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MONOTONE ITERATIVE TECHNIQUE FOR FINITE SYSTEM OF FRACTIONAL DIFFERENCE EQUATIONS

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Abstract: In this paper, we consider non-linear fractional finite difference system and establish the existence of solutions using monotone iterative technique.

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1. Introduction

Fractional calculus gained importance during the past three decades due to its applicability in diverse fields of science and engineering. The notions of fractional calculus may be traced back to the works of Euler, but the idea of fractional difference is very recent. G.V.S.R. Deekshitulu and J. Jagan Mohan [2] modified the definition of fractional difference given by Atsushi Nagai [11] and discussed some basic inequalities, comparison theorems and qualitative properties of the solutions of fractional difference equations [2,3,4,5,6].

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Being an important tool in the study of existence, uniqueness, boundedness, stability and other qualitative properties of solutions of finite systems of differential equations and difference equations, comparison theorems and their applications have attracted great interests of many mathematicians. The investigation of scalar fractional order difference inequalities was initiated by G.V.S.R. Deekshitulu and J. Jagan Mohan [2].

In this paper, we use monotone iterative technique for finite system of fractional difference equations and derive the convergence of monotone sequences to upper and lower solutions. Then using these upper and lower solutions, existence of solutions to finite system of fractional difference inequalities to obtain more general results.

2. Preliminaries

In this section, we introduce some basic definitions and results concerning nabla discrete fractional calculus.

Definition 2.1. The backward difference operator Δ_{-n} is defined as $\Delta_{-n} = \varepsilon^{-1}(1-B)$ where $Bf_n = f_{n-1}$ is standard backward shift operator and ε is interval length.

Henry L Gray and Nien fan Zhang gave a definition of the fractional difference as follows:

Definition 2.2. For any complex number α and f defined over the integer set $\{a-p, a-p+1, ...n\}$, the α^{th} order difference of $f(n)$ over $\{a, a+1, ...n\}$ is defined by

$$\nabla^{\alpha} f(n) = \frac{\nabla^p}{\Gamma(p-\alpha)} \sum_{k=0}^{n-a} \frac{\Gamma(k+p-\alpha)}{\Gamma(k+1)} f(n-k). \tag{2.1}$$

Later, Hirota took the first n terms of Taylor series of $\Delta_{-n}^{\alpha} = \varepsilon^{-\alpha}(1-B)^{\alpha}$ and gave the following definition.

Definition 2.3. Let $\alpha \in \mathbb{R}$. Then difference operator of order α is defined by

$$\Delta_{-n}^{\alpha} u_n = \begin{cases} \varepsilon^{-\alpha} \sum_{j=0}^{n-1} \binom{\alpha}{j} (-1)^j u_{n-j}, & \alpha \neq 1,2,.... \\ \varepsilon^{-m} \sum_{j=0}^{m} \binom{m}{j} (-1)^j u_{n-j}, & \alpha = m \in \mathbb{Z}_{>0}. \end{cases} \tag{2.2}$$

Here $\binom{a}{n}$, $(a \in \mathbb{R}, n \in \mathbb{Z})$ stands for a binomial coefficient defined by

$$\binom{a}{n} = \begin{cases} \frac{\Gamma(a+1)}{\Gamma(a-n+1)\Gamma(n+1)} & \text{n} > 0 \\ 1 & \text{n} = 0 \\ 0 & \text{n} < 0. \end{cases} \tag{2.3}$$

In 2002, Atsushi Nagai [1] introduced another definition of fractional difference which is a slight modification of Hirota's fractional difference operator.

Definition 2.4. Let $\alpha \in \mathbb{R}$ and m be an integer such that $m-1 < \alpha \leq m$. The difference operator $\Delta_{*,-n}$ of order α is defined as

$$\Delta^{\alpha}_{*,-n} u_n = \Delta^{\alpha-m}_{-n} \Delta^{m}_{-n} u_n = \varepsilon^{m-\alpha} \sum_{j=0}^{n-1} \binom{\alpha-m}{j} (-1)^j \Delta^{m}_{-(n-j)} u_{n-j}. \tag{2.4}$$

G.V.S.R.Deekshitulu and J.Jagan Mohan [2] rearranged the terms in Atsushi Nagai's [11] definition for $0 < \alpha < 1$ in such a way that the expression for ∇^{α} does not involve any difference operator and the term $(-1)^j$ inside the summation index as follows.

Definition 2.5. The fractional sum operator of order α is defined as

$$\nabla^{-\alpha} u(n) = \sum_{j=0}^{n-1} \binom{j+\alpha-1}{j} u(n-j) = \sum_{j=1}^{n} \binom{n-j+\alpha-1}{n-j} u(j). \tag{2.5}$$

Definition 2.6. The Caputo type fractional difference operator of order α is defined as

$$\nabla^{\alpha} u(n) = \nabla^{\alpha-1}[\nabla u(n)] = \sum_{j=0}^{n-1} \binom{j-\alpha}{j} \nabla u(n-j). \tag{2.6}$$

Corollary 1. *The equivalent form of (2.6) is*

$$\nabla^{\alpha} u(n) = u(n) - \binom{n-\alpha-1}{n-1} u(0) - \alpha \sum_{j=1}^{n-1} \frac{1}{j-\alpha} \binom{j-\alpha}{j} u(j). \tag{2.7}$$

Theorem 2.1. *(Discrete Langenhop Inequality) Let $y(n)$, $a(n)$ and $b(n)$ be any three non negative functions defined for $n \in \mathbb{N}_0^+$. If for $n, k \in \mathbb{N}_0^+$ such that $k \leq n$ the following inequality be satisfied*

$$y(n) \geq y(k) - b(n) \sum_{j=k+1}^{n} [a(j)y(j)]$$

where $y(n)$ is not necessarily non negative. Then, for all $n, k \in \mathbb{N}_0^+$ such that $k \leq n$

$$y(n) \geq y(k) \prod_{j=k+1}^{n} [1 + b(n)a(j)]^{-1}.$$

Theorem 2.2. *Let $u(n)$ and $v(n)$ be non-negative function defined for $n \in N_0^+$ such that $u(0) = v(0)$.For $0 < \alpha < 1$,if $u(n) \leq v(n)$ for $n \in N_1^+$ then $\nabla^{-\alpha} u(n) \leq \nabla^{-\alpha} v(n)$.*

3. Main Results

In this section, we consider the following finite system of non linear fractional difference equations of order α, $0 < \alpha < 1$

$$\nabla^{\alpha} u(n+1) = f(n, u(n)), u(0) = u_0. \tag{3.1}$$

where u and f are n-vectors with components $u_i(n) : \mathbb{N}_0^+ \longrightarrow \mathbb{R}$ and $f_i : \mathbb{N}_0^+ \times \mathbb{R} \longrightarrow \mathbb{R}$, $1 \leq i \leq n$. We first establish some basic vectorial inequalities. Vectorial inequalities mean that the same inequalities hold between the corresponding components. We need the following properties in this study.

Definition 3.1. A function $f_i = f_i(u_1, u_2)$ is said to be quasi monotone non-decreasing (non-increasing) if for any fixed $u_i (i = 1, 2)$ f_i is non-decreasing (non-increasing) in u_j for $j \neq i$.

Definition 3.2. A function $f \in C[\mathbb{N}_0^+ \times \mathbb{R}^n, \mathbb{R}^n]$ is said to be quasi monotone non decreasing (non increasing) if , for each i such that $1 \leq i \leq n$, $u \leq v$ and $u_i = v_i$, $f_i(t, u) \leq f_i(t, v)$ $(f_i(t, u) \geq f_i(t, v))$.

It means that if for all i,f_i are quasi monotone non-decreasing(quasi monotone non increasing), then f is said to be quasi monotone non decreasing quasi monotone non increasing).

Definition 3.3. A function $v(n) : \mathbb{N}_0^+ \to \mathbb{R}^n$ is said to be a lower solution of (3.1) if

$$\nabla^{\alpha} v(n+1) \leq f(n, v(n)), \qquad v_0 \leq u_0. \tag{3.2}$$

Similarly a function $w(n) : \mathbb{N}_0^+ \to \mathbb{R}^n$ is said to be an upper solution of (3.1) if

$$\nabla^{\alpha} w(n+1) \geq f(n, w(n)), \quad w_0 \geq u_0. \tag{3.3}$$

Theorem 3.1. *Let the function $f(n, v(n))$ be quasi monotone non-decreasing and $v(n)$ and $w(n)$ be lower and upper solutions of (3.1) defined on $\mathbb{N}_0^+$ such that $\mathbf{v}(0) \leq \mathbf{w}(0)$. Then for all $n \in \mathbb{N}_0^+$, $v(n) \leq w(n)$.*

Proof. Suppose there exists a $k \in N_0^+$ such that $v(k) \leq w(k)$ and $v(k+1) > w(k+1)$.
Since $0 < \alpha < 1 \leq j$ and $\binom{j-\alpha}{j} > 0$, we have

$$\begin{aligned}
\nabla^\alpha v(k+1) &= v(k+1) - \binom{k-\alpha}{k} v(0) - \alpha \sum_{j=1}^{k} \frac{1}{j-\alpha}\binom{j-\alpha}{j} v(k+1-j) \\
&> w(k+1) - \binom{k-\alpha}{k} w(0) - \alpha \sum_{j=1}^{k} \frac{1}{j-\alpha}\binom{j-\alpha}{j} w(k+1-j) \\
&= \nabla^\alpha w(k+1).
\end{aligned}$$

Or $f(k, w(k)) \leq \nabla^\alpha w(k+1) < \nabla^\alpha v(k+1) \leq f(k, v(k))$
i.e., $f(k, w(k)) < f(k, v(k))$. This is a contradiction to the quasi monotone property of $f(n, u(n))$. Hence the proof. □

Further, if $u(n)$ is a solution of (3.1), then the by repeatedly applying the above theorem, we obtain

$$v(n) \leq u(n) \leq w(n).$$

Theorem 3.2. *Suppose that $\nabla^\alpha m_{n+1} \leq -Mm_n$ for $M > 0$ and $m_0 \leq 0$, then $m_n \leq 0$.*

Proof. If it is false, then there exists a $k \in \mathbb{N}_0^+$ such that $m_k > 0$ and $m_n \leq 0$ for $n < k$. Consider

$$\begin{aligned}
\nabla^\alpha m_k = m_k - \binom{k-1-\alpha}{k-1} m_0 + \sum_{j=1}^{k-1} \binom{j-\alpha-1}{j} m_{k-j} &\leq -Mm_k \\
\text{or} \quad (M+1)m_k + \sum_{j=1}^{k-1} \binom{j-\alpha-1}{j} m_{k-j} - \binom{k-1-\alpha}{k-1} m_0 &\leq 0
\end{aligned}$$

Since $m_k > 0$, $m_n \leq 0$ for $n = 0, 1, 2, ..., k-1$, $\binom{j-\alpha}{j} > 0$ and $\binom{j-\alpha-1}{j} < 0$ implies that it is a contradiction. Hence $m_n \leq 0$ for $n \in \mathbb{N}_0^+$. □

Monotone iterative technique:

Now we shall apply a general theory of monotone iterative technique for finite system of fractional difference equations of order $\alpha (0 < \alpha < 1.)$ We need the following notions.

For each $i, 1 \leq i \leq n$, let p_i, q_i be two non-negative integers such that $p_i + q_i = n - 1$ so that we can split the vector u into $u = (u_i, [u]_{pi}, [u]_{qi})$. Then the system (3.1)can be written as

$$\nabla^\alpha u_{i+1} = f_i(n, u_i, [u]_{pi}, [u]_{qi})), u(0) = u_0, \tag{3.4}$$

for $1 \leq i \leq n$.

Definition 3.4. Let $v, w \in C^1[N_0^+, R^n]$, then v, w are said to be coupled lower and upper quasi solutions of (3.4), if

$$\begin{aligned}
\nabla^\alpha v_{i+1} &\leq f_i(n, v_i, [v]_{pi}, [w]_{qi})), v(0) \leq u_0, \\
\nabla^\alpha w_{i+1} &\geq f_i(n, w_i, [w]_{pi}, [v]_{qi})), w(0) \geq u_0.
\end{aligned}$$

v and w are said to be coupled quasi solutions of (3.4) if

$$\begin{aligned}
\nabla^\alpha v_{i+1} &= f_i(n, v_i, [v]_{pi}, [w]_{qi})), v(0) = u_0, \\
\nabla^\alpha w_{i+1} &= f_i(n, w_i, [w]_{pi}, [v]_{qi})), w(0) = u_0.
\end{aligned}$$

Definition 3.5. A function $f \in C[N_0^+ \times R^n, R^n]$ is said to possess a mixed quasi monotone property (mqmp), if for each i, $f_i(n, u_i, [u]_{pi}, [u]_{qi})$ is monotone non-decreasing in $[u]_{pi}$ components and monotone non-increasing in $[u]_{qi}$ components.

In particular $f(n, u)$ is said to possess non-decreasing property if $f_i(n, u)$ is non-decreasing in $u_1, u_2, ..., u_n$ for all fixed $n \in N_0^+$.

Theorem 3.3. *Let the function $f(n, u)$ possess mixed monotonic property. Further, let there exists functions $v(n)$ and $w(n)$ defined on N_0^+ such that*

$$\begin{aligned}
\nabla^\alpha v_p(n+1) &\leq f_p(n, v(n)), \\
\nabla^\alpha v_q(n+1) &\geq f_q(n, v(n)), \\
\nabla^\alpha w_p(n+1) &\geq f_p(n, w(n)), \\
\nabla^\alpha w_q(n+1) &\leq f_q(n, w(n)).
\end{aligned}$$

Then for all $n \in N_0^+$, $v_p(n) \leq w_p(n), v_q(n) \geq w_q(n)$ provided $v_p(0) \leq w_p(0), v_q(0) \geq w_q(0)$.

Proof. Define a function $z(n)$ as follows:

$$z_p(n) = w_p(n) - v_p(n) \quad \textit{and} \quad z_q(n) = v_q(n) - w_q(n). \tag{3.5}$$

By using principle of mathematical induction we shall show that $z_i(n) \geq 0$ for all $n \in N_0^+$. We have, $v_p(0) \leq w_p(0)$, $v_q(0) \geq w_q(0)$ i.e.,$z_p(0) \geq 0$, $z_q(0) \geq 0$. Let $v_p(m) \leq w_p(m)$, $v_q(m) \geq w_q(m)$ for some fixed $m \in N_0^+$ or $z_p(m) \geq 0$, $z_q(m) \geq 0$. Now consider

$$\begin{aligned} v_p(m+1) &\leq \binom{m-\alpha}{m} v_p(0) + \alpha \sum_{j=1}^{m} \frac{1}{j-\alpha} \binom{j-\alpha}{j} v_p(m+1-j) \\ &\quad +F_p(m, v(m)) \\ &\leq \binom{m-\alpha}{m} w_p(0) + \alpha \sum_{j=1}^{m} \frac{1}{j-\alpha} \binom{j-\alpha}{j} w_p(m+1-j) \\ &\quad +F_p(m, w(m)) \\ &\leq w_p(m+1). \end{aligned}$$

$$\begin{aligned} v_q(m+1) &\geq \binom{m-\alpha}{m} v_q(0) + \alpha \sum_{j=1}^{m} \frac{1}{j-\alpha} \binom{j-\alpha}{j} v_q(m+1-j) \\ &\quad +F_q(m, v(m)) \\ &\geq \binom{m-\alpha}{m} w_q(0) + \alpha \sum_{j=1}^{m} \frac{1}{j-\alpha} \binom{j-\alpha}{j} w_q(m+1-j) \\ &\quad +F_q(m, w(m)) \\ &\geq w_q(m+1). \end{aligned}$$

By principle of mathematical induction, $z_i(n) \geq 0$ for all $n \in N_0^+$ i.e., $z(n) \geq 0$ for all $n \in N_0^+$ implies $v_p(n) \leq w_p(n), v_q(n) \geq w_q(n)$. Hence the proof. □

Theorem 3.4. *Let the assumptions of the Theorem 3.2 hold good and $u(n)$ is the solution of (3.1). If $v(0) = w(0) = u(0) = u_0$, then for all $n \in \mathbb{N}_0^+$,*

$$v_p(n) \leq u_p(n) \leq w_p(n) \quad \text{and} \quad w_q(n) \leq u_q(n) \leq v_q(n).$$

If u is a solution of (3.4), then the result follows from Theorems (3.1) and (3.2).

Theorem 3.5. *Let $f \in C[N_0^+ \times R^n, R^n]$ possess mixed quasi monotone property,and let v_0, w_0 be coupled lower and upper quasi solutions of system (3.4) on N_0^+.Suppose further that*

$$f_i(n, u_i, [u]_{pi}, [u]_{qi}) - f_i(n, s_i, [u]_{pi}, [u]_{qi}) \geq -M(u_i - s_i) \tag{3.6}$$

whenever $v_0 \leq u \leq w_0$ *and* $v_{0,i} \leq s_i \leq u_i \leq w_{0,i}$ *and* $M > 0$. *Then there exists monotone sequences* $\{v_n\}$, $\{w_n\}$ *such that* $v_n \to v$ *and* $w_n \to w$ *as* $n \to \infty$ *uniformly and monotonically to coupled minimal and maximal quasi solutions of (3.4)on* N_0^+ *provided* $v_0(0) \leq u_0 \leq w_0(0)$. *Further if u is any solution of (3.4) such that* $v_0 \leq u \leq w_0$ *on* N_0^+ *then* $v \leq u \leq w$ *on* N_0^+.

Proof. For any $y, z \in C[N_0^+, R^n]$ such that $v_0 \leq y, z \leq w_0$ on N_0^+,we define

$$\nabla^\alpha u_{i+1} = f_i(n, y_i, [y]_{pi}, [z]_{qi}) - M[u_i - y_i], u(0) = u_0. \tag{3.7}$$

Clearly, (3.7) is a non-homogeneous equation in u_i and has an unique solution [8]. In order to construct and establish the convergence of the monotone sequences $\{v_n\}$, $\{w_n\}$, we define a mapping A such that $A[y, z] = u$, where u is the unique solution of (3.7). This sequences $\{v_n\}$, are $\{w_n\}$ derived in the following steps.

(a). $v_0 \leq A[v_0, w_0], w_0 \geq A[w_0, v_0]$.

(b). 'A' possesses the mixed quasi monotone property on the segment $[v_0, w_0]$, where the segment $[v_0, w_0] = \{u \in C[N_0^+, R^n] : v_0 \leq u \leq w_0\}$.

To prove (a): Let v_1 be the unique solution of (3.7) with $y = v_0, z = w_0$. Then $A[v_0, w_0] = v_1$. Now let $p_i = v_{0,i} - v_{1,i}$ and consider

$$\begin{aligned}
\nabla^\alpha p_{i+1} &= \nabla^\alpha[v_{0,i+1} - v_{1,i+1}], \\
&\leq f_i(n, v_{0,i}, [v_0]_{pi}, [w_0]_{qi}) \\
&\quad -f_i(n, v_{0,i}, [v_0]_{pi}, [w_0]_{qi}) + M[v_{1,i} - v_{0,i}] \\
&\leq -Mp_i. \\
\text{And also} \quad p_i(0) &\leq v_{0,i}(0) - v_{1,i}(0) \leq u(0) - u(0) = 0.
\end{aligned}$$

Hence by Theorem 3.2, we have $p_i \leq 0$. Thus $v_{0,i} \leq v_{1,i}$, i.e., $v_{0,i} \leq A[v_{0,i}, w_{0,i}]$.

Similarly we can prove $w_{0,i} \geq A[w_{0,i}, v_{0,i}]$.

To prove (b): Let us take $y_1, y_2, z \in [v_0, w_0]$ be such that $y_1 \leq y_2$. Suppose $A[y_1, z] = u_1$ and $A[y_2, z] = u_2$. Take $p_i = u_{1,i} - u_{2,i}$.

$$\begin{aligned}
\nabla^\alpha p_{i+1} &= \nabla^\alpha[u_{1,i+1} - u_{2,i+1}] \\
&= f_i(n, y_{1,i}, [y_1]_{pi}, [z]_{qi}) - M[u_{1,i} - y_{1,i}] - f_i(n, y_{2,i}, [y_2]_{pi}, [z]_{qi}) \\
&\quad +M[u_{2,i} - y_{2,i}] \\
&\leq M[y_{2,i} - y_{1,i}] - M[u_{1,i} - y_{1,i}] + M[u_{2,i} - y_{2,i}] \\
&\leq -M[u_{1,i} - u_{2,i}]
\end{aligned}$$

$$\leq \quad -Mp_i.$$

Using similar arguments as above, we have $p_{i+1} \leq 0$. Thus $u_{1,i} \leq u_{2,i}$ or $A[y_1, z] \leq A[y_2, z]$. Similarly if $y, z_1, z_2, \in [v_0, w_0]$ such that $z_1 \leq z_2$, then as before we can prove that $A[y, z_1] \geq A[y, z_2]$.

Consequently this implies that $A[y, z] \leq A[z, y]$ for $y \leq z$. Or A satisfies mixed quasi monotone property.

Continuing this way using (a) and (b) ,we can define the sequences

$$v_n = A[v_{n-1}, w_{n-1}], \qquad w_n = A[w_{n-1}, v_{n-1}]$$

satisfying

$$v_0 \leq v_1 \leq ... \leq v_n \leq w_n \leq ... \leq w_1 \leq w_0.$$

Since the sequences are monotonic and hence by Dini's theorem they converge uniformly to coupled quasi solutions of (v, w) of (3.7).

Therefore $v =^{\lim}_{n\to\infty} v_n, w =^{\lim}_{n\to\infty} w_n$. Hence $v_i(n), w_i(n)$ satisfy

$$\nabla^\alpha v_{i+1} = f_i(n, v_i, [v]_{pi}, [w]_{qi}), v(0) = u_0.$$
$$\nabla^\alpha w_{i+1} = f_i(n, v_i, [w]_{pi}, [v]_{qi}), w(0) = u_0.$$

Now we show that (v, w) are coupled minimal and maximal quasi solutions of (3.4) respectively. Let (u_1, u_2) be any coupled quasi solutions of (3.4) such that $u_1, u_2 \in [v_0, w_0]$.

Let us suppose that for some integer $k > 0$,$v_k \leq u_1, u_2 \leq w_k$ on N_0^+.

Now we take $p_i = v_{k+1,i} - u_{1,i}$ and using the mixed quasi monotone property of f we have

$$\begin{aligned}
\nabla^\alpha p_{i+1} &= \nabla^\alpha [v_{k+1,i+1} - u_{1,i+1}], \\
&= f_i(n, v_k(i), [u_1]_{pi}, [u_2]_{qi}) - M(v_{k+1,i} - v_{k,i}) - f_i(n, u_{1,i}, [u_1]_{pi}, [u_2]_{qi}), \\
&\leq M(u_{1,i} - v_{k,i}) - M(v_{k+1,i} - v_{k,i}), \\
&\leq -Mp_i.
\end{aligned}$$

Also $p_i(0) \leq 0$ implies that $p_i \leq 0$. Thus $v_k \leq u_1$. Similarly we can prove $u_1 \leq w_{k+1}$ also. By the principle of mathematical induction, it can be proved that, for every $m \in \mathbb{N}_0^+$, $v_m(n) \leq u_1(n), u_2(n) \leq w_m(n)$. As $m \to \infty$, we have $v(n) \leq u_1(n), u_2(n) \leq w(n)$. Hence the functions $v(n)$ and $w(n)$ defined on $n \in \mathbb{N}_0^+$ are minimal and maximal quasi solutions of (3.4) respectively.

Since any solution $u \in [v_0, w_0]$ can be considered as (u, u) coupled quasi solutions of (3.4), we also have $v \leq u \leq w$ on $n \in \mathbb{N}_0^+$.

This completes the proof. □

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