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APPROXIMATION BY EXPONENTIAL TYPE
VECTORS OF POSITIVE OPERATORS

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Abstract: We establish the estimations of best approximations of elements of a Banach space by exponential type vectors associated with the positive operator. The corresponding estimations are expressed in terms of quasi-norms of the approximation spaces as Bernstein-Jackson-type inequalities. Such inequalities are applied to spectral approximations in the case of the positive operator with the point spectrum.

1. Introduction and preliminaries

One of the problems in the approximation theory is to characterize the approximation of elements of Banach space by different classes of smooth vectors associated with a closed unbounded operator. A characterization of some classes of infinitely differentiable vectors of a normal operator in a Hilbert space is given in terms of the rate with which the best approximation of the vectors by entire exponential type vectors tend to zero [1]. The spaces of exponential type entire vectors associated with the closed unbounded linear operator in a Banach space are defined in [2]. The direct and inverse theorems in the theory of approximations are established in [3] and [4].

The motivation of our work is to investigate a best approximation problem by invariant subspaces of exponential type vectors of a positive operator in a Banach space. Such problem for any closed unbounded linear operator has been studied in [5] and [6]. In this paper, we define the Lorentz-type interpolation spaces of exponential type vectors of positive operators and appropriate approximation spaces. Interpolation properties of such spaces are explored (Theorems 1, 2).

The inequalities estimating the minimal distance from a given element to a subspace of exponential type vectors are established (Theorem 4). Such inequalities are extended to cases of best spectral approximations of the positive operator in a Banach space (Theorem 5).

In a Banach complex space $(\mathsf{X}, \|\cdot\|)$ we consider a positive operator A with the norm dense domain $\mathsf{D}^1(A)$. It means that $(-\infty, 0] \in \rho(A)$ and there exists a number $c > 0$ such that $\|(A - \lambda I)^{-1}\| \leq c/(1 + |\lambda|)$ for all $\lambda \in (-\infty, 0]$ ([7], Section 1.14.1). Let $\mathsf{D}^{k+1}(A) = \{x \in \mathsf{D}^k(A) \colon A^k x \in \mathsf{D}(A)\}$ and $\mathsf{D}^\infty(A) = \bigcap_{k \in \mathbb{N}} \mathsf{D}^k(A)$. We accept that A^0 is the unit operator on X.

Recall the real (K-functional) and complex interpolation methods (for more details see [7] and [8]). Let $0 < \theta < 1$ and $1 \leq q \leq \infty$. For the quasi-normed spaces X_0 and X_1 we define the interpolation space $(X_0, X_1)_{\theta,q} = \{x \in X_0 + X_1 \colon |x|_{(X_0,X_1)_{\theta,q}} < \infty\}$ with the quasi-norm

$$|x|_{(X_0,X_1)_{\theta,q}} = \begin{cases} \left(\int_0^\infty \left[t^{-\theta} K(t, x; X_0, X_1)\right]^q \frac{dt}{t}\right)^{1/q}, & 1 \leq q < \infty, \\ \sup_{0 < t < \infty} t^{-\theta} K(t, x; X_0, X_1), & q = \infty. \end{cases}$$

where $K(t, x; X_0, X_1) = \inf\limits_{x = x^0 + x^1} \left(|x^0|_{X_0} + t\,|x^1|_{X_1}\right)$, $t > 0$.

For any Banach spaces X_0 and X_1 we define the interpolation space $[X_0, X_1]_\theta = \{x \in X_0 + X_1 \colon \exists\, f(z) \in \mathcal{F}(X_0, X_1), f(\theta) = x\}$ with the norm $\|x\|_{[X_0,X_1]_\theta} = \inf\limits_{f(\theta) = x} \|f(z)\|_{\mathcal{F}(X_0,X_1)}$, where inf takes on all functions $f \in \mathcal{F}(X_0, X_1)$ such that $f(\theta) = x$. By $\mathcal{F}(X_0, X_1)$ we denote the space of $(X_0 + X_1)$-valued functions with the norm $\|f\|_{\mathcal{F}(X_0,X_1)} = \max\left(\sup_t \|f(it)\|_{X_0}, \sup_t \|f(1 + it)\|_{X_1}\right)$, $f(it) \in X_0$, $f(1 + it) \in X_1$ for all $-\infty < t < \infty$.

2. Lorentz-Type Spaces

Let $m, k \in \mathbb{Z}$, $m \geq 0$. It is well known ([7], Section 1.15.1) that for all $\alpha \in \mathbb{C}$ such that $-m < \operatorname{Re}\alpha \leq \sigma - m$, $0 < \sigma < k$ and all $x \in \left(\mathsf{X}, \mathsf{D}^k(A)\right)_{\sigma/k,1}$ is defined

an operator

$$A_\sigma^\alpha x = \frac{\Gamma(k)}{\Gamma(\alpha+m)\Gamma(k-m-\alpha)} \int_0^\infty t^{\alpha+m-1} A^{k-m}(A+tI)^{-k}x\, dt.$$

The operator A_σ^α has a closure A^α in X which is independent of σ. The domain $\mathsf{D}^\alpha := \mathsf{D}^\alpha(A)$ of A^α we consider as a Banach space with the norm $\|x\|_{\mathsf{D}^\alpha} = \|A^\alpha x\|$, $x \in \mathsf{D}^\alpha$.

Let $x_{k,\nu} := (A/\nu)^k x$ ($\nu > 0$) and a sequence $(x_{k,\nu}^*)_{k=0}^\infty$ consists of elements $x_{k,\nu}$ which are numbered in according to decrease of norms

$$\|x_{0,\nu}^*\|_{\mathsf{D}^\alpha} \geq \|x_{1,\nu}^*\|_{\mathsf{D}^\alpha} \geq \ldots \geq \|x_{k,\nu}^*\|_{\mathsf{D}^\alpha} \geq \ldots.$$

For any $0 < \nu < \infty$ and $1 \leq q, p \leq \infty$ we define the space $\mathcal{E}_{q,p}^{\nu,\alpha}(A) = \left\{ x \in \mathsf{D}^\infty(A) \colon \|x\|_{\mathcal{E}_{q,p}^{\nu,\alpha}(A)} < \infty \right\}$, where

$$\|x\|_{\mathcal{E}_{q,p}^{\nu,\alpha}(A)} = \begin{cases} \left(\sum_{k \in \mathbb{N}} \|x_{k-1,\nu}^*\|_{\mathsf{D}^\alpha}^p k^{\frac{p}{q}-1} \right)^{1/p}, & 1 \leq p < \infty, \\ \sup_{k \in \mathbb{N}} \|x_{k-1,\nu}^*\|_{\mathsf{D}^\alpha} k^{\frac{1}{q}}, & p = \infty. \end{cases}$$

We can call the space $\mathcal{E}_{q,p}^{\nu,\alpha}(A)$ endowed with the norm $\|x\|_{\mathcal{E}_{q,p}^{\nu,\alpha}(A)}$ a Lorentz-type space of exponential type entire vectors of A. If $q = p$ then $\mathcal{E}_{q,q}^{\nu,\alpha}(A) = \mathcal{E}_q^{\nu,\alpha}(A)$ and $\mathcal{E}_{1,1}^{\nu,\alpha}(A) = \mathcal{E}_1^{\nu,\alpha}(A)$, $\mathcal{E}_{\infty,\infty}^{\nu,\alpha}(A) = \mathcal{E}_\infty^{\nu,\alpha}(A)$.

Applying [6, Theorem 2.1] for $1 < q < \infty$ and $1 \leq p \leq \infty$, we obtain

$$(\mathcal{E}_1^{\nu,\alpha}(A), \mathcal{E}_\infty^{\nu,\alpha}(A))_{1-1/q,p} = \mathcal{E}_{q,p}^{\nu,\alpha}(A), \tag{1}$$

with equivalent norms. Moreover, the embedding $\mathcal{E}_{q,p}^{\nu,\alpha}(A) \subset \mathcal{E}_{q,p}^{\mu,\alpha}(A)$ with $\mu > \nu$ holds and every space $\mathcal{E}_{q,p}^{\nu,\alpha}(A)$ is complete.

Theorem 1. *Let $0 < \nu < \infty$, $\alpha \in \mathbb{C}$, $1 < q_0, q_1 < \infty$, $q_0 \neq q_1$, $1 \leq p, p_0, p_1 \leq \infty$ and $0 < \theta < 1$. Then the following equality holds:*

$$\left(\mathcal{E}_{q_0,p_0}^{\nu,\alpha}(A), \mathcal{E}_{q_1,p_1}^{\nu,\alpha}(A) \right)_{\theta,p} = \mathcal{E}_{q,p}^{\nu,\alpha}(A), \ \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}. \tag{2}$$

Proof. By Theorem 3.11.5 [8] and (1) we have

$$(\mathcal{E}_{q_0,p_0}^{\nu,\alpha}(A), \mathcal{E}_{q_1,p_1}^{\nu,\alpha}(A))_{\theta,p} = (\mathcal{E}_1^{\nu,\alpha}(A), \mathcal{E}_\infty^{\nu,\alpha}(A))_{\lambda,p} = \mathcal{E}_{1/(1-\lambda),p}^{\nu,\alpha}(A), \tag{3}$$

where $q_0 = 1/(1-\theta_0)$, $q_1 = 1/(1-\theta_1)$, $\lambda = (1-\theta)\theta_0 + \theta\theta_1$, $0 \leq \theta_0 < \theta_1 \leq 1$. Putting $q = 1/(1-\lambda)$ in (3), we obtain (2). $\qed$

For any $1 \le q \le \infty$, $\alpha, \beta \in \mathbb{C}$ and $0 \le \operatorname{Re}\alpha < \operatorname{Re}\beta < \infty$ we define the spaces $\mathcal{E}_{q,\theta}^{\nu,(\alpha,\beta)}(A) = \left\{ x \in \mathbf{D}^\infty(A) : \|x\|_{\mathcal{E}_{q,\theta}^{\nu,(\alpha,\beta)}(A)} < \infty \right\}$,

$$
\|x\|_{\mathcal{E}_{q,\theta}^{\nu,(\alpha,\beta)}(A)} = \begin{cases} \left(\sum_{k \in \mathbb{Z}_+} \|x_{k,\nu}\|_{(\mathbf{D}^\alpha, \mathbf{D}^\beta)_{\theta,q}}^q \right)^{1/q}, & 1 \le q < \infty, \\ \sup_{k \in \mathbb{Z}_+} \|x_{k,\nu}\|_{(\mathbf{D}^\alpha, \mathbf{D}^\beta)_{\theta,q}}, & q = \infty. \end{cases}
$$

and $\mathcal{E}_{q,\theta}^{\nu,[\alpha,\beta]}(A) = \left\{ x \in \mathbf{D}^\infty(A) : \|x\|_{\mathcal{E}_{q,\theta}^{\nu,[\alpha,\beta]}(A)} < \infty \right\}$,

$$
\|x\|_{\mathcal{E}_{q,\theta}^{\nu,[\alpha,\beta]}(A)} = \begin{cases} \left(\sum_{k \in \mathbb{Z}_+} \|x_{k,\nu}\|_{[\mathbf{D}^\alpha, \mathbf{D}^\beta]_\theta}^q \right)^{1/q}, & 1 \le q < \infty, \\ \sup_{k \in \mathbb{Z}_+} \|x_{k,\nu}\|_{[\mathbf{D}^\alpha, \mathbf{D}^\beta]_\theta}, & q = \infty. \end{cases}
$$

Theorem 2. *Let $0 < \theta < 1$, $\alpha, \beta \in \mathbb{C}$, $0 \le \operatorname{Re}\alpha < \operatorname{Re}\beta < \infty$ and $1 \le q, q_0, q_1 \le \infty$. If $0 < \nu_0, \nu_1 < \infty$, $\nu_0 \ne \nu_1$ then for $\nu = \nu_0^{1-\theta}\nu_1^\theta$ the following equality holds:*

$$
\left(\mathcal{E}_{q_0}^{\nu_0,\alpha}(A), \mathcal{E}_{q_1}^{\nu_1,\alpha}(A) \right)_{\theta,q} = \mathcal{E}_q^{\nu,\alpha}(A). \tag{4}
$$

For $1 \le q_0, q_1 < \infty$ such that $1/q = (1-\theta)/q_0 + \theta/q_1$ the following equalities hold:

$$
\left(\mathcal{E}_{q_0}^{\nu,\alpha}(A), \mathcal{E}_{q_1}^{\nu,\beta}(A) \right)_{\theta,q} = \mathcal{E}_{q,\theta}^{\nu,(\alpha,\beta)}(A). \tag{5}
$$

$$
\left[\mathcal{E}_{q_0}^{\nu_0,\alpha}(A), \mathcal{E}_{q_1}^{\nu_1,\alpha}(A) \right]_\theta = \mathcal{E}_q^{\nu,\alpha}(A), \tag{6}
$$

$$
\left[\mathcal{E}_{q_0}^{\nu,\alpha}(A), \mathcal{E}_{q_1}^{\nu,\beta}(A) \right]_\theta = \mathcal{E}_{q,\theta}^{\nu,[\alpha,\beta]}(A). \tag{7}
$$

Proof. The space $\mathcal{E}_q^{\nu,\alpha}(A)$ is isometric to the space of sequences $l_q^{\nu,\alpha} = \big\{ \bar{x} := (A^k x)_{k=0}^\infty : x \in \mathcal{E}_q^{\nu,\alpha}(A) \big\}$ with the norm $\|\bar{x}\|_{l_q^{\nu,\alpha}} = \|x\|_{\mathcal{E}_q^{\nu,\alpha}(A)}$. Let us replace $\nu = 2^{-\sigma}$ in which the condition $\nu = \nu_0^{1-\theta}\nu_1^\theta$ turns into equality $\sigma = (1-\theta)\sigma_0 + \theta\sigma_1$. Moreover, we have $K(t, \bar{x}, l_\infty^{\nu_0,\alpha}, l_\infty^{\nu_1,\alpha}) \sim \sup_{k \in \mathbb{Z}_+} \min(2^{k\sigma_0}, t2^{k\sigma_1})\|A^k x\|_{\mathbf{D}^\alpha}$ for $q_0 = q_1 = \infty$ and $K(t, \bar{x}, l_1^{\nu_0,\alpha}, l_1^{\nu_1,\alpha}) \sim \sum_{k \in \mathbb{N}} \min(2^{k\sigma_0}, t2^{k\sigma_1})\|A^k x\|_{\mathbf{D}^\alpha}$ for $q_0 = q_1 = 1$.

For $\bar{x} \in (l_\infty^{\nu_0,\alpha}, l_\infty^{\nu_1,\alpha})_{\theta,q}$ and $\sigma_0 > \sigma_1$ we obtain

$$
\|\bar{x}\|_{(l_\infty^{\nu_0,\alpha}, l_\infty^{\nu_1,\alpha})_{\theta,q}}^q = \int_0^\infty \left[t^{-\theta} K(t, \bar{x}; l_\infty^{\nu_0,\alpha}, l_\infty^{\nu_1,\alpha}) \right]^q \frac{dt}{t}
$$

$$\sim \sum_{j\in\mathbb{Z}} 2^{-\theta q j(\sigma_0-\sigma_1)} \sup_k \big[\min(2^{k\sigma_0}, 2^{j(\sigma_0-\sigma_1)+k\sigma_1})\|A^k x\|_{\mathsf{D}^\alpha}\big]^q$$

$$\geq \sum_{j\in\mathbb{Z}} 2^{qj[\sigma_0(1-\theta)+\sigma_1\theta]}\|A^j x\|_{\mathsf{D}^\alpha}^q = c\|\bar{x}\|_{l_q^{\nu,\alpha}}^q,$$

which shows the embedding

$$(l_\infty^{\nu_0,\alpha}, l_\infty^{\nu_1,\alpha})_{\theta,q} \subset l_q^{\nu,\alpha}. \tag{8}$$

Using the Hölder's inequality, for $\bar{x} \in l_q^{\nu,\alpha}$ we have

$$\|\bar{x}\|_{(l_1^{\nu_0,\alpha}, l_1^{\nu_1,\alpha})_{\theta,q}}^q = \int_0^\infty \big[t^{-\theta} K(t, \bar{x}; l_1^{\nu_0,\alpha}, l_1^{\nu_1,\alpha})\big]^q \frac{dt}{t}$$

$$\sim \sum_{j\in\mathbb{Z}} 2^{-\theta q j(\sigma_0-\sigma_1)} \Big[\sum_{k\in\mathbb{Z}_+} \min(2^{k\sigma_0}, 2^{j(\sigma_0-\sigma_1)+k\sigma_1})\|A^k x\|_{\mathsf{D}^\alpha}\Big]^q$$

$$\leq c' \sum_{j\in\mathbb{Z}} 2^{qj(\sigma-\sigma_0)} \Big(\sum_{k\leq j} 2^{q'k(\sigma_0-\mu_0)}\Big)^{q/q'} \Big(\sum_{k\leq j} 2^{qk\mu_0}\|A^k x\|_{\mathsf{D}^\alpha}^q\Big)$$

$$+c' \sum_{j\in\mathbb{Z}} 2^{qj(\sigma-\sigma_1)} \Big(\sum_{k>j} 2^{q'k(\sigma_1-\mu_1)}\Big)^{q/q'} \Big(\sum_{k>j} 2^{qk\mu_1}\|A^k x\|_{\mathsf{D}^\alpha}^q\Big)$$

$$\leq c'' \Big(\sum_{k\in\mathbb{Z}} 2^{qk\mu_0}\|A^k x\|_{\mathsf{D}^\alpha}^q \sum_{j\geq k} 2^{qj(\sigma-\mu_0)}$$

$$+\sum_{k\in\mathbb{Z}} 2^{qk\mu_1}\|A^k x\|_{\mathsf{D}^\alpha}^q \sum_{j<k} 2^{qj(\sigma-\mu_1)}\Big) \leq c \sum_{k\in\mathbb{Z}} 2^{k\sigma q}\|A^k x\|_{\mathsf{D}^\alpha}^q = c\|\bar{x}\|_{l_q^{\nu,\alpha}}^q.$$

It follows that

$$l_q^{\nu,\alpha} \subset (l_1^{\nu_0,\alpha}, l_1^{\nu_1,\alpha})_{\theta,q}. \tag{9}$$

From (8) and (9) we obtain

$$l_q^{\nu,\alpha} \subset (l_1^{\nu_0,\alpha}, l_1^{\nu_1,\alpha})_{\theta,q} \subset (l_{q_0}^{\nu_0,\alpha}, l_{q_1}^{\nu_1,\alpha})_{\theta,q} \subset (l_\infty^{\nu_0,\alpha}, l_\infty^{\nu_1,\alpha})_{\theta,q} \subset l_q^{\nu,\alpha}.$$

Thus we have the required equality (4).

The space $\mathcal{E}_q^{\nu,\alpha}(A)$ is isometric embedding in the space $l_q^\alpha = \big\{(\xi_k)_{k=0}^\infty : \xi_k \in \mathsf{D}^\alpha, \|(\xi_k)\|_{l_q^\alpha} = \big(\sum_{k\in\mathbb{Z}_+} \|\xi_k\|_{\mathsf{D}^\alpha}^q\big)^{1/q} < \infty\big\}$. Denote $\mathcal{I}: \mathcal{E}_q^{\nu,\alpha}(A) \ni x \hookrightarrow (\xi_k = x_{k,\nu}) \in l_q^\alpha$. For $x \in \big(l_{q_0}^\alpha, l_{q_1}^\beta\big)_{\theta,q}$ we have

$$K(t, \mathcal{I}(x), l_{q_0}^\alpha, l_{q_1}^\beta) \leq \inf_{x=x_0+x_1} \big(\|\mathcal{I}(x_0)\|_{l_{q_0}^\alpha} + t\|\mathcal{I}(x_1)\|_{l_{q_1}^\beta}\big)$$

$$\leq \|\mathcal{I}\|_{\mathcal{E}_{q_0}^{\nu,\alpha}(A)\to l_{q_0}^\alpha} K\Big(\frac{t\|\mathcal{I}\|_{\mathcal{E}_{q_1}^{\nu,\beta}(A)\to l_{q_1}^\beta}}{\|\mathcal{I}\|_{\mathcal{E}_{q_0}^{\nu,\alpha}(A)\to l_{q_0}^\alpha}}, x, \mathcal{E}_{q_0}^{\nu,\alpha}(A), \mathcal{E}_{q_1}^{\nu,\beta}(A)\Big).$$

Through a substitution $\tau = t\|\mathcal{I}\|_{\mathcal{E}_{q_1}^{\nu,\beta}(A)\to l_{q_1}^{\beta}} \|\mathcal{I}\|_{\mathcal{E}_{q_0}^{\nu,\alpha}(A)\to l_{q_0}^{\alpha}}^{-1}$ we obtain

$$\|\mathcal{I}(x)\|_{\left(l_{q_0}^{\alpha}, l_{q_1}^{\beta}\right)_{\theta,q}} \leq \|\mathcal{I}\|_{\mathcal{E}_{q_0}^{\nu,\alpha}(A)\to l_{q_0}^{\alpha}} \frac{\|\mathcal{I}\|_{\mathcal{E}_{q_1}^{\nu,\beta}(A)\to l_{q_1}^{\beta}}^{\theta}}{\|\mathcal{I}\|_{\mathcal{E}_{q_0}^{\nu,\alpha}(A)\to l_{q_0}^{\alpha}}^{\theta}} \|x\|_{\left(\mathcal{E}_{q_0}^{\nu,\alpha}(A), \mathcal{E}_{q_1}^{\nu,\beta}(A)\right)_{\theta,q}}.$$

Applying the similar estimate for the inverse mapping $\mathcal{I}^{-1}$ defined on the set of values of $\mathcal{I}$, we obtain

$$\|\mathcal{I}(x)\|_{\left(l_{q_0}^{\alpha}, l_{q_1}^{\beta}\right)_{\theta,q}} = \|x\|_{\left(\mathcal{E}_{q_0}^{\nu,\alpha}(A), \mathcal{E}_{q_1}^{\nu,\beta}(A)\right)_{\theta,q}}, x \in \left(\mathcal{E}_{q_0}^{\nu,\alpha}(A), \mathcal{E}_{q_1}^{\nu,\beta}(A)\right)_{\theta,q}.$$

Every space $\mathcal{E}_{q,\theta}^{\nu,(\alpha,\beta)}(A)$ is isometric to the space of sequences $l_{q,\theta}^{\nu,(\alpha,\beta)} = \{\bar{x} := (A^k x)_{k=0}^{\infty} : x \in \mathcal{E}_{q,\theta}^{\nu,(\alpha,\beta)}(A)\}$ endowed with the norm $\|\bar{x}\|_{l_{q,\theta}^{\nu,(\alpha,\beta)}} = \|x\|_{\mathcal{E}_{q,\theta}^{\nu,(\alpha,\beta)}(A)}$. Applying [7, Theorem 1.18.1], we obtain $\left(l_{q_0}^{\alpha}, l_{q_1}^{\beta}\right)_{\theta,q} = l_{q,\theta}^{(\alpha,\beta)}$, where $l_{q,\theta}^{(\alpha,\beta)} = \{(\xi_k)_{k=0}^{\infty} : \xi_k \in (\mathbf{D}^{\alpha}, \mathbf{D}^{\beta})_{\theta,q}, \|(\xi_k)\|_{l_{q,\theta}^{(\alpha,\beta)}}^q = \sum_{k\in\mathbb{Z}_+} \|\xi_k\|_{(\mathbf{D}^{\alpha},\mathbf{D}^{\beta})_{\theta,q}}^q < \infty\}$ and $1/q = (1-\theta)/q_0 + \theta/q_1$. It follows that (5) is true.

The equality (6) follows immediately from (4) and Theorem 4.7.2 [8].

Let $x \in [\mathcal{E}_{q_0}^{\nu,\alpha}(A), \mathcal{E}_{q_1}^{\nu,\beta}(A)]_{\theta}$, $f(z) \in \mathcal{F}(\mathcal{E}_{q_0}^{\nu,\alpha}(A), \mathcal{E}_{q_1}^{\nu,\beta}(A))$ and $f(\theta) = x$. Then

$$g(z) = \left(\frac{\|\mathcal{I}\|_{\mathcal{E}_{q_0}^{\nu,\alpha}(A)\to l_{q_0}^{\alpha}}}{\|\mathcal{I}\|_{\mathcal{E}_{q_1}^{\nu,\beta}(A)\to l_{q_1}^{\beta}}}\right)^{z-\theta} \mathcal{I}f(z) \in \mathcal{F}(l_{q_0}^{\alpha}, l_{q_1}^{\beta}), \ g(\theta) = \mathcal{I}x.$$

It follows that

$$\|g(z)\|_{\mathcal{F}(l_{q_0}^{\alpha}, l_{q_1}^{\beta})} \leq \|\mathcal{I}\|_{\mathcal{E}_{q_0}^{\nu,\alpha}(A)\to l_{q_0}^{\alpha}}^{(1-\theta)} \|\mathcal{I}\|_{\mathcal{E}_{q_1}^{\nu,\beta}(A)\to l_{q_1}^{\beta}}^{\theta} \|f(z)\|_{\mathcal{F}(\mathcal{E}_{q_0}^{\nu,\alpha}(A), \mathcal{E}_{q_1}^{\nu,\beta}(A))},$$

$$\|\mathcal{I}x\|_{[l_{q_0}^{\alpha}, l_{q_1}^{\beta}]_{\theta}} \leq \|\mathcal{I}\|_{\mathcal{E}_{q_0}^{\nu,\alpha}(A)\to l_{q_0}^{\alpha}}^{(1-\theta)} \|\mathcal{I}\|_{\mathcal{E}_{q_1}^{\nu,\beta}(A)\to l_{q_1}^{\beta}}^{\theta} \|x\|_{[\mathcal{E}_{q_0}^{\nu,\alpha}(A), \mathcal{E}_{q_1}^{\nu,\beta}(A)]_{\theta}}.$$

Applying the similar estimate for the inverse mapping $\mathcal{I}^{-1}$, we obtain

$$\|\mathcal{I}(x)\|_{[l_{q_0}^{\alpha}, l_{q_1}^{\beta}]_{\theta}} = \|x\|_{[\mathcal{E}_{q_0}^{\nu,\alpha}(A), \mathcal{E}_{q_1}^{\nu,\beta}(A)]_{\theta}} \text{ for all } x \in [\mathcal{E}_{q_0}^{\nu,\alpha}(A), \mathcal{E}_{q_1}^{\nu,\beta}(A)]_{\theta}.$$

The space $\mathcal{E}_{q,\theta}^{\nu,[\alpha,\beta]}(A)$ is isometric to the space of sequences $l_{q,\theta}^{\nu,[\alpha,\beta]} = \{\bar{x} := (A^k x)_{k=0}^{\infty} : x \in \mathcal{E}_{q,\theta}^{\nu,[\alpha,\beta]}(A)\}$ with the norm $\|\bar{x}\|_{l_{q,\theta}^{\nu,[\alpha,\beta]}} = \|x\|_{\mathcal{E}_{q,\theta}^{\nu,[\alpha,\beta]}(A)}$.

By Theorem 1.18.1 [7] we have $[l_{q_0}^{\alpha}, l_{q_1}^{\beta}]_{\theta} = l_{q,\theta}^{[\alpha,\beta]}$, where $l_{q,\theta}^{[\alpha,\beta]} = \{(\xi_k)_{k=0}^{\infty} : \xi_k \in [\mathbf{D}^{\alpha}, \mathbf{D}^{\beta}]_{\theta}, \|(\xi_k)\|_{l_{q,\theta}^{[\alpha,\beta]}}^q = \sum_{k\in\mathbb{Z}_+} \|\xi_k\|_{[\mathbf{D}^{\alpha},\mathbf{D}^{\beta}]_{\theta}}^q < \infty\}$ and $1/q = (1-\theta)/q_0 + \theta/q_1$. It follows that (7) is true. $\square$

3. Best Approximations

On the linear manifold $\mathcal{E}_{q,p}^{\alpha}(A) = \bigcup_{\nu>0} \mathcal{E}_{q,p}^{\nu,\alpha}(A)$ we define the quasi-norm $|x|_{\mathcal{E}_{q,p}^{\alpha}(A)}$ $= \|x\| + \inf \{\nu > 0 \colon x \in \mathcal{E}_{q,p}^{\nu,\alpha}(A)\}$. Given a pair of numbers $\{0 < s < \infty, \ 0 < \tau \le \infty\}$ or $\{0 \le s < \infty, \ \tau = \infty\}$ we consider the scale of approximation spaces $\mathcal{B}_{q,p,\tau}^{s,\alpha}(A) = \{x \in \mathsf{X} \colon |x|_{\mathcal{B}_{q,p,\tau}^{s,\alpha}(A)} < \infty\}$,

$$|x|_{\mathcal{B}_{q,p,\tau}^{s,\alpha}(A)} = \begin{cases} \left(\int_0^\infty \left[t^\alpha E_{q,p}^\alpha(t,x)\right]^\tau \frac{dt}{t}\right)^{1/\tau}, & 0 < \tau < \infty, \\ \sup_{t>0} t^\alpha E_{q,p}^\alpha(t,x), & \tau = \infty, \end{cases}$$

where $E_{q,p}^\alpha(t,x) = \inf\{\|x - x^0\| \colon x^0 \in \mathcal{E}_{q,p}^\alpha(A), |x^0|_{\mathcal{E}_{q,p}^\alpha(A)} \le t\}$, $x \in \mathsf{X}$.

Let $\left[\mathcal{B}_{q,p,\tau}^{s,\alpha}(A)\right]^\theta$ be the space $\mathcal{B}_{q,p,\tau}^{s,\alpha}(A)$ endowed with the quasi-norm $|x|_{\mathcal{B}_{q,p,\tau}^{s,\alpha}(A)}^\theta$, $x \in \mathcal{B}_{q,p,\tau}^{s,\alpha}(A)$. Applying Theorem 7.1.7 [8], we obtain

$$\left[\mathcal{B}_{q,p,\tau}^{s,\alpha}(A)\right]^\theta = \left(\mathcal{E}_{q,p}^\alpha(A), \mathsf{X}\right)_{\theta,g}, \quad \theta = 1/(s+1), \ \tau = g\theta, \tag{10}$$

with equivalent quasi-norms.

Lemma 3. *Every space $\mathcal{B}_{q,p,\tau}^{s,\alpha}(A)$ is complete.*

Proof. From the inequality $\|x\| \le |x|_{\mathcal{E}_{q,p}^\alpha(A)}$, $x \in \mathcal{E}_{q,p}^\alpha(A)$ it follows that $\|x\| = \inf_{x=x^0+x^1} \left(|x^0|_{\mathcal{E}_{q,p}^\alpha(A)} + \|x^1\|\right)$ with $x^0 \in \mathcal{E}_{q,p}^\alpha(A)$ and $x^1 \in \mathsf{X}$. Thus, every series $\sum_{n\in\mathbb{N}} x_n$ with $x_n \in (\mathcal{E}_{q,p}^\alpha(A), \mathsf{X})_{\theta,g}$ such that $\sum_{n\in\mathbb{N}} |x_n|_{(\mathcal{E}_{q,p}^\alpha(A),\mathsf{X})_{\theta,g}} < \infty$ is convergent to $x \in \mathsf{X}$. Using the inequality $\left|\sum_{n\in\mathbb{N}} x_n\right|_{(\mathcal{E}_{q,p}^\alpha(A),\mathsf{X})_{\theta,g}} \le \sum_{n\in\mathbb{N}} |x_n|_{(\mathcal{E}_{q,p}^\alpha(A),\mathsf{X})_{\theta,g}}$ we obtain $x \in (\mathcal{E}_{q,p}^\alpha(A), \mathsf{X})_{\theta,g}$. So, $(\mathcal{E}_{q,p}^\alpha(A), \mathsf{X})_{\theta,g}$ is complete. The isomorphism (10) implies that the space $\mathcal{B}_{q,p,\tau}^{s,\alpha}(A)$) is complete. $\quad\square$

Theorem 4. *There are constants c_1 and c_2 such that the following inequalities hold:*

$$|x|_{\mathcal{B}_{q,p,\tau}^{s,\alpha}(A)} \le c_1 |x|_{\mathcal{E}_{q,p}^\alpha(A)}^s \|x\|, \quad x \in \mathcal{E}_{q,p}^\alpha(A), \tag{11}$$

$$E_{q,p}^\alpha(t,x) \le c_2 t^{-s} |x|_{\mathcal{B}_{q,p,\tau}^{s,\alpha}(A)}, \quad x \in \mathcal{B}_{q,p,\tau}^{s,\alpha}(A). \tag{12}$$

Proof. Applying [8, Theorem 3.11.4(b)], for some constant $c(\theta, g)$ we obtain

$$|x|_{(\mathcal{E}_{q,p}^\alpha(A),\mathsf{X})_{\theta,g}} \le c |x|_{\mathcal{E}_{q,p}^\alpha(A)}^{1-\theta} \|x\|^\theta, \quad x \in \mathcal{E}_{q,p}^\alpha(A).$$

This inequality and the isomorphism (10) imply that there is a constant $c_1(s, \tau)$ such that the inequality (11) is true.

By [8, Theorem 3.11.4(a)] for some constant $c(\theta, g)$ we have

$$K(t, x; \mathcal{E}_{q,p}^{\alpha}(A), \mathsf{X}) \leq c\, t^{\theta}|x|_{\left(\mathcal{E}_{q,p}^{\alpha}(A),\mathsf{X}\right)_{\theta,g}}, \ \ x \in \left(\mathcal{E}_{q,p}^{\alpha}(A), \mathsf{X}\right)_{\theta,g}.$$

Using (10), we have the inequality

$$K(t, x; \mathcal{E}_{q,p}^{\alpha}(A), \mathsf{X}) \leq c_0\, t^{\theta}|x|_{\mathcal{B}_{q,p,\tau}^{s,\alpha}(A)}^{\theta}, \ \ x \in \mathcal{B}_{q,p,\tau}^{s,\alpha}(A).$$

with a constant $c_0(s, \tau)$.

Let $K_{\infty}(t, x; \mathcal{E}_{q,p}^{\alpha}(A), \mathsf{X}) = \inf_{x=x^0+x^1} \max\{|x^0|_{\mathcal{E}_{q,p}^{\alpha}(A)}, t\|x^1\|\}$. The inequality
$K_{\infty}(t, x; \mathcal{E}_{q,p}^{\alpha}(A), \mathsf{X}) \leq K(t, x; \mathcal{E}_{q,p}^{\alpha}(A), \mathsf{X})$ yields

$$t^{-\theta}K_{\infty}(t, x; \mathcal{E}_{q,p}^{\alpha}(A), \mathsf{X}) \leq c_0|x|_{\mathcal{B}_{q,p,\tau}^{s,\alpha}(A)}^{\theta}, \ \ x \in \mathcal{B}_{q,p,\tau}^{s,\alpha}(A). \tag{13}$$

By Lemma 7.1.2 [8] for every $t > 0$ there exists $v > 0$ such that

$$K_{\infty}(t, x; \mathcal{E}_{q,p}^{\alpha}(A), \mathsf{X}) = v, \ \lim_{\mu\downarrow v} E_{q,p}^{\alpha}(\mu, x) = E_{q,p}^{\alpha}(v + 0, x) \leq v/t.$$

As a result, $v^{1-\theta}[E_{q,p}^{\alpha}(v, x)]^{\theta} \leq t^{-\theta}K_{\infty}(t, x; \mathcal{E}_{q,p}^{\alpha}(A), \mathsf{X})$. Using (13), we have

$$v^{1-\theta}[E_{q,p}^{\alpha}(v, x)]^{\theta} \leq c_0|x|_{\mathcal{B}_{q,p,\tau}^{s,\alpha}(A)}^{\theta}.$$

If $s = (1 - \theta)/\theta$ then $v^s E_{q,p}^{\alpha}(v, x) \leq c_0^{1/\theta}|x|_{\mathcal{B}_{q,p,\tau}^{s,\alpha}(A)}$ for all $x \in \mathcal{B}_{q,p,\tau}^{s,\alpha}(A)$. Taking
$c_2 = c_0^{1/\theta}$, we obtain the required inequality (12). $\qquad\square$

Now we consider a spectral approximation problem for the positive operator with the point spectrum. We say that A is the operator with the point spectrum if $\sigma(A)$ consists of isolated eigenvalues λ_j $(j \in \mathbb{N})$ with a unique boundary point at infinity. Moreover, every root subspace $\mathcal{R}_{\lambda_j}(A)$ corresponding to $\lambda_j \in \sigma(A)$ is finite dimensional.

For $m, n \in \mathbb{N}$ we denote by $\mathcal{R}_{m,n}^{\nu}(A)$ the complex linear span in X of all $\{\mathcal{R}_{\lambda_j}(A) : |\lambda_j| < \min(\nu^{1/(m+1)}, \nu^{1/(n+1)})\}$. Let $\mathcal{H}_{\lambda_j}(A) = \{x \in \mathsf{D}^{\infty}(A) : (\lambda_j I - A)x = 0\}$ be a subspace of eigenvectors corresponding to $\lambda_j \in \sigma(A)$ and $\mathcal{H}_{m,n}^{\nu}(A)$ be a complex linear span of all $\mathcal{H}_{\lambda_j}(A)$ such that

$$|\lambda_j| = \min(\nu^{1/(m+1)}, \nu^{1/(n+1)}),$$

$\lambda_j \in \sigma(A)$.

Theorem 5. *Let A be a positive operator with the point spectrum, $m, n \in \mathbb{N}$, $\alpha = m(1-\theta) + n\theta$ and $0 < \theta < 1$. Then the following inequalities hold:*

$$\inf \left\{ \|x - x^0\| : x^0 \in \mathcal{R}^{\nu}_{m,n}(A) \right\} \leq c\nu^{-s} |x|_{\mathcal{B}^{s,\alpha}_{q,q,\tau}(A)}, \tag{14}$$

$$\inf \left\{ \|x - x^0\| : x^0 \in (\mathcal{R}^{\nu}_{m,n}(A), \mathcal{Q}^{\nu}_{m,n}(A))_{1-1/q,p} \right\} \leq c\nu^{-s} |x|_{\mathcal{B}^{s,\alpha}_{q,p,\tau}(A)}, \tag{15}$$

where $\mathcal{Q}^{\nu}_{m,n}(A) = \mathcal{R}^{\nu}_{m,n}(A) \oplus \mathcal{H}^{\nu}_{m,n}(A)$ and $\|\cdot\|_{\mathcal{R}^{\nu}_{m,n}(A)} := \|\cdot\|_{\mathcal{E}^{\nu,\alpha}_{1}(A)}$, $\|\cdot\|_{\mathcal{Q}^{\nu}_{m,n}(A)} := \|\cdot\|_{\mathcal{E}^{\nu,\alpha}_{\infty}(A)}$.

Proof. If $|x^0|_{\mathcal{E}^{\alpha}_{q,p}(A)} = r(x^0) + \|x^0\| \leq v$, then $r(x^0) \leq v - \|x^0\|$, where $r(x^0) = \inf\{\nu > 0 \colon x^0 \in \mathcal{E}^{\nu,\alpha}_{q,p}(A)\}$. So, $x^0 \in \mathcal{E}^{\nu,\alpha}_{q,p}(A)$ for all $\nu > 0$ such that $r(x^0) \leq \nu \leq v - \|x^0\|$. Since $\mathcal{E}^{\nu,\alpha}_{q,p}(A) \subset \mathcal{E}^{v,\alpha}_{q,p}(A)$ that $x^0 \in \mathcal{E}^{v,\alpha}_{q,p}(A)$. Hence, the inequality

$$\inf\{\|x - x^0\| : x^0 \in \mathcal{E}^{v,\alpha}_{q,p}(A)\} \leq E^{\alpha}_{q,p}(v, x), \ x \in \mathsf{X}, \ v > 0 \tag{16}$$

holds. By Theorem 2.2 [9] for any $1 \leq q < \infty$ and $k \in \mathbb{Z}_+$ we have

$$\mathcal{E}^{\nu,k}_{q}(A) = \operatorname{span}\left\{ \mathcal{R}_{\lambda_j}(A) : |\lambda_j|^{k+1} < \nu \right\}. \tag{17}$$

Applying (17), (7) and Theorem 1.15.3 [7], we obtain

$$\begin{aligned}
\mathcal{E}^{\nu,\alpha}_{q}(A) &= \mathcal{E}^{\nu,[m,n]}_{q,\theta}(A) = \left[\mathcal{E}^{\nu,m}_{q_0}(A), \mathcal{E}^{\nu,n}_{q_1}(A) \right]_{\theta} \\
&= \mathcal{E}^{\nu,m}_{q_0}(A) \cap \mathcal{E}^{\nu,n}_{q_1}(A) = \mathcal{R}^{\nu}_{m,n}(A)
\end{aligned} \tag{18}$$

with $\alpha = m(1-\theta) + n\theta$. Using (18), (16) and (12), we obtain (14).

Now let us prove the equality

$$\mathcal{E}^{\nu,\alpha}_{\infty}(A) = \mathcal{Q}^{\nu}_{m,n}(A). \tag{19}$$

Using [10, Lemma 1], we have

$$\mathcal{E}^{\nu,\alpha}_{\infty}(A) \subset \operatorname{span}\{\mathcal{R}_{\lambda_j}(A) : |\lambda_j| \leq \min(\nu^{1/(m+1)}, \nu^{1/(n+1)})\}.$$

Then it is sufficient to prove the equality $\mathcal{H}_{\lambda_j}(A) = \mathcal{E}^{\nu,0}_{\infty}(A) \cap \mathcal{R}_{\lambda_j}(A)$ for indices j with $|\lambda_j| = \nu$. Assume that this equality is not true. Then there exist root vectors $x_0, \ldots, x_r$ corresponding to λ_j such that $|\lambda_j| = \nu$ and $x_r \in \mathcal{E}^{\nu,0}_{\infty}(A)$, $r \geq 1$. From the equality

$$A^k x_r = \sum_{i=0}^{r} \binom{k}{i} \lambda_j^{k-i} x_{r-i}, \ k \geq r$$

it follows that

$$\lim_{k \to \infty} \frac{\|A^k x_r\|}{\binom{k}{r} \nu^k} = \nu^{-r} \|x_0\|.$$

Since $x_0 \neq 0$ then $\nu^{-r} \|x_0\| \neq 0$ and $x_r \notin \mathcal{E}_\infty^{\nu,0}(A)$. So, the equality $\mathcal{H}_{\lambda_j}(A) = \mathcal{E}_\infty^{\nu,0}(A) \cap \mathcal{R}_{\lambda_j}(A)$ for $j : |\lambda_j| = \nu$ holds. Thus, the equality (19) holds as well.

By (1), (18) and (19) we have

$$\left(\mathcal{R}_{m,n}^\nu(A), \mathcal{Q}_{m,n}^\nu(A) \right)_{1-1/q,p} = \mathcal{E}_{q,p}^{\nu,\alpha}(A). \tag{20}$$

Using (20), (16) and (12), we obtain (15).   □

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