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SUBSPACE SUPERCYCLICITY OF THE TUPLES OF OPERATORS

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Abstract: In this paper, we investigate subspace supercyclicity for tuples of operators and we give conditions under which a tuple of operators satisfying the subspace supercyclicity criterion.

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1. Introduction

By an n-tuple of operators we mean a finite sequence of length n of commuting continuous linear operators on a Banach space X.

Definition 1.1. Let $\mathcal{T} = (T_1, T_2, ..., T_n)$ be an n-tuple of operators acting on a separable infinite dimensional Banach space X over $\mathbb{C}$ and let M be a nonzero subspace of X. We will let $\mathcal{F} = \{{T_1}^{k_1}{T_2}^{k_2}...{T_n}^{k_n} : k_i \geq 0, i = 1, ..., n\}$ be the semigroup generated by $\mathcal{T}$. For $x \in X$, the orbit of x under the tuple $\mathcal{T}$ is the set $Orb(\mathcal{T}, x) = \{Sx : S \in \mathcal{F}\}$. A vector x is called a M-supercyclic (M-hypercyclic) vector for $\mathcal{T}$ if $\mathbb{C}Orb(\mathcal{T}, x) \cap M$ ($Orb(\mathcal{T}, x) \cap M$) is dense in M and in this case the tuple $\mathcal{T}$ is called M-supercyclic (M-hypercyclic). The

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set of all M-supercyclic vectors of $\mathcal{T}$ is denoted by $SC(\mathcal{T}, M)$. Also, for all $k \geq 2$, by $\mathcal{T}_d^{(k)}$ we will refer to the set of all k copies of an element of $\mathcal{F}$, i.e. $\mathcal{T}_d^{(k)} = \{S_1 \oplus ... \oplus S_k : S_1 = ... = S_k \in \mathcal{F}\}$. We say that $\mathcal{T}_d^{(k)}$ is subspace-supercyclic, with respect to M, provided there exist $x_1, ..., x_k \in X$ such that $\mathbf{C}\{W(x_1 \oplus ... \oplus x_k) : W \in \mathcal{T}_d^{(k)}\} \cap M$ is dense in the k copies of M, $M \oplus ... \oplus M$.

Suprisingly, there are something that does not happen for single operators. For example, supercyclic tuples can arise in finite dimensional, and there are operators that have somewhere dense orbits that are not everywhere dense. Also, we note that there are subspace-supercyclic operators that are not supercyclic. For some topics we refer to [1]-[2].

2. Main Results

In this section, we investigate an equivalent condition for subspace-supercyclicity criterion for tuples of operators.

Theorem 2.1. *Let M be a nonzero subspace of separable infinite dimensional Banach space X and $\mathcal{T} = (T_1, T_2, ..., T_n)$ be a tuple of operators T_1, T_2, ..., T_n. Then $\mathcal{T}$ is M-supercyclic if and only if for any two nonempty relatively open sets U and V in M, there exist $m_i \geq 1$ for $i = 1, ..., n$ and and $\lambda \in \mathbf{C}\backslash\{0\}$ such that $\lambda T_1^{m_1} ... T_n^{m_n}(U) \cap V$ is nonempty and $T_1^{m_1} ... T_n^{m_n} M \subset M$.*

Proof. Suppose that x is a M-supercyclic vector for $\mathcal{T}$, hence $\mathbf{C}Orb(\mathcal{T}, x) \cap M$ is dense in M and so there exist $\lambda_1 \in \mathbf{C}\backslash\{0\}$ and $m_i \in \mathbb{N}$; $i = 1, ..., n$ such that $\lambda_1 T_1^{m_1} ... T_n^{m_n} x \in U$. Let $y = \lambda_1 T_1^{m_1} ... T_n^{m_n} x$. Since $\mathbf{C}Orb(\mathcal{T}, x) \cap M$ is also dense in M, thus $\lambda T_1^{m_1} ... T_n^{m_n} y \in V$ for some $\lambda \in \mathbf{C}\backslash\{0\}$ and $m_i \in \mathbb{N}$; $i = 1, ..., n$, hence $\lambda T_1^{m_1} ... T_n^{m_n}(U) \cap V$ is nonempty. To show that $T_1^{m_1} T_2^{m_2} ... T_n^{m_n} M \subset M$, let $x \in M$ and let W be a relatively open nonempty subset of $\lambda^{-1} T_1^{-m_1} ... T_n^{-m_n}(V) \cap U$ in M. Note that

$$\lambda T_1^{m_1} T_2^{m_2} ... T_n^{m_n} W \subset V \cap \lambda T_1^{m_1} T_2^{m_2} ... T_n^{m_n} U \subset V \subset M.$$

Hence $T_1^{m_1} T_2^{m_2} ... T_n^{m_n} W \subset M$. If $x_0 \in W$, then there exists $r > 0$ small enough such that $x_0 + rx \in W$, since W is relatively open. Thus

$$T_1^{m_1} T_2^{m_2} ... T_n^{m_n}(x_0 + rx) = T_1^{m_1} T_2^{m_2} ... T_n^{m_n} x_0 + r T_1^{m_1} T_2^{m_2} ... T_n^{m_n} x \in M,$$

which implies that $T_1^{m_1} T_2^{m_2} ... T_n^{m_n} x \in M$. Thus $T_1^{m_1} T_2^{m_2} ... T_n^{m_n} M \subset M$. For the converse, fix an enumeration $\{B_n : n \in \mathbb{N}\}$ of relatively open balls in M with rational radii, and centers in a countable dense subset of M. By the continuity of the operators T_1, T_2,..., T_n, the sets

$$G_k = \bigcup\{\lambda T_1^{-m_1}...T_n^{-m_n}(B_k) \cap M : m_i \geq 0; i = 1, ..., n, \lambda \in \mathbf{C}\}$$

are relatively open in M. Since for any two nonempty relatively open sets U and V in M, there exist $m_i \geq 1$; $i = 1, ..., n$, and $\lambda \in \mathbf{C}\backslash\{0\}$ such that $\lambda T_1^{m_1}...T_n^{m_n}(U) \cap V$ contains a relatively open nonempty subset of M, thus one can see that $SC(\mathcal{T})$ is exactly equal to $\bigcap\{G_k : k \in \mathbb{N}\}$ that is dense in M. □

Theorem 2.2. (Subspace-Supercyclicity Criterion for Tuples) *Let* $\mathcal{T} = (T_1, T_2, ..., T_n)$ *be a tuple of continuous operators acting on a separable infinite dimensional Banach space* X *and* M *be a nonzero closed subspace of* X. *Suppose that there exist two dense subsets* Y *and* Z *in* M, *and strictly increasing sequences of positive integers* $\{m_{j(i)}\}_j$ *for* $i = 1, ..., n$, *and a sequence of mappings* $S_j : Z \to M$ *such that:*

1. *For every* $z \in Z$, $T_1^{m_{j(1)}}...T_n^{m_{j(n)}}S_jz \to z$,
2. $\|T_1^{m_{j(1)}}...T_n^{m_{j(n)}}y\|\|S_jz\| \to 0$ *for all* $y \in Y$ *and all* $z \in Z$,
3. M *is invariant subspace for* $T_1^{m_{j(1)}}...T_n^{m_{j(n)}}$ *for all* j.

Then $\mathcal{T}$ *is subspace-supercyclic with respect to* M.

Proof. Let U and V be two nonempty relatively open subsets of M. Choose $v \in Y \cap V$ and $u \in Z \cap U$. Since U and V are relatively open, there exists $\epsilon > 0$ such that $B(v, \epsilon) \cap M \subset V$ and $B(u, \epsilon) \cap M \subset U$. By the hypothesis we have:
(i). $T_1^{m_{j(1)}}...T_n^{m_{j(n)}}S_ju \to u$, (ii). $\|T_1^{m_{j(1)}}...T_n^{m_{j(n)}}v\|\|S_ju\| \to 0$,
(iii). M is invariant subspace for $T_1^{m_{j(1)}}...T_n^{m_{j(n)}}$ for all j.
Hence for large j, we can find $\lambda_j \in \mathbf{C}\backslash\{0\}$ such that

$$\|\lambda_jT_1^{m_{j(1)}}...T_n^{m_{j(n)}}v\| < \epsilon/2, \ \|\lambda_j^{-1}S_ju\| < \epsilon, \ \|T_1^{m_{j(1)}}...T_n^{m_{j(n)}}S_ju - u\| < \epsilon/2.$$

Note that since $v \in M$ and $S_ju \in M$, we have $v + \lambda_j^{-1}S_ju \in M$ and $\|(v + \lambda_j^{-1}S_ju) - v\| = \|\lambda_j^{-1}S_ju\| < \epsilon$. Hence $v + S_ju \in V$. Also we note that $T_1^{m_{j(1)}}...T_n^{m_{j(n)}}(v + \lambda_j^{-1}S_ju) \in M$ and $\|\lambda_jT_1^{m_{j(1)}}...T_n^{m_{j(n)}}(v + \lambda_j^{-1}S_ju) - u\| < \epsilon$. Thus $\lambda_jT_1^{m_{j(1)}}...T_n^{m_{j(n)}}(v + \lambda_j^{-1}S_ju) \in B(u, \epsilon) \cap M$ and so $\lambda_jT_1^{m_{j(1)}}...T_n^{m_{j(n)}}(v + \lambda_j^{-1}S_ju) \in U$. Hence $v + \lambda_j^{-1}S_ju \in \lambda_j^{-1}T_1^{-m_{j(1)}}...T_n^{-m_{j(n)}}(U) \cap V$ and so by Theorem 2.1, $\mathcal{T}$ is subspace-supercyclic with respect to M. □

Theorem 2.3. *Let M be a nonzero subspace of a separable infinite dimensional Banach space X and $\mathcal{T} = (T_1, T_2, ..., T_n)$ be a tuple of operators $T_1, T_2, ..., T_n$. If $\mathcal{T}_d^{(2)}$ is $M \oplus M$-supercyclic, then $\mathcal{T}$ satisfies the Subspace-Supercyclicity Criterion with respect to M.*

Proof. Suppose that $\mathcal{T}_d^{(2)}$ is $M \oplus M$-supercylcic and let (x, y) be a $M \oplus M$-supercyclic vector for $\mathcal{T}_d^{(2)}$. In particular x and y are supercyclic vectors for $\mathcal{T}$. For all $k \in \mathbb{N}$, put $U_k = B(0, \frac{1}{k}) \cap M$. Then there exist $\{m_{ji}\}_j \subset \mathbb{N}$ for $i = 1, ..., n$ and $\lambda_j \in \mathbb{C}$ such that

$$\lambda_j (T_1^{m_{j1}} T_2^{m_{j2}} ... T_n^{m_{jn}} \oplus T_1^{m_{j1}} T_2^{m_{j2}} ... T_n^{m_{jn}})(x, y) \in U_j \oplus (x + U_j),$$

$$T_1^{m_{j1}} T_2^{m_{j2}} ... T_n^{m_{jn}} \oplus T_1^{m_{j1}} T_2^{m_{j2}} ... T_n^{m_{jn}} M \oplus M \subset M \oplus M.$$

Therefore, $\lambda_j T_1^{m_{j1}} T_2^{m_{j2}} ... T_n^{m_{jn}} x \in U_j$, $\lambda_j T_1^{m_{j1}} T_2^{m_{j2}} ... T_n^{m_{jn}} y \in x + U_j$, and $T_1^{m_{j1}} T_2^{m_{j2}} ... T_n^{m_{jn}} M \subset M$ for all $j \in \mathbb{N}$. This implies that

$$\lambda_j T_1^{m_{j1}} T_2^{m_{j2}} ... T_n^{m_{jn}} x \to 0, \ \lambda_j T_1^{m_{j1}} T_2^{m_{j2}} ... T_n^{m_{jn}} y \to x.$$

Let $Y = Z = \mathbb{C}Orb(\mathcal{T}, x)$ which is dense in M. Also for all $j \in \mathbb{N}, \lambda \in \mathbb{C}$ and $k_{r_i} \in \mathbb{N}$ $forall$ $i = 1, ..., n$, define $S_j(\lambda T_1^{k_{r1}} ... T_n^{k_{rn}} x) = \lambda \lambda_j T_1^{k_{r1}} ... T_n^{k_{rn}} y$. Note that $T_1^{m_{j1}} ... T_n^{m_{jn}} S_j(\lambda T_1^{k_{r1}} ... T_n^{k_{rn}} x) = \lambda T_1^{k_{r1}} ... T_n^{k_{rn}} (\lambda_j T_1^{m_{j1}} ... T_n^{m_{jn}} y)$ which tends to $\lambda T_1^{k_{r1}} T_2^{k_{r2}} ... T_n^{k_{rn}} x$ as $j \to \infty$. So $T_1^{m_{j1}} T_2^{m_{j2}} ... T_n^{m_{jn}} S_j z \to z$ for all $z \in Z$. Also for all $\lambda, w \in \mathbb{C}$ and t_{s_i}, $k_{r_i} \in \mathbb{N}$ for $i = 1, ..., n$, clearly we can see that $||T_1^{m_{j1}} T_2^{m_{j2}} ... T_n^{m_{jn}} (\lambda T_1^{t_{s1}} T_2^{t_{s2}} ... T_n^{t_{sn}} x)|| \ ||S_j(w T_1^{k_{r1}} T_2^{k_{r2}} ... T_n^{k_{rn}} x)|| \to 0$ as $j \to \infty$. Thus for all $y \in Y$ and $z \in Z$, we get $||T_1^{m_{j1}} T_2^{m_{j2}} ... T_n^{m_{jn}} y|| \ ||S_j z|| \to 0$ and so $\mathcal{T}$ satisfies the M-Supercyclicity Criterion. □

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