x^3 - 10, \quad g(x) = \sqrt{\frac{10}{x}},\\
&f(x) = xe^{x^2} - \sin^2 x + 3\cos x + 5, \quad g(x) = e^{-x^2}(\sin^2 x + 3\cos x + 5),\\
&f(x) = e^{x^2+7x-30} - 1, \quad g(x) = \frac{1}{7}(30 - x^2).
\end{aligned}
$$

Table 1. Comparison of NM, AM, HM, CM, NR and NA
($f(x) = \sin^2 x - x^2 + 1$, $g(x) = \sin x + \frac{1}{x+\sin x}$, $x_0 = -1$)

Method	Number of iterations	x_n	$f(x_n)$
NM	7	1.404491648315341226350868177	-1.04e-50
AM	5	1.404491648315341226350868177	-5.81e-55
HM	4	1.404491648315341226350868178	-5.4e-62
CM	5	1.404491648315341226350868178	-2.0e-63
NR	5	1.404491648315341226350868176	1.2e-28
NA	3	1.404491648315341226350868176	6.08e-28

Table 2. Comparison of NM, AM, HM, CM, NR and NA
($f(x) = x^2 - e^x - 3x + 2$, $g(x) = \frac{e^x - 2}{x-3}$, $x_0 = 2$)

Method	Number of iterations	x_n	$f(x_n)$
NM	6	0.257530285439860760455367303	2.93e-55
AM	5	0.257530285439860760455367304	1.0e-63
HM	5	0.257530285439860760455367305	0
CM	4	0.257530285439860760455367304	1.0e-63
NR	5	0.257530285439860760455367306	-6.0e-24
NA	5	0.257530285439860760455367302	1.63e-28

Table 3. Comparison of NM, AM, HM, CM, NR and NA
($f(x) = \cos x - x$, $g(x) = \cos x$, $x_0 = 1.7$)

Method	Number of iterations	x_n	$f(x_n)$
NM	5	0.739085133215160641655372084	-2.03e-32
AM	4	0.739085133215160641655372085	-7.14e-47
HM	4	0.739085133215160641655372086	-5.02e-59
CM	4	0.739085133215160641655372087	0
NR	4	0.739085133215160641655372089	1.4e-15
NA	4	0.739085133215160641655312088	8.74e-73

Table 4. Comparison of NM, AM, HM, CM, NR and NA
($f(x) = (x-1)^3 - 1$, $g(x) = 1 + \sqrt{\frac{1}{x-1}}$, $x_0 = 3.5$)

Method	Number of iterations	x_n	$f(x_n)$
NM	8	2.000000000000000000000000023	2.06e-42
AM	5	2	0
HM	5	2	0
CM	5	2	0
NR	5	2.000000000000000000000000058	1.03e-23
NA	4	2.000000000000000000000000002	4.85e-45

Table 5. Comparison of NM, AM, HM, CM, NR and NA
($f(x) = x^3 - 10$, $g(x) = \sqrt{\frac{10}{x}}$, $x_0 = 1.5$)

Method	Number of iterations	x_n	$f(x_n)$
NM	7	2.154434690031883721759235664	2.06e-54
AM	5	2.154434690031883721759235667	-5.0e-63
HM	5	2.154434690031883721759235667	-5.0e-63
CM	5	2.154434690031883721759235667	-5.0e-63
NR	5	2.154434690031883721759235663	3.1e-42
NA	4	2.154434690031883721759235665	2.55e-70

Table 6. Comparison of NM, AM, HM, CM, NR and NA
$(f(x) = xe^{x^2} - \sin^2 x + 3\cos x + 5,$
$g(x) = e^{-x^2}(\sin^2 x + 3\cos x + 5),\ x_0 = -2)$

Method	Number of iterations	x_n	$f(x_n)$
NM	9	-1.207647827130918927009416758**5**	-2.27e-40
AM	6	-1.2076478271309189270094167584	-4.0e-63
HM	6	-1.2076478271309189270094167584	-4.0e-63
CM	6	-1.2076478271309189270094167584	-4.0e-63
NR	5	-1.2076478271309189270094167583	-3.0e-31
NA	4	-1.2076478271309189270094167582	-1.3e-26

Table 7. Comparison of NM, AM, HM, CM, NR and NA
$(f(x) = e^{x^2+7x-30} - 1,\ g(x) = \frac{1}{7}(30 - x^2),\ x_0 = 3.5)$

Method	Number of iterations	x_n	$f(x_n)$
NM	13	3.0000000000000000000006	1.52e-47
AM	7	3.0000000000000000000007	-4.33e-48
HM	8	3.0000000000000000000001	2.0e-62
CM	8	3.0000000000000000000001	2.0e-62
NR	7	3.0000000000000000000011	-2.1e-29
NA	3	3.0000000000000000000009	1.93e-33

5. Conclusions

In this paper, we have suggested a two-step iterative method for solving nonlinear equations. Derivation technique of this method is very simple as compared with the adomain decomposition method. It is clear from above examples that our two-step iterative method is better than the fourth-order of Chun [4] and it also performs better than the method presented in [7]. Noor et al. [7] has established free of second derivatives three-step iterative method with cubic convergence. The method established in this paper, is also free of second derivatives and is a two-step iterative method of third order convergence. This method can be considered as an improvement and refinement of previously existing method of third order convergence.

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