International Journal of Pure and Applied Mathematics
Volume 111 No. 2 2016, 249-272
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijjpam.eu
doi: 10.12732/ijpam.v111i2.10

ijpam.eu

SOLUTION AND STABILITY OF TWO TYPES OF n-DIMENSIONAL QUARTIC FUNCTIONAL EQUATION IN GENERALIZED 2-NORMED SPACES

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Abstract: In this paper, we investigate the general solution and generalized Ulam-Hyers stability of a n-dimensional quartic functional equation of the form

$$(8-n)f\left(\sum_{i=1}^{n} x_i\right)+\sum_{j=1}^{n} f\left(-x_j+\sum_{i=1;i\neq j} x_i\right)=4\sum_{1\leq i<j<k\leq n}(x_i+x_j+x_k)+(-4n+14)$$
$$\sum_{i=1;i\neq j} f(x_i+x_j)+2\sum_{i=1;i\neq j} f(x_i-x_j)+\sum_{j=1}^{n} f(2x_j)+(2n^2-14n+14)\sum_{i=1}^{n} f(x_i),$$

with $n\geq 3$ in generalized 2-normed space using two types of different methods.

AMS Subject Classification: 39B52, 39B82, 46B99
Key Words: Ulam-Hyers stability, 2-normed space, quartic functional equation and fixed point

Received: August 31, 2016
Revised: October 7, 2016
Published: December 11, 2016

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1. Introduction

In 1940, before the guidance of the mathematics club of the University of Wisconsin S. M. Ulam [37] presented a list of unsolved problems. One of these problem can be considered as the starting point of a new line of investigations. The stability problem, suppose that a group G and a metric group H are given. For any $\epsilon > 0$, does there exists a $\delta > 0$ such that if a function $f : G \to H$ satisfies the inequality $d(f(xy), f(x)f(y)) < \delta$ for all $x, y \in G$, then a homomorphism $a : G \to H$ exists with $d(f(x), a(x)) < \epsilon$ for all $x \in G$? These kind of questions form the material of the stability theory [36]. In 1941, D. H. Hyers [15] answered the Ulam's problems for the case of approximately additive functions under the assumption that G and H are Banach spaces. Taking this fact into account, the additive functional equation

$$f(x+y) = f(x) + f(y)$$

is said to have Ulam-Hyers stability [14]. For more information see [1, 2, 3, 8, 9, 10, 11, 12, 13, 16, 28, 32, 34, 38].

The U-type stability result for the quartic functional equation

$$f(x+2y) + f(x-2y) + 6f(x) = 4\left[f(x+y) + f(x-y)\right] + 24f(y) \qquad (1.1)$$

was first invented by J.M. Rassias in [27]. Subsequently, P.K. Sahoo and J.K. Chung [33], S.H. Lee et. al., [20] modified the J. M. Rassias equation (1.1) as

$$f(2x+y) + f(2x-y) = 4f(x+y) + 4f(x-y) + 24f(x) - 6f(y). \qquad (1.2)$$

Many other types of quartic functional equations were introduced and investigated in [30, 35].

Motivated by this observation, in this paper the authors investigate the general solution and generalized Ulam-Hyers stability of a n-dimensional quartic functional equation of the form,

$$\begin{aligned}(8-n)f\left(\sum_{i=1}^{n} x_i\right) + \sum_{j=1}^{n} f\left(-x_j + \sum_{i=1; i\neq j} x_i\right. \\ = 4 \sum_{1\leq i<j<k\leq n} (x_i + x_j + x_k) + (-4n+14) \\ \sum_{i=1; i\neq j} f(x_i + x_j) + 2 \sum_{i=1; i\neq j} f(x_i - x_j)\end{aligned}$$

$$+\sum_{j=1}^{n} f(2x_j) + (2n^2 - 14n + 14)\sum_{i=1}^{n} f(x_i), \quad (1.3)$$

with $n \geq 3$ in generalized 2-normed space using two types of different methods.

2. Preliminaries of 2-Normed Spaces

Now, we present some basic definitions related to Generalized 2-normed spaces.

Definition 2.1. [24]. Let X be a linear space. A function $N(.,.) : X \times X \to [0, \infty)$ is called a generalized 2-normed space if it is satisfies the following:

$(G2N1)$ $N(x, y) = 0$ if and only if x and y are linearly independent vectors.

$(G2N2)$ $N(x, y) = N(y, x)$ for all $x, y \in X$.

$(G2N3)$ $N(\lambda x, y) = |\lambda|\, N(x, y)$ for all $x, y \in X$ and $X = \phi$, ϕ is real or complex field.

$(G2N4)$ $N(x + y, z) \leq N(x, z) + N(y, z)$; if for all $x, y, z \in X$.
The generalized 2-normed space is denoted by $(X, N(.,.))$.

Definition 2.2. [24]. A sequence $\{x_n\}$ in a generalized 2-normed space $(X, N\,(.,.))$ is called a Cauchy sequence if there exists two linearly independent elements y and z in X such that $\{N(xn, y)\}$ and $\{N(xn, z)\}$ are real Cauchy sequences.

Definition 2.3. [29, 7]. A sequence $\{x_n\}$ in a generalized 2-normed space $(X, N(.,.))$ is called convergent if there exists $x \in X$ such that

$$\lim_{n\to\infty} N\,(x_n - x, y) = 0,$$

then for all $y \in X$, we have

$$\lim_{n\to\infty} N\,(x_n, y) = N(x, y).$$

Definition 2.4. [24, 21]. A generalized 2-normed space $(X, N\,(.,.))$ is called generalized 2-Banach space if every Cauchy sequence is convergent.

3. A Solution of quartic functional equation

It is well known [3, 4, 5, 6, 17, 18, 19, 22, 23, 25, 26, 31] that a function $f : X \to Y$ between a real vector spaces is quadratic if and only if there exists a unique symmetric biadditive function B such that $f(x) = B(x, x)$ for all $x \in X$. The biadditive function B is given by

$$B(x,y) = \frac{1}{4}\left(f(x+y) - f(x-y)\right).$$

In this section, we prove that a function $f : X \to Y$ between the real vector spaces satisfies the functional equation (1.2) if and only if there exists a symmetric biquadratic function F such that $f(x) = F(x, x)$ for all $x \in X$. Throughout this section X and Y will be real vector spaces.

Theorem 3.1. *Let X and Y be a Real Vector Space, the mapping $f : X \to Y$ satisfies the functional equation (1.2), for all $x, y \in X$ if and only if the mapping $f : X \to Y$ satisfies the functional equation (1.3) for all $x_1, x_2, ..., x_n \in X$.*

Proof. Assume that f satisfies the functional equation (1.2). Putting $x = y = 0$ in (1.2), we have $f(0) = 0$. Putting $x = 0$ in (1.2) $f(y) = f(-y)$ for all $y \in X$. Putting $y = 0$ and $y = x$ in (1.2), we obtain that

$$f(2x) = 16f(x),$$

and for all $x \in X$, we have

$$f(3x) = 81f(x),$$

respectively. Actually, we can lead to

$$f(nx) = n^4 f(x),$$

for all $x \in X$ and all $n \in N$. Replacing x and y by $x_1 + x_2$ and $x_1 - x_2$ in (1.2), respectively, we get

$$\begin{aligned} & f(2x_1 + 2x_2 + x_1 - x_2) + f(2x_1 + 2x_2 - x_1 + x_2) \\ &\ = 4f(x_1 + x_2 + x_1 - x_2) + 4f(x_1 + x_2 - x_1 + x_2) \\ &\quad + \ 24f(x_1 + x_2) - 6f(x_1 - x_2)f(3x_1 + x_2) + f(x_1 + 3x_2) \\ &\ = 4f(2x_1) + 4f(2x_2) + 24f(x_1 + x_2) - 6f(x_1 - x_2)f(3x_1 + x_2) + f(x_1 + 3x_2) \\ &\ = 64f(x_1) + 64f(x_2) + 24f(x_1 + x_2) - 6f(x_1 - x_2), \end{aligned} \tag{3.1}$$

for all $x_1, x_2 \in X$. Replacing x_1 and x_2 by x_1+x_2 and $2x_2$ in (1.2), respectively, we arrive to

$$\begin{aligned}
&f(2x_1+2x_2+2x_2)+f(2x_1+2x_2-2x_2)\\
&\quad=4f(x_1+x_2+2x_2)+4f(x_1+x_2-2x_2)\\
&\qquad+24f(x_1+x_2)-6f(2x_2)f\left(2(x_1+2x_2)\right)+f(2x_1)\\
&\quad=4f(x_1+3x_2)+4f(x_1-x_2)+24f(x_1+x_2)-6f(2x_2)16f\left(x_1+2x_2\right)+16f(x_1)\\
&\quad=4f(x_1+3x_2)+4f(x_1-x_2)+24f(x_1+x_2)-24f(x_2)4f\left(x_1+2x_2\right)+4f(x_1)\\
&\quad=f(x_1+3x_2)+f(x_1-x_2)+6f(x_1+x_2)-24f(x_2),
\end{aligned} \tag{3.2}$$

for all $x_1, x_2 \in X$. Interchanging x_1 and x_2 in (3.2), we obtain that

$$4f\left(x_2+2x_1\right)+4f(x_2)=f(x_2+3x_1)+f(x_2-x_1)+6f(x_2+x_1)-24f(x_1), \tag{3.3}$$

for all $x_1, x_2 \in X$. Adding (3.2) and (3.3) and also using (3.1), we obtain

$$\begin{aligned}
&4f\left(x_1+2x_2\right)+4f(x_1)+4f\left(x_2+2x_1\right)+4f(x_2)\\
&\quad=f(x_1+3x_2)+f(x_1-x_2)+6f(x_1+x_2)-24f(x_2)+f(x_2+3x_1)\\
&\qquad+f(x_2-x_1)+6f(x_2+x_1)-24f(x_1)4f\left(x_1+2x_2\right)+4f\left(2x_1+x_2\right)\\
&\quad=f(x_1+3x_2)+f(3x_1+x_2)+12f(x_1+x_2)+f(x_1-x_2)\\
&\qquad+f(x_2-x_1)-24f(x_1)-24f(x_2)-4f(x_1)-4f(x_2).
\end{aligned}$$

Then using the result (3.1), we have

$$\begin{aligned}
&4f\left(x_1+2x_2\right)+4f\left(2x_1+x_2\right)=64f(x_1)+64f(x_2)+24f(x_1+x_2)\\
&-6f(x_1-x_2)+12f(x_1+x_2)+f(x_1-x_2)+f(x_2-x_1)-28f(x_1)-28f(x_2).
\end{aligned}$$

Using $f(-x)=f(x)$, we have

$$\begin{aligned}
4f\left(x_1+2x_2\right)+4f\left(2x_1+x_2\right)&=36f(x_1)+36f(x_2)+36f(x_1+x_2)-6f(x_1-x_2)\\
&\quad+2f(x_1-x_2)f\left(x_1+2x_2\right)+f\left(2x_1+x_2\right)\\
&=9f(x_1)+9f(x_2)+9f(x_1+x_2)-f(x_1-x_2)
\end{aligned} \tag{3.4}$$

for all $x_1, x_2 \in X$. Setting $x=x_1$; $x_2=x_3$ in (1.2), we receive

$$f\left(2x_1+x_3\right)+f\left(2x_1-x_3\right)=24f(x_1)-6f(x_3)+4f(x_1+x_3)+4f(x_1-x_3), \tag{3.5}$$

for all $x_1, x_3 \in X$. Replacing $x_1=x_2$ and $x_2=x_3$ in (1.2), we obtain

$$f\left(2x_2+x_3\right)+f\left(2x_2-x_3\right)=24f(x_2)-6f(x_3)+4f(x_2+x_3)+4f(x_2-x_3), \tag{3.6}$$

for all $x_2, x_3 \in X$. Adding (3.5) and (3.6), that the resultant equation, we obtain

$$\begin{aligned} &9f\left(2x_1+x_3\right)+9f\left(2x_1-x_3\right)+9f\left(2x_2+x_3\right)+9f\left(2x_2-x_3\right)\\ &\quad=36f(x_1+x_3)+36f(x_1-x_3)+36f(x_2+x_3)\\ &\qquad+36f(x_2-x_3)+216f(x_1)+216f(x_2)-108f(x_3) \end{aligned} \tag{3.7}$$

for all $x_1, x_2, x_3 \in X$. Replacing $x = 2x_1+x_3$ and $x_2 = 2x_2+x_3$ in the equation (3.4), gives that

$$\begin{aligned} &f(2x_1+x_3+4x_2+2x_3)+f(4x_1+2x_2+2x_3+x_3)\\ &\quad=9f(2x_1+x_3)+9f(2x_2+x_3)+9f(2x_1+x_3+2x_2+x_3)\\ &\qquad-f(2x_1+x_3-2x_2-x_3)f(2x_1+4x_2+3x_3)+f(4x_1+2x_2+3x_3)\\ &\quad=9f(2x_1+x_3)+9f(2x_2+x_3)+9f(2x_1+2x_2+2x_3)\\ &\qquad-f(2x_1-2x_2)f(2x_1+4x_2+3x_3)+f(4x_1+2x_2+3x_3)\\ &\qquad-9f(2x_1+2x_2+2x_3)+f(2x_1-2x_2)\\ &\quad=9f(2x_1+x_3)+9f(2x_2+x_3)9f(2x_1+x_3)+9f(2x_2+x_3)\\ &\quad=f(2x_1+4x_2+3x_3)+f(4x_1+2x_2+3x_3)\\ &\qquad-9f(2x_1+2x_2+2x_3)+f(2x_1-2x_2) \end{aligned} \tag{3.8}$$

for all $x_1, x_2, x_3 \in X$. Replacing x_1 and x_2 by $2x_1 - x_3$ and $x_2 = 2x_2 - x_3$ in the equation (3.4), we arrive that

$$\begin{aligned} &f(2x_1-x_3+4x_2-2x_3)+f(4x_1-2x_3+2x_2-x_3)\\ &\quad=9f(2x_1-x_3)+9f(2x_2-x_3)+9f(2x_1-x_3+2x_2-x_3)\\ &\qquad-f(2x_2-x_3-2x_2+x_3)f(2x_1+4x_2-3x_3)+f(4x_1+2x_2-3x_3)\\ &\quad=9f(2x_1-x_3)+9f(2x_2-x_3)+9f(2x_1+2x_2-2x_3)\\ &\qquad-f(2x_1-2x_2)f(2x_1+4x_2-3x_3)+f(4x_1+2x_2-3x_3)\\ &\qquad-9f(2x_1+2x_2-2x_3)+f(2x_1-2x_2)\\ &\quad=9f(2x_1-x_3)+9f(2x_2-x_3)9f(2x_1-x_3)+9f(2x_2-x_3)\\ &\quad=f(2x_1+4x_2-3x_3)+f(4x_1+2x_2-3x_3)\\ &\quad-9f(2x_1+2x_2-2x_3)+f(2x_1-2x_2) \end{aligned} \tag{3.9}$$

for all $x_1, x_2, x_3 \in X$. Adding (3.8) and (3.9), and using the result (1.2), we obtain

$$\begin{aligned} &9f(2x_1+x_3)+9f(2x_2+x_3)+9f(2x_1-x_3)+9f(2x_2-x_3)\\ &\quad=4f(x_1+2x_2+3x_3)+4f(x_1+2x_2-3x_3)\ +24f(x_1+2x_2)\\ &\qquad-6f(3x_3)+4f(2x_1+x_2+3x_3)+4f(2x_1+x_2-3x_3)\\ &\qquad+24f(2x_1+x_2)-6f(3x_3)-144f(x_1+x_2+x_3)\\ &\qquad-144f(x_1+x_2-x_3)+32f(x_1-x_2), \end{aligned} \tag{3.10}$$

for all $x_1, x_2, x_3 \in X$. By the equations (3.7) and (3.10), and rearranging the equation, we obtain

$$\begin{aligned}36f(x_1+x_3)+36f(x_1-x_3)+216f(x_1)+54f(x_3)+36f(x_2+x_3)+36f(x_2-x_3)\\ +216f(x_2)-54f(x_3)=4f(x_1+2x_2+3x_3)+4f(x_1+2x_2-3x_3)+24f(x_1+2x_2)\\ -6f(3x_3)+4f(2x_1+x_2+3x_3)+4f(2x_1+x_2-3x_3)+24f(2x_1+x_2)\\ -6f(3x_3)-144f(x_1+x_2+x_3)-144f(x_1+x_2-x_3)+32f(x_1-x_2)\end{aligned} \tag{3.11}$$

for all $x_1, x_2, x_3 \in X$. Replacing x_1 and x_2 by $2x_1 + x_3$ and $x_2 = 2x_2 - x_3$ in the equation (3.4), we arrive that

$$\begin{aligned}&f(2x_1-x_3+4x_2-2x_3)+f(4x_1+2x_3+2x_2-x_3)\\ &\quad=9f(2x_1+x_3)+9f(2x_2-x_3)+9f(2x_1+x_3+2x_2-x_3)\\ &\qquad-f(2x_1+x_3-2x_2+x_3)f(2x_1+4x_2-x_3)+f(4x_1+2x_2+x_3)\\ &\quad=9f(2x_1+x_3)+9f(2x_2-x_3)+9f(2x_1+2x_2)-f(2x_1-2x_2+2x_3)\end{aligned}$$

which implies that,

$$\begin{aligned}9f(2x_1+x_3)+9f(2x_2-x_3)=f(2x_1+4x_2-x_3)+f(4x_1+2x_2+x_3)\\ -9f(2x_1+2x_2)+f(2x_1-2x_2+2x_3)\end{aligned} \tag{3.12}$$

for all $x_1, x_2, x_3 \in X$. Replacing $x_1 = 2x_1 - x_3$ and $x_2 = 2x_2 + x_3$ in the equation (3.4), we obtain that

$$\begin{aligned}&f(2x_1-x_3+4x_2+2x_3)+f(4x_1-2x_3+2x_2+x_3)\\ &\quad=9f(2x_1-x_3)+9f(2x_2+x_3)+9f(2x_1-x_3+2x_2+x_3)\\ &\qquad-f(2x_1-x_3-2x_2-x_3)f(2x_1+4x_2+x_3)+f(4x_1+2x_2-x_3)\\ &\quad=9f(2x_1-x_3)+9f(2x_2+x_3)+9f(2x_1+2x_2)-f(2x_1-2x_2-2x_3)\end{aligned}$$

which gives that,

$$\begin{aligned}9f(2x_1-x_3)+9f(2x_2+x_3)=f(2x_1+4x_2+x_3)+f(4x_1+2x_2-x_3)\\ -9f(2x_1+2x_2)+f(2x_1-2x_2-2x_3)\end{aligned} \tag{3.13}$$

for all $x_1, x_2, x_3 \in X$. Adding (3.12) and (3.13), we have,

$$\begin{aligned}&9f(2x_1+x_3)+9f(2x_2-x_3)+9f(2x_1-x_3)+9f(2x_2+x_3)\\ &\quad=f(2x_1+4x_2-x_3)+f(4x_1+2x_2+x_3)-9f(2x_1+2x_2)+f(2x_1-2x_2+2x_3)\\ &\qquad+f(2x_1+4x_2+x_3)+f(4x_1+2x_2-x_3)-9f(2x_1+2x_2)+f(2x_1-2x_2-2x_3)\end{aligned} \tag{3.14}$$

for all $x_1, x_2, x_3 \in X$. Using the equation (1.2) in the above equation, we get

$$\begin{aligned}
&9f(2x_1+x_3)+9f(2x_2-x_3)+9f(2x_1-x_3)+9f(2x_2+x_3)\\
&\quad=4f(x_1+2x_2+x_3)+4f(x_1+2x_2-x_3)+24f(x_1+2x_2)-6f(x_3)\\
&\qquad+4f(2x_1+x_2+x_3)+4f(2x_1+x_2-x_3)+24f(2x_1+x_2)-6f(x_3)\\
&\qquad-288f(x_1+x_2)+16f(x_1-x_2+x_3)+16f(x_1-x_2-x_3)
\end{aligned}\tag{3.15}$$

for all $x_1, x_2, x_3 \in X$. The above equation implies that

$$\begin{aligned}
&9f(2x_1+x_3)+9f(2x_2-x_3)+9f(2x_1-x_3)+9f(2x_2+x_3)\\
&\quad=4f(x_1+2x_2+x_3)+4f(x_1+2x_2-x_3)+24f(x_1+2x_2)-6f(x_3)\\
&\qquad+4f(2x_1+x_2+x_3)+4f(2x_1+x_2-x_3)+24f(2x_1+x_2)-6f(x_3)\\
&\qquad-288f(x_1+x_2)+16f(x_1-x_2+x_3)+16f(x_1-x_2-x_3)
\end{aligned}\tag{3.16}$$

for all $x_1, x_2, x_3 \in X$. Replacing x_3 by $3x_3$ in the above equation (3.16), we arrive that

$$\begin{aligned}
&9f(2x_1+3x_3)+9f(2x_2-3x_3)+9f(2x_1-3x_3)+9f(2x_2+3x_3)\\
&\quad=4f(x_1+2x_2+3x_3)+4f(x_1+2x_2-3x_3)+24f(x_1+2x_2)-6f(3x_3)\\
&\qquad+4f(2x_1+x_2+3x_3)+4f(2x_1+x_2-3x_3)+24f(2x_1+x_2)-6f(3x_3)\\
&\qquad-288f(x_1+x_2)+16f(x_1-x_2+3x_3)+16f(x_1-x_2-3x_3)
\end{aligned}\tag{3.17}$$

for all $x_1, x_2, x_3 \in X$. Using the equation (3.11), from the resultant equation, we obtain

$$\begin{aligned}
&=4f(x_1+2x_2+3x_3)+4f(x_1+2x_2-3x_3)+24f(x_1+2x_2)-6f(3x_3)\\
&\quad+4f(2x_1+x_2+3x_3)+4f(2x_1+x_2-3x_3)+24f(2x_1+x_2)-6f(3x_3)
\end{aligned}\tag{3.18}$$

for all $x_1, x_2, x_3 \in X$. Substitute (3.18) in (3.17), we get

$$\begin{aligned}
&9f(2x_1+x_3)+9f(2x_2-x_3)+9f(2x_1-x_3)+9f(2x_2+x_3)\\
&\quad=36f(x_1+x_3)+36f(x_1-x_3)+216f(x_1)-54f(x_3)+36f(x_2+x_3)\\
&\qquad+36f(x_2-x_3)+216f(x_2)-54f(x_3)+144f(x_1+x_2+x_3)+144f(x_1+x_2-x_3)\\
&\qquad-32f(x_1-x_2)-288f(x_1+x_2)+16f(x_1-x_2+3x_3)+16f(x_1-x_2-3x_3)
\end{aligned}\tag{3.19}$$

for all $x_1, x_2, x_3 \in X$. Replacing $x_1 = x_1 - x_2 + 3x_3$ and $x_2 = x_1 - x_2 - 3x_3$ in (3.4), which implies that

$$\begin{aligned} & f(3x_1 - 3x_2 - 3x_3) + f(3x_1 - 3x_2 + 3x_3) \\ & \quad = 9f(x_1 - x_2 + 3x_3) + 9f(x_1 - x_2 - 3x_3) + 9f(2x_1 - 2x_3) \\ & \qquad - f(6x_3)9f(x_1 - x_2 + 3x_3) + 9f(x_1 - x_2 - 3x_3) \\ & \quad = f(3x_1 - 3x_2 - 3x_3) + f(3x_1 - 3x_2 + 3x_3) - 9f(2x_1 - 2x_3) \\ & \qquad + f(6x_3)9f(x_1 - x_2 + 3x_3) + 9f(x_1 - x_2 - 3x_3) \\ & \quad = 81f(x_1 - x_2 - x_3) + 81f(x_1 - x_2 + x_3) - 144f(x_1 - x_3) + 1296f(x_3) \end{aligned} \tag{3.20}$$

for all $x_1, x_2, x_3 \in X$. Divided by $(\frac{16}{9})$, we obtain

$$\begin{aligned} & 16f(x_1 - x_2 + 3x_3) + 16f(x_1 - x_2 - 3x_3) \\ & \quad = 144f(x_1 - x_2 - x_3) + 144f(x_1 - x_2 + x_3) - 256f(x_1 - x_3) + 2304f(x_3) \end{aligned} \tag{3.21}$$

for all $x_1, x_2, x_3 \in X$. using the equation (3.19), (3.20) and (3.21), we will have

$$\begin{aligned} & 9f(2x_1 + 3x_3) + 9f(2x_2 - 3x_3) + 9f(2x_1 - 3x_3) + 9f(2x_2 + 3x_3) \\ & \quad = 36f(x_1 + x_3) + 36f(x_1 - x_3) + 216f(x_1) - 54f(x_3) + 36f(x_2 + x_3) \\ & \qquad + 36f(x_2 - x_3) + 216f(x_2) - 54f(x_3) + 144f(x_1 + x_2 + x_3) \\ & \qquad + 144f(x_1 + x_2 - x_3) - 32f(x_1 - x_2) - 288f(x_1 + x_2) \\ & \qquad + 144f(x_1 - x_2 - x_3) + 144f(x_1 - x_2 + x_3) - 256f(x_1 - x_2) + 2304f(x_3) \end{aligned} \tag{3.22}$$

for all $x_1, x_2, x_3 \in X$. Setting $x_1 = 2x_1 + 3x_3$ and $x_2 = 2x_1 - 3x_3$ in (3.4), which gives that

$$\begin{aligned} & f(2x_1 + 3x_3 + 4x_1 - 6x_3)f(4x_1 + 6x_3 + 2x_1 - 3x_3) \\ & \quad = 9f(2x_1 + 3x_3) + 9f(2x_1 - 3x_3) + 9f(2x_1 + 3x_3 + 2x_1 - 3x_3) \\ & \qquad - f(2x_1 + 3x_3 - 2x_1 - 3x_3)f(6x_1 - 3x_3) + f(6x_1 + 3x_3) \\ & \quad = 9f(2x_1 + 3x_3) - 9f(2x_1 - 3x_3) + 9f(4x_1) - f(6x_3) \end{aligned} \tag{3.23}$$

for all $x_1, x_2, x_3 \in X$. Replacing $x_1 = 2x_2 - 3x_3$ and $x_2 = 2x_2 + 3x_3$ in (3.4), we arrive that

$$\begin{aligned} & f(6x_2 + 3x_3) + f(6x_2 - 3x_3) \\ & \qquad = 9f(2x_2 - 3x_3) - 9f(2x_2 + 3x_3) + 9f(4x_2) - f(6x_3) \end{aligned} \tag{3.24}$$

for all $x_1, x_2, x_3 \in X$. Adding (3.23) and (3.24), we obtain

$$\begin{aligned} &9f(2x_1+3x_3)+9f(2x_1-3x_3)+9f(2x_2+3x_3)+9f(2x_2-3x_3)\\ &\quad+9f(2x_2+3x_3)=324f(x_1+x_3)+324f(x_1-x_3)+1944f(x_1)\\ &\qquad-486f(x_3)+324f(x_2+x_3)+324f(x_2-x_3)+1944f(x_2)\\ &\qquad-486f(x_3)-2304f(x_1)-2304f(x_2)+2592f(x_3) \end{aligned} \tag{3.25}$$

for all $x_1, x_2, x_3 \in X$. From equation (3.22) and (3.25), L. H. S are equal, that

$$\begin{aligned} &36f(x_1+x_3)+36f(x_1-x_3)+216f(x_1)-54f(x_3)+36f(x_2+x_3)+36f(x_2-x_3)\\ &\quad+216f(x_2)-54f(x_3)+144f(x_1+x_2+x_3)+144f(x_1+x_2-x_3)\\ &\quad-32f(x_1-x_2)-288f(x_1+x_2)+144f(x_1-x_2-x_3)+144f(x_1-x_2+x_3)\\ &\quad-256f(x_1-x_2)+2304f(x_3)=324f(x_1+x_3)+324f(x_1-x_3)\\ &\qquad+1944f(x_1)-486f(x_3)+324f(x_2+x_3)+324f(x_2-x_3)\\ &\qquad+1944f(x_2)-486f(x_3)-2304f(x_1)-2304f(x_2)+2592f(x_3) \end{aligned} \tag{3.26}$$

for all $x_1, x_2, x_3 \in X$. From the resultant equation (3.26), we get

$$\begin{aligned} &144\left[f(x_1+x_2+x_3)+f(x_1+x_2-x_3)+f(x_1-x_2-x_3)+f(x_1-x_2+x_3)\right]\\ &\quad=324f(x_1+x_3)+324f(x_1-x_3)+1944f(x_1)-486f(x_3)\\ &\qquad+324f(x_2+x_3)+324f(x_2-x_3)+1944f(x_2)-486f(x_3)-2304f(x_1)\\ &\qquad-2304f(x_2)+2592f(x_3)36f(x_1+x_3)+36f(x_1-x_3)+216f(x_1)\\ &\qquad-54f(x_3)+36f(x_2+x_3)+36f(x_2-x_3)+216f(x_2)-54f(x_3)\\ &\qquad-32f(x_1-x_2)-288f(x_1+x_2)-256f(x_1-x_2)+2304f(x_3) \end{aligned}$$

which implies that

$$\begin{aligned} &f(x_1+x_2+x_3)+f(x_1+x_2-x_3)+f(x_1-x_2-x_3)+f(x_1-x_2+x_3)\\ &\quad=2\left(f(x_1+x_2)+\ f(x_1+x_3)+f(x_2+x_3)\right)\\ &\qquad+2\left(f(x_1-x_2)+f(x_1-x_3)+f(x_2-x_3)\right)-4\left\{f(x_1)+f(x_2)+f(x_3)\right\} \end{aligned} \tag{3.27}$$

for all $x_1, x_2, x_3 \in X$. From the equation (3.27), rearranging we get

$$\begin{aligned} &f(x_1+x_2+x_3)+f(-x_1+x_2-x_3)+f(x_1+x_2-x_3)+f(x_1-x_2+x_3)\\ &\quad=\left(4\sum_{1\le i<j<k\le 3} f(x_i+x_j+x_k)\right)+(-4\times 3+14)\sum_{1\le i<j\le 3} f(x_i+x_j)\\ &\qquad+2\sum_{1\le i<j\le 3} f(x_i-x_j)\sum_{j=1}^{3} f(2x_j)+(2\times 3^2-14\times 3+4)\sum_{i=1}^{3} f(x_i) \end{aligned} \tag{3.28}$$

for all $x_1, x_2, x_3 \in X$. Similarly one can easily verify that this result is true for four variables, we get

$$\begin{aligned} &f(x_1+x_2+x_3+x_4)+f(-x_1+x_2-x_3+x_4)+f(x_1+x_2-x_3+x_4) \\ &\quad + f(x_1-x_2+x_3+x_4) = \left(4 \sum_{1\le i<j<k\le 4} f(x_i+x_j+x_k)\right) \\ &\qquad + (-4\times 4+14)\sum_{1\le i<j\le 4} f(x_i+x_j) \\ &\qquad + 2\sum_{1\le i<j\le 4} f(x_i-x_j)\sum_{j=1}^{4} f(2x_j) + (2\times 4^2 - 14\times 4+4)\sum_{i=1}^{4} f(x_i) \end{aligned} \tag{3.29}$$

for all $x_1, x_2, x_3 \in X$. Similarly one can easily verify that this result is true for n variables, Extending this result, for any positive integer n, we arrive

$$f\left(\sum_{i=1}^{n} x_i\right)+\sum_{j=1}^{n} f\left(-x_j + \sum_{i=1; i\neq j} x_i\right) = 4\sum_{1\le i<j<k\le n}(x_i+x_j+x_k)+(-4n+14)$$

$$\sum_{i=1;i\neq j} f\left(x_i+x_j\right)+2\sum_{i=1;i\neq j} f\left(x_i-x_j\right)+\sum_{j=1}^{n} f(2x_j)+(2n^2-14n+14)\sum_{i=1}^{n} f(x_i)$$

for all $x_1, x_2, x_3 \in X$. This completes the proof of the first part of the theorem.

Conversely, assume that $f : X \to Y$ satisfies the functional equation (3.29). Now we prove that the function $f : X \to Y$ satisfies the functional equation (1.2). Now replacing (x_1, x_2, x_3) by (x, x, y) in three variables that

$$\begin{aligned} &(8-n)f(2x+y)+f(2x-y)+2f(y) \\ &\quad = 4f(2x+y)+2(-4n+14)f(x+y)+(2.2)f(x-y) \\ &\qquad + 4f(y)(8-n)f(2x+y)+f(2x-y)+2f(y) \\ &\quad = 4f(2x+y)+2(-4n+14)f(x+y)+(2.2)f(x-y) \\ &\qquad + (2.2)f(x-y)+(-4n+14)f(2x)+2.(2n^2-14n+14)f(x) \\ &\qquad + (2n^2-14n+14)f(y)+(-4n+14)f(2x)+f(2x)+f(2y) \end{aligned} \tag{3.30}$$

for all $x, y \in X$. And again from the equation (3.29) for four variables, which gives that

$$\begin{aligned}
&(8-n)f(2x+y)+f(2x+y)+f(2x-y)+2f(y)\\
&\quad=4f(2x+y)+2(-4n+14)f(x+y)+(2.4)f(x-y)+(2.2)f(x-y)\\
&\qquad+2(-4n+14)f(x)+(-4n+14)f(2x)+2.(2n^2-14n+14)f(x)\\
&\qquad+(2n^2-14n+14)f(y)+2(-4n+14)f(x)+(-4n+14)f(2x)\\
&\qquad+(-4n+14)f(y)+f(2x)+f(2y)+(2.2)f(2x)+2f(2y)
\end{aligned}\tag{3.31}$$

for all $x,y\in X$. And again from the equation (3.29) for five variables, gives

$$\begin{aligned}
&(8-n)f(2x+y)+f(2x-y)+2f(2x+y)+2f(y)\\
&\quad=4f(2x+y)+(4.4)f(x+y)+2(-4n+14)f(x+y)\\
&\qquad+(2.4)f(x-y)+(2.2)f(x-y)+4(-4n+14)f(x)+(2.4)f(x)\\
&\qquad+(1.4)f(y)+(-4n+14)f(2x)+2.(2n^2-14n+14)f(x)\\
&\qquad+(2n^2-14n+14)f(y)+4(-4n+14)f(x)+2(-4n+14)f(y)\\
&\qquad+(-4n+14)f(2x)+f(2x)+f(2y)+(2.4)f(2x)+(2.2)f(2y)
\end{aligned}\tag{3.32}$$

for all $x,y\in X$. Combining the equations (3.30), (3.31) and (3.32) and for n variables, gives that

$$\begin{aligned}
&(8-n)f(2x+y)+(n-3)f(2x+y)+f(2x-y)+2f(y)\\
&\quad=4f(2x+y)+2(n-3)[4f(x+y)]+2[2f(x-y)]+(-4n+14)[2f(x+y)]\\
&\qquad+2(n-3)(-4n+14)f(x)+(n-3)(n-4)[4f(x)]+\frac{(n-3)(n-4)}{2}[4f(y)]\\
&\qquad+(-4n+14)f(2x)+2.(2n^2-14n+4)f(x)+(2n^2-14n+14)f(y)\\
&\qquad+(n-3)(-4n+14)f(y)+4(n-3)f(2x)+(2.2)f(2y)\\
&\qquad+(4n-12)f(x)+2nf(y)-6f(y)+f(2x)+f(2y)+f(2x)
\end{aligned}\tag{3.33}$$

for all $x,y\in X$. Now from the equation (3.33), and rearranging that

$$\begin{aligned}
&8f(2x+y)+3f(2x+y)+f(2x-y)+2f(y)\\
&\quad=4f(2x+y)+(2n-6)[4f(x+y)]+4f(x-y)+(-8n+28)f(x+y)\\
&\qquad+(2n-6)(-4n+14)f(x)+(n^2-7n-12)[4f(x)]\\
&\qquad+(n^2-7n-12)[2f(y)]+(-4n+14)f(2x)+(4n^2-28n+8)f(x)\\
&\qquad+(2n^2-14n+4)f(y)+(n^2-7n-42)f(y)+(4n-12)f(2x)\\
&\qquad+(4n-12)f(x)+2nf(2y)+-6f(y)+16f(x)+16f(y)+16f(x)
\end{aligned}\tag{3.34}$$

for all $x, y \in X$. Now from the equation (3.34), modified as

$$\begin{aligned}5f(2x+y)+f(2x-y)+2f(y) = {} & 4f(2x+y)+8nf(x+y)-24f(x+y)\\ & +4f(x-y)-8nf(x+y)+28f(x+y)+\\ & -8n^2f(x)+52f(x)-84f(x)+4n^2f(x)\\ & -28nf(x)-48f(x)+2n^2\ f(y)-14nf(y)\\ & -24f(y)-4nf(2x)+14f(2x)+4n^2f(x)\\ & -28nf(x)-8f(x)+2n^2f(y)-14nf(y)\\ & +4f(y)+n^2f(y)-7nf(y)-42f(y)+4nf(2x)\\ & -12f(2x)+4nf(x)-12f(x)+2nf(2y)\\ & +-6f(y)+16f(x)+16f(y)+16f(x)\end{aligned} \tag{3.35}$$

$$\begin{aligned}&5f(2x+y)+f(2x-y)+2f(y)\\ &\qquad = 4f(2x+y)+4f(x+y)+4f(x-y)+24f(x)-4f(y),\end{aligned}$$

$$\begin{aligned}&5f(2x+y)-4f(2x+y)+f(2x-y)\\ &\qquad = -2f(y)+4f(x+y)+4f(x-y)+24f(x)-4f(y),\end{aligned}$$

which implies that,

$$f(2x+y)+f(2x-y)=4f(x+y)+4f(x-y)+24f(x)-6f(y) \tag{3.36}$$

for all $x, y \in X$. Setting y by $-y$ in the equation (3.36), we obtain

$$f(2x-y)+f(2x+y)=4f(x-y)+4f(x+y)+24f(x)-6f(-y) \tag{3.37}$$

for all $x, y \in X$. Using the even function then we have from (3.37), we obtain that

$$f(2x-y)+f(2x+y)=4f(x-y)+4f(x+y)+24f(x)-6f(y) \tag{3.38}$$

for all $x, y \in X$. Adding the equations (3.36) and (3.38), we get

$$2f(2x-y)+2f(2x+y)=8f(x-y)+8f(x+y)+24f(x)-6f(y)$$

Dividing the above by 2, we arrive that

$$f(2x+y)+f(2x-y)=4f(x+y)+4f(x-y)+24f(x)-6f(y)$$

for all $x, y \in X$. Which completes the proof of the Theorem. □

4. Stability of n-Dimensional Quartic Functional Equation (1.3)

In this section, let us consider X be a generalized 2-normed space and Y be a generalized 2-Banach space respectively. Define a mapping $Q_f : X^n \to Y$ by

$$Q_f(x) = Q_f\left(x_1, x_2, x_3, ..., x_n\right) = (8-n)f\left(\sum_{i=1}^{n} x_i\right) + \sum_{j=1}^{n} f\left(-x_j + \sum_{i=1; i\neq j} x_i\right)$$
$$-4 \sum_{1\leq i<j<k\leq n} \left(x_i + x_j + x_k\right) - (-4n+14) \sum_{i=1; i\neq j} f\left(x_i + x_j\right) - 2 \sum_{i=1; i\neq j} f\left(x_i - x_j\right)$$
$$-\sum_{j=1}^{n} f(2x_j) - (2n^2 - 14n + 14)\sum_{i=1}^{n} f(x_i)$$

for all $x_1, x_2, x_3, ..., x_n \in X^n$ and $n \geq 3$.

Theorem 4.1. *Let $\psi : X^n \to [0, \infty)$ be a function such that*

$$\sum_{m=0}^{\infty} \frac{\psi\left(\left(m^{nj}x_1, z\right), \left(m^{nj}x_2, z\right)\right), \left(m^{nj}x_3, z\right), ..., \left(m^{nj}x_n, z\right)}{m^{4nj}}$$

converges and

$$\lim_{n\to\infty} \left\{ \frac{\psi\left(\left(m^{nj}x_1, z\right), \left(m^{nj}x_2, z\right), \left(m^{nj}x_3, z\right), ..., \left(m^{nj}x_n, z\right)\right)}{m^{4nj}} \right\} = 0 \tag{4.1}$$

for all $x_1, x_2, x_3, ..., x_n \in X^n$, $z \in X$ and let $f : X \to Y$ be a function satisfying the inequality

$$N\left(Q_f(x), z\right) \leq \psi\left(\left(x_1, z\right), \left(x_2, z\right), ..., \left(x_n, z\right)\right) \tag{4.2}$$

for all $x_1, x_2, x_3, ..., x_n \in X^n$, $z \in X$. Then there exists a unique quartic function $F : X \to Y$ such that

$$N\left(f(x) - F(x), z\right) \leq \sum_{n=\frac{1-j}{2}}^{\infty} \frac{\varphi\left(2^{nj}x, z\right)}{16^{nj}} \tag{4.3}$$

where $\varphi\left(2^{nj}x, z\right) = \psi(x, 0, 0, ..., z)$ for all $x, z \in X$. The function $F(x)$ is defined by

$$N\left(F(x), z\right) = \lim_{n\to\infty} N\left(\frac{f\left(2^{nj}x\right)}{16^{nj}}, z\right) \tag{4.4}$$

for all $x, z \in X$ and $j = \pm 1$.

Proof. Assume that $j = 1$. Replacing $(x_1, x_2, x_3, ..., x_n)$ by $(x, 0, 0, ...0)$ in (4.2), we get

$$N\big((8-n)f(x) + f(-x) + (n-1)f(x) - 2[n^2+3n+2]f(x) - (-4n+14)$$

$$(n-1)f(x) - 2(n-1)f(x) - (2n^2-14n+14)f(x), z\big) \leq \psi(x,0,0,...,0)$$

$$N\left(\frac{f(2x)}{16} - f(x), z\right) \leq \frac{\psi(x,0,0,...,0)}{16} \tag{4.5}$$

for all $x, z \in X$. Letting

$$\varphi(x,z) = \frac{\psi(x,0,0,...,0)}{16}$$

in the equation (4.5), we arrive

$$N\left(\frac{f(2x)}{16} - f(x), z\right) \leq \varphi(x,z) \tag{4.6}$$

for all $x, z \in X$. Now setting x by $2x$ and dividing by 16 in (4.6), we obtain

$$N\left(\frac{f(2^2x)}{16^2} - \frac{f(2x)}{16}, z\right) \leq \varphi(2x,z) \tag{4.7}$$

for all $x, z \in X$. Using the inequality (4.6) and (4.7), and the definition of (G2N4), we obtain

$$\begin{aligned} N\left(\frac{f(2^2x)}{16^2} - f(x), z\right) &= N\left(\frac{f(2^2x)}{16^2} - \frac{f(2x)}{16} + \frac{f(2x)}{16} - f(x), z\right) \\ &\leq N\left(\frac{f(2^2x)}{16^2} - \frac{f(2x)}{16}, z\right) + N\left(\frac{f(2x)}{16} - f(x), z\right) \\ &\leq \varphi(x,z) + \frac{\varphi(2x,z)}{16} \end{aligned} \tag{4.8}$$

for all $x, z \in X$. In general for any positive integer m, we have

$$N\left(\frac{f(2^mx)}{16^m} - f(x), z\right) \leq \sum_{n=0}^{m-1} \frac{\varphi(2^nx,z)}{16^n} \leq \sum_{n=0}^{\infty} \frac{\varphi(2^nx,z)}{16^n} \tag{4.9}$$

for all $x, z \in X$. In order to prove the convergence of the sequence $\left\{\frac{f(2^n x,z)}{16^n}\right\}$ replace x by $2^l x$ and dividing by 16^l in (4.9), for any $n, l \geq 0$, we obtain

$$N\left(\frac{f(2^{l+n}x)}{16^{l+n}} - \frac{f(2^l x)}{16^l}, z\right) \leq \frac{1}{16^l} N\left(\frac{f(2^l.2^n x)}{16^n} - f(2^l x), z\right)$$

$$\leq \sum_{n=0}^{\infty} \frac{\varphi(2^{l+n}x, z)}{16^{l+n}} = 0 as l \to \infty. \tag{4.10}$$

for all $x, z \in X$. Also, we have

$$N\left(\frac{f(2^{l+n}x)}{16^{l+n}} - \frac{f(2^l x)}{16^l}, z_1\right) \leq \frac{1}{16^l} N\left(\frac{f(2^l.2^n x)}{16^n} - f(2^l x), z_1\right)$$

$$\leq \sum_{n=0}^{\infty} \frac{\varphi(2^{l+n}x, z_1)}{16^{l+n}} = 0 as l \to \infty. \tag{4.11}$$

for all $x, z \in X$. Hence there exists two linearly independent elements z and z_1 in X such that $\left\{\frac{f(2^n x)}{16^n}, z\right\}$ and $\left\{\frac{f(2^n x)}{16^n}, z_1\right\}$ are real Cauchy sequences. Hence the sequence $\left\{\frac{f(2^n x)}{16^n}\right\}$ is Cauchy sequence, since Y is complete, there exists a mapping $F : X \to Y$ such that

$$N\left(F(x), z\right) = \lim_{n\to\infty} N\left(\frac{f(2^n x)}{16^n}, z\right)$$

for all $x, z \in X$. Letting $n \to \infty$ in (4.9), we see that (4.4) holds for all $x, z \in X$. To prove F satisfies (4.4), setting $(x_1, x_2, ..., x_n)$ by $(2^n x_1, 2^n x_2, 2^n x_3, ..., 2^n x_n)$ and dividing by 16^n in (4.2) and allow $n \to \infty$, we arrive

$$N\left(D_F(x_1, x_2, ..., x_n), z\right) = 0.$$

Hence F satisfies (4.3) for all $x_1, x_2, ..., x_n, z \in X$. To prove F is unique. Let $J(x)$ be another quartic mapping satisfying (4.4) and (4.3). Then

$$N\left(F(x) - J(x), z\right) = \frac{1}{16^n} N\left(F(2^n x) - J(2^n x), z\right)$$

$$\leq \frac{1}{16^n}\left\{N\left(F(2^n x) - f(2^n x), z\right) + N\left(f(2^n x) - J(2^n x), z\right)\right\}$$

$$\leq \sum_{n=0}^{\infty} \frac{\varphi(2^{l+n}x, z)}{16^{l+n}} = 0 as l \to \infty.$$

The right hand side of the above inequality converges to 0 as $n \to \infty$ for all $x, z \in X$. Hence F is Unique, for $j = -1$, we can prove the similar type pf stability result. This completes the proof of the Theorem. $\square$

The following corollary is an immediate consequence of Theorem **??** concerning the stability of (4.4).

Corollary 4.2. *Let δ and s be the non-negative real numbers. If a function $f : X \to Y$ satisfying the inequality*

$$N\left(Q_f(x), z\right) \leq \begin{cases} \delta \\ \delta \sum\limits_{k=1}^{n} \|x_k, z\|^s, \quad s > 4(or)s < 4 \\ \delta \left\{ \sum\limits_{k=1}^{n} \|x_k, z\|^{ns} + \prod\limits_{k=1}^{n} \|x_k, z\|^{ns} \right\}, s > \frac{4}{n}(or)s < \frac{4}{n} \end{cases}$$

for all $x_1, x_2, ..., x_n, z \in X$. Then there exists a unique quartic function $F : X \to Y$ such that

$$N\left(f(x) - F(x), z\right) \leq \begin{cases} \frac{16\delta}{15}, \\ \frac{16\delta}{|16-2^s|} \|x, z\|^s, \\ \frac{16\delta}{|16-2^{ns}|} \|x, z\|^{ns}, \end{cases}$$

for all $x \in X$ and $z \in X$.

5. Stability of n-Dimensional Quartic Functional Equation (1.3): Fixed Point Method

The following Theorem provide the stability result of (1.3) in fixed point method. For proving the stability result, we define the following, m_i is a constant such that

$$m_i = \begin{cases} 2, if i = 0, \\ \frac{1}{2} if i = 1 \end{cases}$$

and η is the set such that

$$\eta = \{g/g : X \to Y; g(0) = 0\}$$

Theorem 5.1. *Let $f : X \to Y$ be a mapping for which there exists a function $\psi : X^n \to (0, \infty)$ with the condition*

$$\sum_{m=0}^{\infty} \frac{\psi\left(\left(m_i^n x_1, z\right), \left(m_i^n x_2, z\right)\right), \left(m_i^n x_3, z\right), ..., \left(m_i^n x_n, z\right)}{m_i^{4n}} \tag{5.1}$$

converges and

$$\lim_{n\to\infty}\left\{\frac{\psi\left(\left(m_i^n x_1, z\right), \left(m_i^n x_2, z\right), \left(m_i^n x_3, z\right), ..., \left(m_i^n x_n, z\right)\right)}{m_i^{4n}}\right\} = 0 \tag{5.2}$$

for all $x_1, x_2, x_3, ..., x_n \in X$, $z \in X$ satisfying the functional inequality

$$N\left(Q_f(x), z\right) \le \psi\left(\left(x_1, z\right), \left(x_2, z\right), ..., \left(x_n, z\right)\right) \tag{5.3}$$

for all $x_1, x_2, x_3, ..., x_n \in X$, $z \in X$. If there exists $L = L(i) < 1$ such that the function

$$x \to \varphi(x, z) = 16\varphi\left(\left(\frac{x}{2}, z\right), (0, z), (0, z), ..., (0, z)\right)$$

has the property, that for all $x \in X$,

$$\frac{\varphi\left(m_i x, z\right)}{m_i^4} = L\varphi(x, z) \tag{5.4}$$

Then there exists a unique quartic function $F : X \to Y$ satisfying the functional equation (4.3) and for all $x \in X$, we have

$$N\left(f(x) - F(x), z\right) \le \frac{L^{1-i}}{1-L}\varphi(x, z). \tag{5.5}$$

Proof. Let ϕ be a general metric on η such that

$$d(g, h) = inf\left\{k \in (0, \infty) : N\left(g(x) - h(x), z\right) \le k\phi(x, z), x, z \in X\right\}$$

It is easy to see that (η, d) is complete. Define $T : \eta \to \eta$ by

$$T_f(x) = \frac{1}{m_i^4} f(m_i x)$$

for all $x \in X$. For $f, h \in \eta$, we have $d(f, h) = k$.

$$N\left(f(x) - h(x), z\right) k\varphi x, z$$

$$N\left(\frac{f\left(m_i x\right)}{m_i^4} - \frac{h\left(m_i x\right)}{m_i^4}, z\right) \le \frac{1}{m_i^4} k\varphi(m_i x, z)$$

which implies that

$$d\left(T_f(x), T_h(x)\right) \le KL.$$

Hence, $d(Tg, Th) \leq Ld(f, h)$, for all $f, h \in \eta$. Therefore, T is strictly contractive mapping on η with Lipschitz constant L. Replacing $(x_1, x_2, ..., x_n)$ by $(x, 0, 0, ..., 0)$ in (5.3) and using the definition of $\varphi(x, z)$, we get

$$N\left(f(2x) - 16f(x), z\right) \leq \varphi(2x, z) \tag{5.6}$$

for all $x, z \in X$. Applying the equation (5.4) in the above equation for $i = 0$, we have

$$N\left(f(2x) - 16f(x), z\right) \leq \varphi(2x, z)$$

which implies that

$$N\left(T_f(x) - f(x)\right) \leq L\varphi(x, z)$$

for all $x, z \in X$. Hence, we arrive

$$N\left(T_f(x), f(x)\right) \leq L = L^{1-i} \tag{5.7}$$

for all $x \in X$. Replacing x by $\frac{x}{2}$ in (5.6) and using the definition of $\varphi(x)$ for $i = 1$, we have

$$\|f(2x) - 16f(x/2)\|_P \leq \varphi(x)$$

which gives that

$$\|f(2x) - T_f(x)\|_P \leq \varphi(x)$$

for all $x \in X$. Hence we arrive that

$$N\left(T_f(x), f(x)\right) \leq 1 = L^{1-i} \tag{5.8}$$

for all $x \in X$. Then from (5.7) and (5.8) we can conclude that

$$N\left(T_f(x), f(x)\right) \leq L^{1-i} < \infty$$

for all $x \in X$. Now from the fixed point alternative in both cases, it follows that there exists a fixed point F of T in η such that

$$N\left(F(x), z\right) = \lim_{m\to\infty} N\left(\frac{(m_i^n x, z)}{m_i^{4n}}, z\right) \tag{5.9}$$

for all $x \in X$. To prove F satisfies the functional equation (4.4), we can use the same idea in Theorem 4.1. Since F is unique fixed point of T in the set

$$\zeta = \{f \in \eta : d(f, F)\}$$

therefore, F is unique function such that for all $x, z \in X$, we have

$$N\left(f(x) - F(x), z\right) \leq K\varphi(x, z). \tag{5.10}$$

Again using the fixed point alternative, we obtain

$$d(f,F) \leq \frac{1}{1-L} d(f,T_f) \leq \frac{L^{1-i}}{1-L}$$

gives that

$$N\left(f(x)-F(x),z\right) \leq \frac{L^{1-i}}{1-L}\varphi(x,z). \tag{5.11}$$

This completes the proof of the Theorem. □

The following corollary is an immediate consequence of Theorem 5.1 concerning the stability of (4.4).

Corollary 5.2. *Let $f : X \to Y$ be a mapping there exists a real numbers Ω and s such that*

$$N\left(Q_f(x),z\right) \leq \begin{cases} \Omega, \\ \Omega \sum\limits_{k=1}^{n} \|x_k,z\|^s, \quad s>2(or)s<2 \\ \Omega \left\{ \sum\limits_{k=1}^{n} \|x_k,z\|^{ns} + \prod\limits_{k=1}^{n} \|x_k,z\|^{ns} \right\}, s > \frac{2}{n}(or)s<\frac{2}{n} \end{cases} \tag{5.12}$$

for all $x_1, x_2, ..., x_n, z \in X$. Then there exists a unique quartic function $F : X \to Y$ such that

$$N\left(f(x)-F(x),z\right) \leq \begin{cases} \frac{16|\Omega|}{15}, \\ \frac{16|\Omega|}{|16-2^s|}\|x,z\|^s, \\ \frac{16|\Omega|}{|16-2^{ns}|}\|x,z\|^{ns}, \end{cases} \tag{5.13}$$

for all $x \in X$.

Proof. Set $\varphi(x,z)$ as right hand side of the equation (5.12), for all $x_1, x_2, x_3, ..., x_n \in X$. It is easy to conclude that

$$\frac{1}{m_i^4}\varphi(m_i^4 x) \to 0 \quad as\ n \to \infty.$$

Thus (5.1) holds. But we have

$$\varphi(x,z) = 16\psi\left(\left(\frac{x}{2},z\right),(0,z),(0,z),...,(0,z)\right).$$

Now

$$\frac{1}{m_i^4}\varphi(m_i x, z) = \begin{cases} 16 m_i^{-4}\Omega, \\ m_i^{s-4} 2^{4-s} \|x, z\|^s, \\ m_i^{ns-4} 2^{4-ns} \|x, z\|^{ns}, \end{cases} = L\varphi(x)$$

where, $L = m_i^{-4}$, $L = m_i^{s-4}$ and $L = m_i^{ns-4}$ for the respective assumption of ψ. Hence the inequality (5.4) holds. Now, from (5.5), we prove the following cases for conditions. **Case: (i)**. $L = 2^{-4}$, if $i = 0$, we have

$$N(f(x) - F(x), z) \leq \frac{L^{1-0}}{1-L}\varphi(x, z) = \frac{\left(2^{-4}\right)^{1-0}}{1-2^{-4}} 16\Omega$$

$$N(f(x) - F(x), z) \leq \frac{16}{15}\Omega.$$

Case: (ii). $L = 2^{-4}$ and if $i = 1$

$$N(f(x) - F(x), z) \leq \frac{L^{1-1}}{1-L}\varphi(x, z) = \frac{\left(\left(\frac{1}{2}\right)^{-4}\right)^{1-1}}{1-\left(\frac{1}{2}\right)^{-4}} 16\Omega$$

$$N(f(x) - F(x), z) \leq \frac{-16}{15}\Omega.$$

Case: (iii). $L = 2^{s-4}$ and if $i = 0$

$$N(f(x) - F(x), z) \leq \frac{2^{s-4}}{1-2^{s-4}}\Omega 2^{4-s}\|x, z\|^s = \frac{2^{s-4}}{1-2^s.2^{-4}}\Omega 2^{4-s}\|x, z\|^s$$

$$N(f(x) - F(x), z) \leq \frac{16}{16-2^s}\Omega\|x, z\|^s$$

Case: (iv). $L = \frac{1}{2^{s-4}}$ and if $i = 1$

$$N(f(x) - F(x), z) \leq \frac{L^{1-1}}{1-L}\varphi(x, z) = \frac{\left(\left(\frac{1}{2}\right)^{s-4}\right)^{1-1}}{1-2^{4-s}}\Omega 2^{4-s}\|x, z\|^s$$

$$N(f(x) - F(x), z) \leq \frac{-16}{16-2^s}\Omega\|x, z\|^s$$

Case: (v). $L = 2^{ns-4}$ and if $i = 0$

$$N(f(x) - F(x), z) \leq \frac{L^{1-0}}{1-L}\varphi(x, z)$$

$$= \frac{2^{ns-4}}{1-2^{ns-4}} \Omega 2^{4-ns} \|x, z\|^n s$$

$$= \frac{2^{ns-4}}{1-2^{ns}.2^{-4}} \Omega 2^{4-ns} \|x, z\|^{ns}$$

$$N(f(x) - F(x), z) \leq \frac{16}{16-2^{ns}} \Omega \|x, z\|^{ns}$$

Case: (vi). $L = \left(\frac{1}{2}\right)^{ns-4}$ and if $i = 1$

$$N(f(x) - F(x), z) \leq \frac{L^{1-1}}{1-L} \varphi(x, z)$$

$$= \frac{1}{1-2^{-ns+4}} \Omega 2^{4-ns} \|x, z\|^n s$$

$$= \frac{1}{1-2^{-ns}.2^{4}} \Omega 2^{4-ns} \|x, z\|^{ns}$$

$$N(f(x) - F(x), z) \leq -\frac{16}{16-2^{ns}} \Omega \|x, z\|^{ns}.$$

□

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