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REFLEXIVITY ON HILBERT FUNCTION SPACES

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Abstract: In this paper we present sufficient conditions for reflexivity of any powers of the multiplication operator acting on Hilbert spaces of analytic functions on a finitely connected domain. This improves the main result of [21].

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1. Introduction

By $H(G)$ and $H^{\infty}(G)$ we will mean respectively the set of analytic functions on a plane domain G and the set of bounded analytic functions on G. Assume that Ω is a finitely connected domain. It is well known that Ω is conformally equivalent to a circular domain. By a circular domain we mean any domain that is obtained by removing a finite number of disjoint closed subdisks from the open unit disk $\mathbb{D}$. So we let $\Omega = \mathbb{D}\backslash(\bar{D_1} \cup ... \cup \bar{D_N})$ where $\bar{D_i} = \{z : |z - z_i| \leq r_i\}$ $(i = 1, ..., N)$ are disjoint subdisks of the open unit disk $\mathbb{D}$. We can choose $\epsilon_i > 0$ $(i = 1, ..., N)$ such that the circles

$$\Gamma_i = \{z : |z - z_i| = r_i + \epsilon_i\} \ \ (i = 1, ..., N)$$

and $\Gamma_0 = \{z : |z| = 1 - \epsilon_0\}$ lying in Ω concentrate to the boundary circles of Ω

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so that they don't meet each other. We denote $\Omega_i = \mathbf{C}\backslash\bar{D}_i \ \ (i = 1, ..., N)$. In [3] it is proved that

$$H^\infty(\Omega) = H^\infty(\mathbb{D}) + H_0^\infty(\Omega_1) + ... + H_0^\infty(\Omega_N)$$

where the subscript zero means that the corresponding functions vanish at ∞.

Consider a Hilbert space $\mathcal{H}$ of functions analytic on a plane domain G, such that for each $\lambda \in G$ the linear functional, e_λ, of evaluation at λ is bounded on $\mathcal{H}$. Assume further that $\mathcal{H}$ contains the constant functions and multiplication by the independent variable z defines a bounded linear operator M_z on $\mathcal{H}$. The continuity of point evaluations along with the Riesz representation theorem imply that for each $\lambda \in G$ there is a unique function $k_\lambda \in \mathcal{H}$ such that $e_\lambda(f) = f(\lambda) =< f, k_\lambda >$, $f \in \mathcal{H}$. The function k_λ is called the *reproducing kernel* for the point λ.

A complex valued function φ on G for which $\varphi f \in \mathcal{H}$ for every $f \in \mathcal{H}$ is called a *multiplier* of $\mathcal{H}$ and the collection of all these multipliers is denoted by $\mathcal{M}(\mathcal{H})$. Each multiplier φ of $\mathcal{H}$ determines a multiplication operator M_φ on $\mathcal{H}$ by $M_\varphi f = \varphi f$, $f \in \mathcal{H}$. It is well known that each multiplier is a bounded analytic function on G (see [12]). In fact $\|\varphi\|_G \leq \|M_\varphi\|$, where

$$\|\varphi\|_G = sup\{|\varphi(z)| : \ z \in G\}.$$

We shall use the following notation for the norm of the operator M_φ:

$$\|\varphi\|_\infty = \|M_\varphi\|.$$

We also point out that if φ is a multiplier and $\lambda \in G$, then

$$M_\varphi^* k_\lambda = \overline{\varphi(\lambda)} k_\lambda.$$

Recall that if E is a separable Banach space and $A \in B(E)$, then $Lat(A)$ is by definition the *lattice of all invariant subspaces* of A, and $AlgLat(A)$ is the algebra of all operators B in $B(E)$ such that $Lat(A) \subset Lat(B)$. For the algebra $B(E)$, the *weak operator topology* is the one induced by the family of seminorms $p_{x^*,x}(A) = | < Ax, x^* > |$ where $x \in E$, $x^* \in E^*$ and $A \in B(E)$. Hence $A_\alpha \longrightarrow A$ in the weak operator topology if and only if $A_\alpha x \longrightarrow Ax$ weakly. Also similarly $A_\alpha \longrightarrow A$ in the *strong operator topology* if and only if $A_\alpha x \longrightarrow Ax$ in the norm topology. An operator A in $B(E)$ is said to be *reflexive* if $AlgLat(A) = W(A)$, where $W(A)$ is the smallest subalgebra of $B(E)$ that contains A and the identity I and is closed in the weak operator topology.

2. Main Results

The operator M_z has been the focus of attention for several decades and many of its properties have been studied (e.g. [2,12]). In [1o] Sarason proved that normal operators are reflexive. It was shown by J. Deddens (see [4]) that every isometry is reflexive. Also, R. Olin and J. Thomson (see [8]) have shown that subnormal operators are reflexive. H. Bercovici, C. Foias, J. Langsam, and C. Pearcy (see [1]) have shown that (BCP)-operators are reflexive. The reflexive operators on a finite dimensional space were characterized by J. Deddens and P. A. Fillmore (see [5]). In [7,11,14 – 21] some sufficient conditions for the reflexivity of multiplication operators on some function spaces have been investigated. Also, reflexivity of canonical models were studied in [6]. In [17] it is proved that if M_z is invertible on the space $L^p(\beta)$, then it is reflexive. In this article we would like to give some sufficient conditions so that the powers of the operator M_z, acting on a Hilbert space of analytic functions on a finitely connected domain, becomes reflexive. This extends the main result of the paper [21]. For a good source of reflexivity see [9].

From now on, let Ω be a finitely connected domain in the complex plane and suppose that the Hilbert space $\mathcal{H}$ under consideration satisfy the following axioms:

Axiom 1. $\mathcal{H}$ is a subspace of the space of all analytic functions on Ω.

Axiom 2. For each $\lambda \in \Omega$, the linear functional of evaluation at λ, $e(\lambda)$, is bounded on $\mathcal{H}$.

Axiom 3. The sequence $\{f_k\}_k$ is an orthogonal basis for $\mathcal{H}$ where $f_k(z) = z^k$ for all integers k.

For $h \in H(\mathbb{D}) \cap \mathcal{M}(\mathcal{H})$ and $w \in \partial\mathbb{D}$, define h_w by $h_w(z) = h(wz)$. Thus $\hat{h}_w(n) = w^n \hat{h}(n)$ for all n. Note that since $|w| = 1$, we have

$$\|h_w\|^2 = \sum_n |\hat{h}_w(n)|^2 \|f_n\|^2 = \sum_n |\hat{h}(n)|^2 \|f_n\|^2 = \|h\|^2.$$

Also, we say that $H(\mathbb{D}) \cap \mathcal{M}(\mathcal{H})$ is isometrically rotation invariant if whenever $\varphi \in H(\mathbb{D}) \cap \mathcal{M}(\mathcal{H})$, then $\varphi_{e^{-i\theta}} \in H(\mathbb{D}) \cap \mathcal{M}(\mathcal{H})$ and $\|\varphi\|_\infty = \|\varphi_{e^{-i\theta}}\|_\infty$ for all $\theta \in \mathbb{R}$.

Furthermore, we assume that $\mathcal{H}$ holds in the following axiom:

Axiom 4. $H(\mathbb{D}) \cap \mathcal{M}(\mathcal{H})$ is isometrically rotation invariant.

For the proof of the main result we will need the following lemmas.

Lemma 2.1. *Let* $\varphi \in H(\mathbb{D}) \cap \mathcal{M}(\mathcal{H})$. *Then:*

(i) If $w \longrightarrow 1$, *then* $M_{\varphi_w} \longrightarrow M_\varphi$ *in the strong operator topology.*

(ii) If g *is a continuous complex valued function on* $\partial\mathbb{D}$ *and* $d\lambda = |dw|/2\pi$ *is the normalized Lebesgue measure on* $\partial\mathbb{D}$, *then the operator*

$$\int_{\partial\mathbb{D}} \varphi_w g(w) d\lambda$$

defined by

$$\left(\int_{\partial\mathbb{D}} \varphi_w g(w) d\lambda\right) f = \int_{\partial\mathbb{D}} g(w) M_{\varphi_w} f d\lambda$$

is in $\mathcal{M}(\mathcal{H})$ *and*

$$\left\| \int_{\partial\mathbb{D}} \varphi_w g(w) d\lambda \right\|_\infty \leq \|M_\varphi\| \int_{\partial\mathbb{D}} |g| d\lambda.$$

Proof. (i) For $w \in \partial\mathbb{D}$ we have

$$\begin{aligned} \|(M_{\varphi_w} - M_\varphi) f_m\|^2 &= \|(\varphi_w - \varphi) f_m\|^2 = \|\sum_n (\hat{\varphi}_w(n) - \hat{\varphi}(n)) f_{n+m}\|^2 \\ &= \sum_n |\hat{\varphi}_w(n) - \hat{\varphi}(n)|^2 \|f_{n+m}\|^2 \\ &= \sum_n |\hat{\varphi}(n)|^2 |w^n - 1| \, \|f_{n+m}\|^2. \end{aligned}$$

Thus for all m, $M_{\varphi_w} f_m \longrightarrow M_\varphi f_m$ as $w \longrightarrow 1$. Since $\|M_{\varphi_w}\| = \|M_\varphi\| < \infty$ for all $w \in \partial\mathbb{D}$, indeed $M_{\varphi_w} \longrightarrow M_\varphi$ in the strong operator topology.
(ii) First note that the strong operator continuity of φ_w allows us to define

$$\int_{\partial\mathbb{D}} \varphi_w g(w) f d\lambda$$

for all $f \in \mathcal{H}$. If $f, h \in \mathcal{H}$, then

$$< \int_{\partial\mathbb{D}} \varphi_w g(w) f d\lambda, h >= \int_{\partial\mathbb{D}} g(w) < \varphi_w f, h > d\lambda.$$

Since $\|M_\varphi\| = \|M_{\varphi_w}\|$, we get

$$\left\| \int_{\partial\mathbb{D}} \varphi_w g(w) f d\lambda \right\| \leq \|M_\varphi\| \, \|f\| \int_{\partial\mathbb{D}} |g| d\lambda.$$

Hence

$$\|(\int_{\partial\mathbb{D}} \varphi_w g(w) d\lambda) f\| = \| \int_{\partial\mathbb{D}} g(w) M_{\varphi_w} f d\lambda\| \leq \|M_\varphi\| \, \|f\| \int_{\partial\mathbb{D}} |g| d\lambda.$$

This completes the proof. □

Throughout this paper we suppose that M_z is bounded on $\mathcal{H}$. In the following by $H(G)$ and $H^\infty(G)$ we will mean respectively the set of analytic functions on a plane domain G and the set of bounded analytic functions on G.

Lemma 2.2. *Let $\varphi \in H(\mathbb{D}) \cap \mathcal{M}(\mathcal{H})$ and let p be a polynomial. Then $\varphi * p \in \mathcal{M}(\mathcal{H})$ and*

$$\int_{\partial\mathbb{D}} \varphi_w p(\bar{w}) d\lambda = M_{\varphi * p}$$

where

$$(\varphi * p)(z) = \sum_i \hat{\varphi}(i) \hat{p}(i) f_i.$$

Proof. It is enough to consider the case $p = f_j$. Define the operator L by

$$L = \int_{\partial\mathbb{D}} \varphi_w f_j(\bar{w}) d\lambda.$$

We should prove that the operators L and $M_{\hat{\varphi}(j) f_j}$ have the same matrix entries with respect to the orthogonal basis $\{f_j\}_j$. We have

$$< M_{\hat{\varphi}(j) f_j} f_m, f_n >= \hat{\varphi}(j) < f_{m+j}, f_n >$$

which is equal to $\hat{\varphi}(n-m)\|f_n\|^2$ whenever $n = m + j$, and is 0 else. On the other hand we note that

$$\begin{aligned} < L f_m, f_n > &= \int_{\partial\mathbb{D}} \bar{w}^j < \varphi_w f_m, f_n > d\lambda \\ &= \int_{\partial\mathbb{D}} \bar{w}^j \hat{\varphi}_w(n-m) \|f_n\|^2 d\lambda \\ &= \int_{\partial\mathbb{D}} \bar{w}^j w^{n-m} \hat{\varphi}(n-m) \|f_n\|^2 d\lambda \\ &= \hat{\varphi}(n-m) \|f_n\|^2 \int_{\partial\mathbb{D}} w^{n-m-j} d\lambda \end{aligned}$$

which is equal to $\hat{\varphi}(n-m)\|f_n\|^2$ whenever $n = m + j$, and is 0 else. Hence $L = M_{\hat{\varphi}(j) f_j}$. Since $\{f_j\}_j$ is a basis for $\mathcal{H}$, the proof is complete. □

Lemma 2.3. *If* $\varphi \in H(\mathbb{D}) \cap \mathcal{M}(\mathcal{H})$*, then for the sequence of polynomials* $\{r_n\}$ *where* $\hat{r_n}(j) = (1 - \frac{j}{n+1})\hat{\varphi}(j)$ *whenever* $j = 0, ..., n$ *and is* 0 *else, we have* $M_{r_n} \to M_\varphi$ *in the weak operator topology.*

Proof. Let $\varphi \in H(\mathbb{D}) \cap \mathcal{M}(\mathcal{H})$. Since $\mathbb{D}$ is a Caratheodory domain, φ can be represenyed by the power series $\sum_{k=0}^{\infty} \hat{\varphi}(k)z^k$. Put

$$P_n(\varphi) = \sum_{k=0}^{n}(1 - \frac{k}{n+1})\hat{\varphi}(k)z^k, \qquad n \geq 0$$

and

$$K_n(w) = \sum_{|k| \leq n}(1 - \frac{|k|}{n+1})w^k, \qquad w \in \partial U, \ \ n \geq 0.$$

Then

$$\int_{\partial\mathbb{D}} \varphi_w K_n(\bar{w})d\lambda = M_{\varphi * K_n}, \qquad n \geq 0$$

where

$$(\varphi * K_n)(z) = \sum_{j=0}^{n} \hat{\varphi}(j)\hat{K}_n(j)z^j = P_n(\varphi).$$

Note that $K_n \geq 0$ and

$$\int_{\partial\mathbb{D}} K_n d\lambda = 1.$$

For all $n \geq 0$, $P_n(\varphi) \in H(\mathbb{D}) \cap \mathcal{M}(\mathcal{H})$ and by Lemma 2.1 (ii), we get

$$||M_{P_n(\varphi)}|| = ||M_{\varphi * K_n}|| \leq ||M_\varphi|| \int_{\partial\mathbb{D}} K_n d\lambda = ||M_\varphi||.$$

Put $r_n = P_n(\varphi)$ and note that M_{r_n} is represented by the matrix whose (i,j)-th entry is

$$< M_{r_n} f_j, f_i >= \hat{r}_n(i-j)\|f_i\|^2 = (1 - \frac{i-j}{n})\hat{\varphi}(i-j)\|f_i\|^2.$$

Hence

$$\lim_n < M_{r_n} f_j, f_i >=< M_\varphi f_j, f_i >$$

for all base elements f_j and f_i in $\mathcal{H}$. By the boundedness of the sequence $\{M_{r_n}\}$, we have $M_{r_n} \to M_\varphi$ in the weak operator topology. This completes the proof. □

Corollary 2.4. *If* $\varphi \in H(\mathbb{D}) \cap \mathcal{M}(\mathcal{H})$, *then* $M_\varphi \in W(M_z)$.

Proof. Let $\varphi \in H(\mathbb{D}) \cap \mathcal{M}(\mathcal{H})$. Then by the proof of Lemma 2.3, we get $M_{r_n} \to M_\varphi$ in the weak operator topology where $r_n = P_n(\varphi)$. Since r_n is a polynomial and $M_{r_n} = r_n(M_z)$, we conclude that $M_\varphi \in W(M_z)$. □

Theorem 2.5. *For all* $k \geq 1$, *the operator* M_{z^k} *is reflexive on* $\mathcal{H}$.

Proof. $\Omega = \mathbb{D} \backslash (\bar{D_1} \cup ... \cup \bar{D_N})$ where $\bar{D_i} = \{z : |z - z_i| \leq r_i\}$ $(i = 1, ..., N)$ are disjoint subdisks of the open unit disk $\mathbb{D}$. Choose $\epsilon_i > 0$ $(i = 1, ..., N)$ such that the circles

$$\Gamma_i = \{z : |z - z_i| = r_i + \epsilon_i\} \quad (i = 1, ..., N)$$

and $\Gamma_0 = \{z : |z| = 1 - \epsilon_0\}$ lying in Ω concentrate to the boundary circles of Ω so that they don't meet each other. Denote $\Omega_i = \mathbf{C} \backslash \bar{D_i}$ $(i = 1, ..., N)$.

First, we note that convergence in $\mathcal{H}$ implies uniform convergence on compact subsets of Ω. For this let K be a compact subset of Ω and consider the family of bounded linear functionals $\{e_\lambda : \lambda \in K\}$. If $f \in \mathcal{H}$, then $\|f\|_K < \infty$. So by the Principle of Uniform Boundedness Theorem, the family $\{e_\lambda : \lambda \in K\}$ is bounded. Put $c = sup\{\|e_\lambda\| : \lambda \in K\}$ and let a sequence $\{f_n\}_n$ converges to f in $\mathcal{H}$. Then we have

$$|f_n(\lambda) - f(\lambda)| \leq \|e_\lambda\| \, \|f_n - f\| \leq c\|f_n - f\|.$$

Hence convergence in $\mathcal{H}$ implies uniform convergence on compact subsets of Ω. Set

$$L_0 = \{f \in \mathcal{H} : \int_{\Gamma_0} z^n f(z) dz = 0, \quad n = 0, 1, 2, \cdots\}.$$

Note that L_0 is a subspace of $\mathcal{H}$. To see that L_0 is closed, let $\{g_k\}$ be a sequence in L_0 such that $g_k \to h$ in $\mathcal{H}$. Since Γ_0 is a compact subset of Ω, it is now easy to see that $h \in L_0$ and so L_0 is closed in $\mathcal{H}$. Also, clearly L_0 is invariant under M_z and contains the constants. Let $k \in \mathbb{N}$ and note that $W(M_{z^k}) \subset AlgLat(M_{z^k})$. On the other hand, let $A \in AlgLat(M_{z^k})$. Since $Lat(M_z) \subset Lat(M_{z^k})$, thus we have $Lat(M_z) \subset Lat(A)$. This implies that $A \in AlgLat(M_z)$. Note that since $M_z^* e(\lambda) = \overline{\lambda} e(\lambda)$ for all λ in Ω, the one dimensional span of $e(\lambda)$ is invariant under M_z^*. Therefore it is invariant under A^* and we can write $A^* e(\lambda) = \overline{\varphi(\lambda)} e(\lambda)$, $\lambda \in \Omega$. So

$$< Af, e(\lambda) >=< f, A^* e(\lambda) >= \varphi(\lambda) f(\lambda)$$

for all $f \in \mathcal{H}$ and $\lambda \in \Omega$. This implies that $A = M_\varphi$ and $\varphi \in \mathcal{M}(\mathcal{H})$, hence $\varphi \in H^\infty(\Omega)$. Since $L_0 \in Lat(M_z)$, we have $AL_0 \subset L_0$, so $A1 = \varphi \in L_0$. By applying the Cauchy integral formula we can write $\varphi = \varphi_0 + \varphi_1 + ... + \varphi_N$ where $\varphi_0 \in H(\mathbb{D})$ and $\varphi_i \in H_0(\Omega_i)$ (i=1,...,N) (here $H_0(\Omega_i)$ denotes the space of all functions in $H(\Omega_i)$ that vanishes at ∞). Set $g = \varphi_1 + ... + \varphi_N$. Therefore g is analytic in $ext(\Gamma'_0)$ the unbounded component of $\mathbb{C} \setminus \Gamma'_0$ where the circle Γ'_0 is chosen sufficiently close to Γ_0 with smaller radius so that Γ_0 lies in $ext(\Gamma'_0)$. Now, we can write

$$g(z) = \sum_{n=-\infty}^{-1} \hat{g}(n) z^n, \quad z \in ext(\Gamma'_0).$$

Note that

$$\hat{g}(n) = \frac{1}{2\pi i} \int_{\Gamma_0} g(z) z^{-(n+1)} dz, \quad n < 0.$$

Since $\varphi_0 \in H(\mathbb{D})$, we have

$$\int_{\Gamma_0} z^n \varphi_0(z) dz = 0, \qquad n = 0, 1, , 2, \cdots .$$

But $\varphi \in L_0$, thus we get

$$\int_{\Gamma_0} z^n g(z) dz = 0, \quad n = 0, 1, 2, \cdots .$$

From this it follows that $\hat{g}(n) = 0$ for all integers $n \leq -1$ and so $g(z) = 0$, $z \in ext(\Gamma'_0)$. Hence $g \equiv 0$ which implies that $\varphi = \varphi_0 \in H(\mathbb{D})$. Thus $\varphi \in H(\mathbb{D}) \cap \mathcal{M}(\mathcal{H})$ and so by Lemma 2.3, there exists a sequence of polynomials $\{r_n\}$ such that $M_{r_n} \to M_\varphi$ in the weak operator topology. Now, we use a similar method used in the proof of the main theorem in [21]: let $\mathcal{M}_k$ be the closed linear span of the set $\{f_{nk} : n \geq 0\}$ (recall that $f_i(z) = z^i$ for all i). We have $M_{z^k} f_{nk} = f_{(n+1)k} \in \mathcal{M}_k$ for all $n \geq 0$. Thus $\mathcal{M}_k \in Lat(M_{z^k})$, and so $\mathcal{M}_k \in Lat(M_\varphi)$. Let $\varphi(z) = \sum_{n=0}^{\infty} \hat{\varphi}(n) z^n$. Since $1 \in \mathcal{M}_k$, thus $M_\varphi 1 = \varphi \in \mathcal{M}_k$. Hence $\hat{\varphi}(i) = 0$ for all $i \neq nk$, $n \geq 0$. Now, by a consequence of the particular construction of r_n used in Lemma 2.3, each r_n should be a polynomial in z^k, i.e., $r_n(z) = q_n(z^k)$ for some polynomial q_n. Thus $M_{r_n} = r_n(M_z) = q_n(M_{z^k}) \to A$ in the weak operator topology. Hence $A \in W(M_{z^k})$. Thus M_{z^k} is reflexive and so the proof is complete. $\square$

In the following we give an example of a Hilbert space for which the axioms 1 through 4, hold.

Example 2.6. Let $\{\beta(n)\}_{n=-\infty}^{\infty}$ be a sequence of positive numbers satisfying $\beta(0) = 1$. The space $L^2(\beta)$ consists of all formal Laurent series $f(z) = \sum_{n=-\infty}^{\infty} \hat{f}(n)z^n$ such that the norm

$$\|f\| = \|f\|_\beta = (\sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2\beta(n)^2)^{\frac{1}{2}}$$

is finite. These are Hilbert spaces with the norm $\|\cdot\|_\beta$, see [13]. Let $\hat{f}_k(n) = \delta_k(n)$, so $f_k(z) = z^k$ and $\|f_k\| = \beta(k)$. Let M_z be bounded on $L^2(\beta)$ and consider the following notations:

$$\begin{aligned} r_0 = \overline{\lim}\beta(-n)^{\frac{-1}{n}} \quad &; \quad \Omega_0 = \{z \in \mathbf{C} : |z| > r_0\} \\ r_1 = \underline{\lim}\beta(n)^{\frac{1}{n}} \quad &; \quad \Omega_1 = \{z \in \mathbf{C} : |z| < r_1\} \\ \Omega = \Omega_0 \cap \Omega_1 \quad &= \quad \{z \in \mathbf{C} : r_0 < |z| < r_1\}. \end{aligned}$$

Assume that $r_0 < r_1 = 1$. Then, Ω is clearly a circular domain and $L^2(\beta) \subset H(\Omega)$ (see [13, Theorem 10′(ii), page 79]), so Axiom 1 holds. Also, each point of Ω is a bounded point evaluation on $L^2(\beta)$ (see [13, Theorem 10′(ii), page 79), so Axiom 2 holds. By proposition 6 in [13, page 57], $\{f_k\}_{k\in\mathbf{Z}}$ is an orthogonal basis for $L^2(\beta)$ and so Axiom 3 holds. Furthermore, Axiom 4 holds by Proposition 28 in [13, page 88].

Corollary 2.7. *For all $k \geq 1$, the operator M_{z^k} is reflexive on $L^2(\beta)$.*

Corollary 2.8. *Let M_z be invertible on $L^2(\beta)$. Then M_{z^k} is reflexive for all integers k.*

Proof. By Corollary 2.7, M_{z^k} is reflexive for all positive integers k. Note that since $M_z f_m = f_{m+1}$, we have $M_z^{-1} f_m = f_{m-1}$ for all m. Let $f'_m = f_{-m}$. Then $M_z^{-1} f'_m = f'_{m+1}$ for all m. So $\{f'_m\}$ is shifted (forward) by M_z^{-1}. Hence M_z^{-k} is reflexive for all $k \geq 1$. But the identity operator is also reflexive, thus indeed M_{z^k} is reflexive for all integers k. □

Remark 2.9. Theorem 2.5 can be extended by the same proof to a Banach space setting instead of a Hilbert space and so the main result of [21] is clearly obtained for the Banach spaces $L^p(\beta)$ where $1 < p < \infty$.

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