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ON gW-CONTINUOUS FUNCTIONS INDUCED BY GENERALIZED w-OPEN SETS IN w-SPACES

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Abstract: The purpose of this paper is to introduce the notions of gW-continuous, gW^*-continuous, gW-irresolute, and gW^*-irresolute functions induced by gw-open sets in w-spaces, and to study characterizations and the relationships among them.

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1. Introduction

Siwiec [16] introduced the notions of weak neighborhoods and weak base in a topological space. We introduced the weak neighborhood systems defined by using the notion of weak neighborhoods in [11]. The weak neighborhood system induces a weak neighborhood space which is independent of neighborhood spaces [4] and general topological spaces [2]. The notions of weak structure, w-space, W-continuity and W^*-continuity were investigated in [12].

Levine [5] introduced the notion of g-closed subsets in topological spaces. In fact, the set of all g-closed subsets is a kind of weak structure. In the same way, we introduced the notion of generalized w-closed set (simply, gw-closed set) [14] in weak spaces, and investigated some basic properties of such notions.

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The purpose of this paper is to introduce the notions of gW-continuous, gW^*-continuous, gW-irresolute, and gW^*-irresolute functions induced by gw-open sets in w-spaces, and to study characterizations and the relationships among such notions, W-continuity and W^*-continuity in w-spaces.

2. Preliminaries

Let X be a nonempty set. A subfamily w_X of the power set $P(X)$ is called a *weak structure* on X if it satisfies the following:

(1) $\emptyset \in w_X$ and $X \in w_X$.

(2) For $U_1, U_2 \in w_X$, $U_1 \cap U_2 \in w_X$.

Then the pair (X, w_X) is called a *w-space* on X. Then $V \in w_X$ is called a *w-open* set and the complement of a w-open set is a *w-closed* set. The collection of all w-open sets (resp., w-closed sets) in a w-space X will be denoted by $W(X)$ (resp., $WC(X)$). We set $W(x) = \{U \in W(X) : x \in U\}$.

Let S be a subset of a topological space X. The closure (resp., interior) of S will be denoted by clS (resp., $intS$). A subset S of X is called a *preopen* set [9] (resp., α-open set [15], *semi-open* [6]) if $S \subset int(cl(S))$ (resp., $S \subset int(cl(int(S))), S \subset cl(int(S))$). The complement of a preopen set (resp., α-open set, *semi-open*) is called a *preclosed* set (resp., α-closed set, *semi-closed*). The family of all preopen sets (resp., α-open sets, semi-open sets) in X will be denoted by $PO(X)$ (resp., $\alpha(X), SO(X)$). We know the family $\alpha(X)$ is a topology finer than the given topology on X. And a subset A of X is said to be g-closed [5] (resp., gp-closed [7], gs-closed [1, 3]) if $cl(A)$ (resp., $pCl(A)$, $sCl(A)$) $\subset U$ whenever $A \subset U$ and U is open in X.

Then the family τ, $GO(X)$, $g\alpha O(X)$, and $g\alpha^* O(X)$, are all weak structures on X. But $PO(X)$, $GPO(X)$ and $SO(X)$ are not weak structures on X. A subfamily m_X of the power set $P(X)$ of a nonempty set X is called a *minimal structure* on X [8] if $\emptyset \in w_X$ and $X \in w_X$. Thus clearly every weak structure is a minimal structure.

For a subset A of X, the *w-closure* of A and the *w-interior* of A are defined as follows:

(1) $wC(A) = \cap\{F : A \subseteq F, X - F \in w_X\}$.

(2) $wI(A) = \cup\{U : U \subseteq A, U \in w_X\}$.

Theorem 2.1 ([12]). *Let (X, w_X) be a w-space and $A \subseteq X$.*

(1) $x \in wI(A)$ if and only if there exists an element $U \in W(x)$ such that $U \subseteq A$.

(2) $x \in wC(A)$ if and only if $A \cap V \neq \emptyset$ for all $V \in W(x)$.

(3) If $A \subseteq B$, then $wI(A) \subseteq wI(B)$; $wC(A) \subset wC(B)$.

(4) $wC(X - A) = X - wI(A)$; $wI(X - A) = X - wC(A)$.

(5) If A is w-closed (resp., w-open), then $wC(A) = A$ (resp., $wI(A) = A$).

Let (X, w_X) be a w-space and $A \subseteq X$. Then A is called *a generalized w-closed set* (simply, gw-closed set) [14] if $wC(A) \subseteq U$, whenever $A \subseteq U$ and U is w-open. Then the union of two gw-closed sets is a gw-closed set, but the intersection of two gw-closed sets is not always gw-closed. The family of all w-closed sets (resp., gw-closed sets, gw-open sets) in X will be denoted by $WC(X)$ (resp., $GWC(X), GWO(X)$). We set $gW(x) = \{U \in GWO(X) : x \in U\}$. And A is called *a generalized w-open set* (simply, gw-open set) if $X - A$ is gw-closed. Then A is gw-open if and only if $F \subseteq wI(A)$ whenever $F \subseteq A$ and F is w-closed. For a subset A of X, *gw-closure* of A and *gw-interior* [14] of A are defined as the following:

(1) $gwC(A) = \cap\{F : A \subseteq F, F$ is gw-closed$\}$.

(2) $gwI(A) = \cup\{U : U \subseteq A, U$ is gw-open$\}$.

Theorem 2.2 ([14]). *Let (X, w_X) be a w-space and $A \subseteq X$.*

(1) If A is gw-open (gw-closed), then $gwI(A) = A$ ($gwC(A) = A$).

(2) If $A \subseteq B$, then $gwI(A) \subseteq gwI(B)$; $gwC(A) \subseteq gwC(B)$.

(3) $gwC(X - A) = X - gwI(A)$; $gwI(X - A) = X - gwC(A)$.

(4) $x \in gwI(A)$ iff there exists a gw-open set U containing x such that $U \subseteq A$.

(5) $x \in gwC(A)$ iff $A \cap V \neq \emptyset$ for all gw-open set V containing x.

3. Main Results

Definition 3.1. Let $f : X \to Y$ be a function in w-spaces. Then f is said to be

(1) *gW-continuous* if for $x \in X$ and for each w-open set V containing $f(x)$, there is a gw-open set U containing x such that $f(U) \subseteq V$;

(2) *gW^*-continuous* if for every w-open set V in Y, $f^{-1}(V)$ is a gw-open set in X.

Obviously we obtain the following theorem:

Theorem 3.2. *Every gW^*-continuous function is gW-continuous.*

The following example supports that the converse of the above theorem is not true in general.

Example 3.3. Let $X = \{a, b, c, d\}$ and a w-structure $w = \{\emptyset, \{a, c\}, \{a\}, \{b\}, \{c\}, \{a, d\}, X\}$ in X. Then for the power set $P(X)$ of X, $GWO(X) = P(X) - \{\{b, c, d\}, \{b, d\}\}$ is the set of all gw-open sets. Consider a function $f : (X, w) \to (X, w)$ defined by $f(a) = b; f(b) = a; f(c) = d; f(d) = c$. Then f is gW-continuous. For a w-open set $\{a, c\}$, $f^{-1}(\{a, c\}) = \{b, d\}$ is not gw-open, and so f is not gW^*-continuous.

We recall that: Let $f : X \to Y$ be a function on w-spaces. Then f is said to be

(1) *W-continuous* [12] if for $x \in X$ and for each w-open set V containing $f(x)$, there is a w-open set U containing x such that $f(U) \subseteq V$;

(2) *W^*-continuous* [12] if for every w-open set V in Y, $f^{-1}(V)$ is a w-open set in X.

Obviously, the following things are obtained:

Theorem 3.4. *(1) Every W-continuous function is gW-continuous.*
(2) Every W^*-continuous function is gW^*-continuous.

Proof. Since every w-open set is gw-open, the things are obvious. □

The following example supports that the converses of the above theorem are not true in general.

Example 3.5. In Example 3.3: (1) For $d \in X$ and for a w-open set $V = \{b\}$ containing $f(d)$, there is no any w-open set U containing d such that $f(U) \subseteq V$. So, the gW-continuous function f is not W-continuous.

(2) Consider a function $g : (X, w) \to (X, w)$ defined by $g(a) = b; g(b) = c; g(c) = a; g(d) = d$. Then g is gW^*-continuous but not W^*-continuous.

Definition 3.6. Let $f : (X, w_\tau) \to (Y, w_\mu)$ be a function in two associated w-spaces with τ and μ. Then f is said to be

(1) *gW-irresolute* if for $x \in X$ and for each gw-open set V containing $f(x)$, there is gw-open set U containing x such that $f(U) \subseteq V$;

(2) *gW^*-irresolute* if for every gw-open set V in Y, $f^{-1}(V)$ is gw-open in X.

Theorem 3.7. *(1) Every gW^*-irresolute function is gW-irresolute.*
(2) Every gW-irresolute is gW-continuous.
(3) Every gW^*-irresolute function is gW^*-continuous.

Proof. (1) Obvious.
(2) and (3) Since every w-open set is gw-open, they are obtained. □

The following example supports that the converses of the above theorem are not true in general.

Example 3.8. 1) In Example 3.3, (i) consider a function $f : (X, w) \to (X, w)$ defined by $f(a) = a; f(b) = f(c) = f(d) = b$. Then f is gW-irresolute but not gW^*-irresolute;

(ii) consider the function $g : (X, w) \to (X, w)$ defined by $g(a) = b; g(b) = c; g(c) = a; g(d) = d$. Then g is gW^*-continuous but not gW^*-irresolute.

2) Let $X = \{a, b, c, d\}$ and $w_X = \{\emptyset, \{a\}, \{a, b\}, \{a, c\}, \{a, d\}, \{a, b, d\}, X\}$ be a w-structure in X. Note that:

$WC(X) = \{\emptyset, \{b, c, d\}, \{c, d\}, \{b, d\}, \{b, c\}, \{c\}, X\}$;

$GWC(X) = \{\emptyset, \{b\}, \{c\}, \{d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, b, c\}, \{a, c, d\}, \{b, c, d\}, X\}$;

$GWO(X) = \{\emptyset, \{a\}, \{b\}, \{d\}, \{a, d\}, \{a, c\}, \{a, b\}, \{a, b, c\}, \{a, c, d\}, \{a, b, d\}, X\}$.

Consider a function $h : (X, w) \to (X, w)$ defined by $h(a) = a; h(b) = c; h(c) = b; h(d) = d$. Then h is gW-continuous. But for a gw-open set $\{b\}$, $h^{-1}(\{b\}) = \{c\}$ is not gw-open, so h is not gW-irresolute.

Remark 3.9. For a function from a w-space to a w-space, we have the following diagram:

$$\begin{array}{ccc} W^*\text{-continuity} & \Rightarrow & W\text{-continuity} \\ \Downarrow & & \Downarrow \\ gW^*\text{-continuity} & \Rightarrow & gW\text{-continuity} \\ \Uparrow & & \Uparrow \\ gW^*\text{-irresolute} & \Rightarrow & gW\text{-irresolute} \end{array}$$

Theorem 3.10. *Let $f : X \to Y$ be a function in w-spaces. Then f is gW^*-continuous if and only if for every w-closed set F in Y, $f^{-1}(F)$ is gw-closed in X.*

Proof. It is obvious. □

Theorem 3.11. *Let $f : X \to Y$ be a function in w-spaces. Then the following statements are equivalent:*

(1) f is gW-continuous.

(2) $f(gwC(A)) \subseteq wC(f(A))$ for $A \subseteq X$.

(3) $gwC(f^{-1}(V)) \subseteq f^{-1}(wC(V))$ for $V \subseteq Y$.

(4) $f^{-1}(wI(V)) \subseteq gwI(f^{-1}(V))$ for $V \subseteq Y$

Proof. (1) ⇒ (2) Let $x \in gwC(A)$. Suppose that $f(x)$ is not in $wC(f(A))$; then there exists a w-open set V containing $f(x)$ such that $V \cap f(A) = \emptyset$. By gW-continuity, there is a gw-open set U containing x such that $f(U) \subseteq V$, and so $f(U) \cap f(A) = \emptyset$. Hence $U \cap A = \emptyset$, which is a contradiction to $x \in gwC(A)$. So, $f(gwC(A)) \subseteq wC(f(A))$.

(2) ⇒ (3) Let $A = f^{-1}(B)$ for $B \subseteq Y$. By hypothesis, $f(gwC(A)) \subseteq wC(f(A)) = wC(f(f^{-1}(B))) \subseteq wC(B)$. Hence, $gwC(f^{-1}(B)) \subseteq f^{-1}(wC(B))$.

(3) ⇒ (4) By Theorem 2.1 and Theorem 2.2, it is obvious.

(4) ⇒ (1) Let V be any w-open set containing $f(x)$ for each $x \in X$. Then by hypothesis, $x \in f^{-1}(wI(V)) \subseteq gwI(f^{-1}(V))$. So, there exists a gw-open set U such that $x \in U \subseteq gwI(f^{-1}(V)) \subseteq f^{-1}(V)$. It implies that f is gW-continuous. □

Corollary 3.12. *Let $f : X \to Y$ be a function on w-spaces. Then the following statements are equivalent:*
(1) f is gW-continuous.
(2) $f^{-1}(V) = gwI(f^{-1}(V))$ for every w-open set $V \in Y$.
(3) $f^{-1}(B) = gwC(f^{-1}(B))$ for every w-closed set $B \subseteq Y$.

Proof. From (1) of Theorem 2.2, it is obvious. □

Theorem 3.13. *Let $f : (X, w_\tau) \to (Y, w_\mu)$ be a function in two associated w-spaces with τ and μ. Then the following statements are equivalent:*
(1) f is gW-irresolute.
(2) $f(gwC(A)) \subseteq gwC(f(A))$ for $A \subseteq X$.
(3) $gwC(f^{-1}(V)) \subseteq f^{-1}(gwC(V))$ for $V \subseteq Y$.
(4) $f^{-1}(gwI(V)) \subseteq gwI(f^{-1}(V))$ for $V \subseteq Y$

Proof. Since the family of all gw-open sets is a weak structure in X, it is similar to the proof of Theorem 3.11. □

Let (X, w) be a w-space. Let $gW(x)$ (resp., $W(x)$) denote the set of all gw-open (resp., w-open) set containing x in X. A collection $\mathcal{H}$ of subsets of X is called an *m-family* [10] on X if $\cap\mathcal{H} \neq \emptyset$. Let $\mathcal{H}$ be an m-family on X. Then we say that an m-family $\mathcal{H}$ *gw-converges* (resp., *w-converges*) to $x \in X$ if $\mathcal{H}$ is finer than $gW(x)$ (resp., $W(x)$) i.e., $gW(x) \subseteq \mathcal{H}$ (resp., $W(x) \subseteq \mathcal{H}$). Let $f : X \to Y$ be a function; then it is obvious $f(\mathcal{H}) = \{f(F) : F \in \mathcal{H}\}$ is an m-family on Y.

Theorem 3.14. *Let $f : X \to Y$ be a function in w-spaces. If f is gW-continuous, then for an m-family $\mathcal{H}$ gw-converging to $x \in X$, an m-family $< f(\mathcal{H}) >= \{F : H \subseteq F \text{ for } H \in f(\mathcal{H})\}$ w-converges to $f(x)$.*

Proof. Let f be gW-continuous and let $\mathcal{H}$ be an m-family gw-converging to $x \in X$. By gW-continuity, for a w-open set V containing $f(x)$, there exists a gw-open set U containing x such that $f(U) \subseteq V$. Since $f(gW(x)) \subseteq f(\mathcal{H})$, $V \in< f(\mathcal{H}) >$, and so $W(f(x)) \subseteq< f(\mathcal{H}) >$. Hence the m-family $< f(\mathcal{H}) >$ w-converges to $f(x)$. □

Corollary 3.15. *Let $f : X \to Y$ be a function on w-spaces. Then if f is gW^*-continuous, then for an m-family $\mathcal{H}$ gw-converging to $x \in X$, the m-family $< f(\mathcal{H}) >$ w-converges to $f(x)$.*

Proof. Since every gW^*-continuous function is gW-continuous, it is obtained obviously. □

Corollary 3.16. *Let $f : X \to Y$ be a function on w-spaces. Then if f is W^*-continuous, then for an m-family $\mathcal{H}$ w-converging to $x \in X$, the m-family $< f(\mathcal{H}) >$ w-converges to $f(x)$.*

Proof. Since every W^*-continuous function is gW^*-continuous, from the above corollary, it is obtained obviously. □

Theorem 3.17. *Let $f : X \to Y$ be a bijective function on w-spaces. Then f is gW^*-continuous iff for an m-family $\mathcal{H}$ gw-converging to $x \in X$, the m-family $f(\mathcal{H})$ w-converges to $f(x)$.*

Proof. Suppose f is gW^*-continuous and $\mathcal{H}$ is an m-family gw-converging to $x \in X$. By hypothesis and surjectivity, $W(f(x)) \subseteq f(gW(x)) \subseteq f(\mathcal{H})$, and so the m-family $f(\mathcal{H})$ w-converges to $f(x)$.

For the converse, let $U \in W(f(x))$ for $U \subseteq Y$. Since the family $gW(x)$ clearly gw-converges to x, by hypothesis, we get $W(f(x)) \subseteq f(gW(x))$ for $x \in X$. Since f is injectivity, $f^{-1}(U) \in gW(x)$. □

Corollary 3.18. *Let $f : X \to Y$ be a bijective function on w-spaces. Then f is W^*-continuous iff for an m-family $\mathcal{H}$ w-converging to $x \in X$, the m-family $f(\mathcal{H})$ w-converges to $f(x)$.*

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References

[1] P. Bhattacharyya and B. K. Lahiri; Semi-generalized closed sets in topology, Indian J. Math., 29(3) (1987), 375-382.

[2] Á. Csázár; Generalized Topology, Generalized Continuity, Acta Math. Hungar., 96 (2002), 351-357.

[3] J. Dontchev and H. Maki; On sg-closed sets and semi-λ-closed sets, Q & A in General Topology, 15(1997), 259-266.

[4] D. C. Kent and W. K. Min; Neighborhood Spaces, International Journal of Mathematics and Mathematical Sciences, 32(7) (2002), 387-399.

[5] N. Levine; Generalized closed sets in topology, Rend. Cir. Mat. Palermo, 19(1970), 89-96.

[6] N. Levine; Semi-open sets and semi-continuity in topological spaces, Ams. Math. Monthly, 70(1963), 36-41.

[7] H. Maki, J. Umehara and T. Noiri; Every topological space is pre-$T_{\frac{1}{2}}$, Mem. Fac. Sci. Kochi Univ. Ser. A, 17(1996), 33-42.

[8] H. Maki; On generalizing semi-open and preopen sets, Report for Meeting on Topological Spaces Theory and its Applications, August 1996, Yatsushiro College of Technology, 13-18.

[9] A. S. Mashhour, M. E. Abd El-Monsef and S. N. El-Deeb; On precontinuous and weak precontinuous mappings, Proc. Math. and Phys. Soc. Egypt, 53 (1982), 47-53.

[10] W. K. Min; Some Results on Generalized Topological Spaces and Generalized Systems, Acta Math. Hungar., 108 (1-2) (2005), 171-181.

[11] W. K. Min; On Weak Neighborhood Systems and Spaces, Acta Math. Hungar., 121(3) (2008), 283-292.

[12] W. K. Min and Y. K. Kim; On Weak Structures and w-spaces, Far East Journal of Mathematical Sciences, 97(5) (2015), 549-561.

[13] W. K. Min and Y. K. Kim; On Product Weak Structures and Product w-Spaces, International Journal of Mathematical Analysis, 9(35) (2015), 1715-1721.

[14] W. K. Min and Y. K. Kim; On generalized w-closed sets in w-spaces, International Journal of Pure and Applied Mathematics, 110(2) (2016), 327-335.

[15] O. Njastad; On some classes of nearly open sets, Pacific Journal of Mathematics, 15(3)(1964), 961-970.

[16] F. Siwiec; On Defining a Space by a Weak Base, Pacific Journal of Mathematics, 52(1) (1974), 351-357.