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DYNAMICS OF AN SIS EPIDEMIC MODEL WITH A SATURATED INCIDENCE RATE UNDER TIME DELAY AND STOCHASTIC INFLUENCE

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Abstract: This paper examines an SIS model with saturated incidence rate and latent period. To start with the stability of the disease-free and endemic equilibrium of the model with and without delay is dealt with. The existence of Hopf bifurcation is analysed and then obtained by regarding the time delay as the bifurcation parameter. Further, the stochastic model is derived from the deterministic epidemic model through the introduction of random perturbations around the endemic equilibrium point and stochastic stability properties of the model are investigated. The examples and simulations are supplied to throw light on results arrived at.

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1. Introduction

SIR models are predicated on transitions emerging from susceptible (S) to infective (I) to removed (R), with the removal coming through recovery with full immunity (as in measles) or through death from the disease (as in plague, rabies, and many other animal diseases).

Another type of model is an SIS model [1,2] in which infectives go back to the susceptible class on recovery because the disease guarantees no immunity against reinfection. Such models work extremely well with most diseases transmitted by bacterial or helminth agents, and most sexually transmitted diseases (including gonorrhoea, but not such diseases as AIDS, from which there is no recovery). One important and perhaps crucial aspect in which SIS models differ from SIR models is that in the former there is a continuing flow of new susceptible, namely, recovered infectives.

The simplest SIS model, due to Kermack and McKendrick(1932) [13], is

$$\frac{dS}{dt} = -\beta SI + \gamma I,$$
$$\frac{dI}{dt} = \beta SI - \gamma I.$$

Mathematical modelling apparently is an important tool to understand and predict the spread of infectious diseases [5]. The rate of incidence very naturally has a crucial role to play in comprehending the processes. It is an epidemiological model that looks at the rate at which a susceptible organism begins to show symptoms of infection too. The representation of incidence rate takes the form made so significant through the classical epidemic Kermack McKendrick model (1927) βSI, where S and I denote the number of susceptible and infectious, respectively, while β is the infection coefficient. The standard incidence is $\beta SI/N$ where N is the total population size and β is the daily contact rate. The saturated incidence rate [7-9] assumes the form $\beta SI/1 + \alpha I$, where α is a constant. The effect of saturation factor (α) stems from epidemic control.

This paper sees the adoption of an SIS model with saturated incidence rate described by the following differential system of equations:

$$\begin{aligned} \frac{dS}{dt} &= rS(1 - \frac{S}{k}) - \frac{\beta SI}{1 + \alpha I} - dS + \gamma I \\ \frac{dI}{dt} &= \frac{\beta SI}{1 + \alpha I} - (d + \gamma)I \end{aligned} \tag{1}$$

The growth rate of the susceptible organisms depends logistically on density where r is the maximal per capita birth rate; $1/k$ is the strength of density dependence on birth rates, β is the transmission rate, d is the natural death rate and γ is the recovery rate of the infective individuals. The saturated incidence rate $\beta SI/1+\alpha I$ was employed by Capasso and Serio [6] in their modelling of cholera, where βSI measures the infection force of the disease and $1/1+\alpha I$ measures the inhibition effect from the behavioural change of the susceptible individuals when their number increases or as a result of the crowding effect of the infective individuals.

In many infectious diseases, an individual once infected is unable to immediately infect another, and he/she/it undergoes a certain dormant period (τ) before it can infect others. Mathematically this may be viewed as the delay effect of the infection. Therefore it is interesting to investigate the saturated incidence rates and the latent period.

Hence, system (1) can be represented as:

$$\begin{aligned} \frac{dS}{dt} &= rS(1-\frac{S}{k}) - \frac{\beta SI(t-\tau)}{1+\alpha I(t-\tau)} - dS + \gamma I \\ \frac{dI}{dt} &= \frac{\beta SI(t-\tau)}{1+\alpha I(t-\tau)} - (d+\gamma)I \end{aligned} \tag{2}$$

The dynamics of the system (2) are studies in terms of local stability and of the description of the Hopf bifurcation [3, 4, 14].

We consider a stochastic version of the SIS model by generating a slight disturbance in the deterministic system (1) through a white noise [11, 15]. Population systems are often subject to environmental noise; that is, due to environmental fluctuations. Parameters involved in epidemic models are not absolute constants, and they may oscillate or fluctuate around some average values.

Based on these factors, several mathematicians and scientists began to express interest in stochastic epidemic models since these models are capable of providing an additional degree of realism in comparison with their deterministic counterparts and therefore offer a greater degree of flexibility and accuracy.

In epidemic model, randomness in the coefficients may stem from errors in the observed or measured population data, marked variation in the populations and diseases, expected uncertainties such as missing data, or simply from incomplete or incorrect or little awareness of how the system actually works.

The following stochastic differential equation corresponds to model (1)

$$\begin{aligned} \frac{dS}{dt} &= rS(1-\frac{S}{k}) - \frac{\beta SI}{1+\alpha I} - dS + \gamma I + \sigma_1(S-S^*)d\xi_t^1 \\ \frac{dI}{dt} &= \frac{\beta SI}{1+\alpha I} - (d+\gamma)I + \sigma_2(I-I^*)d\xi_t^2 \end{aligned} \tag{3}$$

where σ_1, σ_2 are real constants and known as the intensity of environmental fluctuations, and $\xi_t^i = \xi_i(t)$, $i = 1, 2$ are independent of each other in standard Wiener processes.

The paper is structured as follows. In Section 2 we obtain the existence of equilibrium points and their stability with and without delay and we carry out the existence of local Hopf bifurcation in Section 3. Furthermore, we study the dynamical properties of stochastic model (3) by means of Lyapunov functions methods in Section 4. Some numerical simulations aimed at justifying the theoretical analysis are presented in Section 5. Finally, conclusions are presented in Section 6.

2. Equilibrium States and Their Stability

The system (2) has always disease free equilibrium E_0 and unique endemic equilibrium E_1.

The disease-free equilibrium is given by $E_0(\frac{k}{r}(r-d), 0)$ and it is exists if $r>d$. The endemic equilibrium is the solution of following equations

$$rS(1-\frac{S}{k}) - \frac{\beta SI}{1+\alpha I} - dS + \gamma I = 0 \tag{4}$$

$$\frac{\beta SI}{1+\alpha I} - (d+\gamma)I = 0 \tag{5}$$

Now from equation (5) $S^* = (d+\gamma)(1+\alpha I)/\beta$ and substituting S^* in (4), we get

$$r(d+\gamma)\alpha^2 I^2 + \left[2r(d+\gamma) + k\beta d + (k\beta^2 d/\alpha) - k\beta r\right]\alpha I + r(d+\gamma) + k\beta d - k\beta r = 0 \tag{6}$$

we define the basic reproduction number as follows

$$R_0 = \frac{k\beta r}{r(d+\gamma) + k\beta d} \tag{7}$$

From equation (6) we see that if $R_0 \leq 1$, there is no positive solution as in that case coefficient of I^2, I and constant term are all positive, but if $R_0 \geq 1$, then by Descartes rule there exists a unique positive solution of (6) and consequently there exists unique positive equilibrium $E_1(S^*, I^*)$ called endemic equilibrium.

Therefore

$$I^* = \frac{-\Delta_2 + \sqrt{\Delta_2^2 - 4\Delta_1\Delta_3}}{2\Delta_1} \tag{8}$$

where

$$\Delta_1 = r(d+\gamma)\alpha^2, \quad \Delta_2 = [2r(d+\gamma) + k\beta d + (k\beta^2 d/\alpha) - k\beta r]\alpha, \\ \Delta_3 = r(d+\gamma) + k\beta d - k\beta r$$

obviously $\Delta_2^2 - 4\Delta_1\Delta_3>0$, when $R_0>1$

2.1. Disease-Free Equilibrium and Its Stability

The Jacobian matrix of the linearized system of model (2) at $E_0(S^*, 0)$ is given by

$$J = \begin{bmatrix} r - \frac{2rS^*}{k} - d & -\beta S^* e^{-\lambda\tau} + \gamma \\ 0 & \beta S^* e^{-\lambda\tau} - d - \gamma \end{bmatrix} \tag{9}$$

with the characteristic equation

$$\left(\lambda - r + \frac{2rS^*}{k} + d\right)\left(\lambda - \beta S^* e^{-\lambda\tau} + d + \gamma\right) = 0.$$

$$\text{At} \quad E_0\left(\frac{k}{r}(r-d), 0\right) \text{ above equation reduces to}$$

$$(\lambda + r - d)\left(\lambda - \frac{\beta k}{r}(r-d)e^{-\lambda\tau} + d + \gamma\right) = 0 \tag{10}$$

In case of $\tau = 0$ (10) has two real and negative roots are if $r>d$ and $R_0<1$. In case of $\tau>0$, we assume that the root of (10) $\tau = \omega i$ must satisfy

$$\omega^2 = \left(\frac{k\beta d + r(d+\gamma)}{r^2}\right)[k\beta(r-d) + r(d+\gamma)]\,[R_0 - 1]$$

Then, when $r>d$ and $R_0<1$, then there is no positive real roots ω. Therefore, we arrive the following theorem to indicate the stability of E_0.

Theorem 1. *The disease-free equilibrium $E_0(\frac{k}{r}(r-d), 0)$ of system (2) is asymptotically stable when $r>d$ and $R_0<1$ and unstable when $R_0>1$.*

2.2. Endemic Equilibrium and Its Stability

Now the Jacobian matrix J at endemic equilibrium $E_1(S^*, I^*)$ is given by

$$J = \begin{bmatrix} r - \frac{2rS^*}{k} - \frac{\beta I^*}{1+\alpha I^*} - d & -\frac{\beta S^* e^{-\lambda\tau}}{(1+\alpha I^*)^2} + \gamma \\ \frac{\beta I^*}{1+\alpha I^*} & \frac{\beta S^* e^{-\lambda\tau}}{(1+\alpha I^*)^2} - d - \gamma \end{bmatrix} \tag{11}$$

The characterstic equation of (11) is given by

$$\lambda^2 + P_1\lambda + P_2 + e^{-\lambda\tau}(Q_1\lambda + Q_2) = 0 \tag{12}$$

where $P_1 = \frac{2rS^*}{k} + \frac{\beta I^*}{1+\alpha I^*} + 2d + \gamma - r$

$$P_2 = r(d+\gamma) + \frac{2rS^*}{k}(d+\gamma) + d(d+\gamma) + \frac{\beta d I^*}{1+\alpha I^*}$$

$$Q_1 = -\frac{\beta S^*}{(1+\alpha I^*)^2} \quad \& \quad Q_2 = \frac{r\beta S^*}{(1+\alpha I^*)^2} - \frac{2r\beta S^{*2}}{k(1+\alpha I^*)^2} - \frac{d\beta S^*}{(1+\alpha I^*)^2}$$

we need to find the necessary and sufficient condition for every root of the characteristic equation (12) having negative real part.

Case 1: For $\tau = 0$, (12) becomes

$$\lambda^2 + (P_1 + Q_1)\lambda + (P_2 + Q_2) = 0 \tag{13}$$

where $R_0 > 1$, we have $P_1 + Q_1 > 0 \quad \& \quad P_2 + Q_2 > 0$.

By Routh-Hurwitz criteria, all roots of (13) are real and negative or complex conjugate with negative real part.

Hence the system (2) without delay is locally asymptotically stable when $R_0 > 1$.

Case 2: If $\tau > 0$, suppose that there is a positive τ_0 such that equation (12) has pair of purely imaginary roots $\pm i\omega$, $\omega > 0$. Then ω satisfies

$$-\omega^2 + P_1\omega i + P_2 + (Q_1\omega i + Q_2)[\cos\omega t - i\sin\omega t] = 0 \tag{14}$$

separating the real and imaginary parts, we have

$$\begin{aligned} \omega^2 - P_2 &= Q_1\omega\sin\omega t + Q_2\cos\omega t \\ -P_1\omega &= -Q_2\sin\omega t + Q_1\omega\cos\omega t \end{aligned} \tag{15}$$

which is equivalent to

$$\omega^4 + (P_1^2 - 2P_2 - Q_1^2)\omega^2 + (P_2^2 - Q_2^2) = 0 \tag{16}$$

If $P_1^2 - 2P_2 - Q_1^2 > 0, \quad P_2^2 - Q_2^2 > 0$, equation (16) has no real root. Thus the real parts of all Eigen values of (12) are negative for all $\tau \geq 0$. Hence endemic equilibrium E_1 is asymptotically stable for all τ if the following conditions hold:

$$\begin{aligned} &i).R_0 > 1 \\ &ii).(P_1 + Q_1) > 0, (P_2 + Q_2) > 0 \\ &iii).P_1^2 - 2P_2 - Q_1^2 > 0, P_2^2 - Q_2^2 > 0 \end{aligned} \tag{17}$$

If $P_2^2 - Q_2^2$ is negative, there is a unique positive ω_0 satisfying (16) and then there is a positiive τ_0 such that equation (12) has pair of purely imaginary roots $\pm i\omega_0$ as $\tau = \tau_0$, and all eigen values with negative real parts as $0<\tau<\tau_0$.

From (15) τ_k corresponding to ω_0 can be obtained

$$\tau_k = \frac{1}{\omega_0} \arccos\left[\frac{(Q_2 - P_1Q_1)\omega^2 - P_2Q_2}{Q_1^2\omega^2 + Q_2^2}\right] + \frac{2n\pi}{\omega_0}, n = 0, 1, 2..... \tag{18}$$

3. Hopf Bifurcation

Based on the above results, we have the following

Theorem 2. *Assume that $R_0>1$ then there is a positive τ_0 such that the following results hold.*

(i) If $0<\tau<\tau_0$, equation (2) has an endemic equilibrium E_1 which is locally asymptotically stable.

(ii) Equation (2) can undergo a Hopf bifurcation if $\tau>\tau_0$, and a periodic orbit exists in the small neighbourhood of the endemic equilibrium.

Proof. To obtain the Hopf bifurcation, we need to check the transversal condition for the complex eigen values of the E_1 at $\tau = \tau_0$. Then, from equation (12), we have □

$$\frac{d\lambda}{dt}\left[2\lambda + P_1 + Q_1e^{-\lambda\tau} - (Q_1\lambda + Q_2)\tau e^{-\lambda\tau}\right] = \lambda(Q_1\lambda + Q_2)e^{-\lambda\tau}$$

$$\left(\frac{d\lambda}{dt}\right)^{-1} = \frac{2\lambda + P_1 + Q_1e^{-\lambda\tau} - (Q_1\lambda + Q_2)\tau e^{-\lambda\tau}}{\lambda(Q_1\lambda + Q_2)e^{-\lambda\tau}}$$

$$\left(\frac{d\lambda}{dt}\right)^{-1} = \frac{2\lambda + P_1}{-\lambda(\lambda^2 + P_1\lambda + P_2)} + \frac{Q_1}{\lambda(Q_1\lambda + Q_2)} - \frac{\tau}{\lambda}$$

$$\left.\frac{dRe(\lambda)}{dt}\right|_{\lambda=i\omega_0} = Re\left.\left(\frac{d\lambda}{dt}\right)^{-1}\right|_{\lambda=i\omega_0}$$

$$= Re\left[\frac{2i\omega_0 + P_1}{-i\omega_0(-\omega_0^2 + P_1i\omega_0 + P_2)} + \frac{Q_1}{i\omega_0(Q_1i\omega_0 + Q_2)} - \frac{\tau}{i\omega_0}\right]$$

$$= Re\left[\frac{1}{\omega_0}\left(\frac{2i\omega_0 + P_1}{P_1\omega_0 + (\omega_0^2 - P_2)i} + \frac{Q_1}{(-Q_1\omega_0 + Q_2i)} + \tau i\right)\right]$$

$$= \frac{1}{\omega_0}\left(\frac{2\omega_0(\omega_0^2 - P_2) + P_1^2\omega_0}{P_1^2\omega_0 + (\omega_0^2 - P_2)^2} - \frac{Q_1^2}{(Q_1^2\omega_0^2 + Q_2^2)}\right)$$
$$= \frac{2\omega_0^2 + (P_1^2 - 2P_2 - Q_1^2)}{(Q_1^2\omega_0^2 + Q_2^2)}$$

Under the condition $P_1^2 - 2P_2 - Q_1^2 > 0$, we have $\frac{dRe(\lambda)}{dt}\Big|_{\lambda = i\omega_0} > 0$.

Therefore, the transversality condition holds and Hopf bifurcation occurs at $\omega = \omega_0, \quad \tau = \tau_0$.

4. Stochastic Stability of The Model at Endemic Equilibrium

To study the environmental fluctuations on model (1), we assume that the stochastic perturbations are of white noise type and that they are proportional to the distances of S and I respectively. The system (3) has the same equilibria as the system (1).

The stochastic differential system (3) can be centered at its positive equilibrium E_1 by the change of variables

$$u_1 = S - S^*, \quad u_2 = I - I^* \tag{19}$$

The linearized Stochastic Differential Equations around E_1 take the form

$$du(t) = f(u(t))dt + g(u(t))d\xi(t) \tag{20}$$

where $u(t) = (u_1(t), u_2(t))^T$

$$f(u(t)) = \begin{bmatrix} r - \frac{2rS^*}{k} - \frac{\beta I^*}{1+\alpha I^*} - d & \frac{-\beta S^*}{(1+\alpha I^*)^2} + \gamma \\ \frac{\beta I^*}{1+\alpha I^*} & \frac{\beta S^*}{(1+\alpha I^*)^2} - d - \gamma \end{bmatrix} \tag{21}$$

$$g(u) = \begin{bmatrix} \sigma_1 u_1 & 0 \\ 0 & \sigma_2 u_2 \end{bmatrix}$$

Let $C^{1,2}([0, +\infty) \times \Re^2, \Re^+)$ be the family of nonnegative functions. $W(t, u)$ defined on $([0, +\infty) \times \Re^2$ is a continuously differentiable function with respect to t and twice with respect to u.

We define the differential operator L for a function $W(t, u)$ by

$$LW(t, u) = \frac{\partial W(t, u)}{\partial t} + f^T(u)\frac{\partial W(t, u)}{\partial u} + \frac{1}{2}Tr\left[g^T(u)\frac{\partial^2 W(t, u)}{\partial u^2}g(u)\right] \tag{22}$$

$$\frac{\partial W}{\partial u} = col\left(\frac{\partial W}{\partial u_1}, \frac{\partial W}{\partial u_2}, \frac{\partial W}{\partial u_3}\right)$$

$$\frac{\partial^2 W(t,u)}{\partial u^2} = \left(\frac{\partial^2 W}{\partial u_j \partial u_i}\right) \quad i,j = 1,2 \quad \text{and} \quad 'T' \quad \text{means transposition.}$$

with reference to the book by Afanas'ev et al. [10], the following theorem holds.

Theorem 3. *Suppose there exists a function* $W(t,u) \in C^{1,2}([0,+\infty) \times \Re^2, \Re^+)$ *satisfying the following inequalities*

$$\begin{aligned} K_1 \mid u \mid^p \leq W(t,u) \leq K_2 \mid u \mid^p \\ LW(t,u) \leq -K_3 \mid u \mid^p, \end{aligned} \tag{23}$$

where K_1, K_2, K_3 and p are positive constants.

Then the trivial solution of (20) is exponentially p - stable for $t \geq 0$. Moreover, if in (23), $p = 2$ the trivial solution of (20) is globally asymptotically stable in probability.

Theorem 4. *Suppose that*

$$\sigma_1^2 \leq 2\left(\frac{2rS^*}{k} + \frac{\beta I^*}{1+\alpha I^*} + d - r\right), \sigma_2^2 \leq 2\left((d+\gamma) - \frac{\beta S^*}{(1+\alpha I^*)^2}\right) \textit{hold.}$$

Then, the trivial solution of (20) is asymptotically mean square stable.

Proof. Let us consider the Lyapunov function □

$$W(u) = \frac{1}{2}\left[w_1 u_1^2 + w_2 u_2^2\right] \tag{24}$$

where w_1, w_2 are non-negative constants to be chosen in the following. It is easy to check that inequalities (23) hold true with $p = 2$.

$$\begin{aligned} LW(u) = w_1 \left[\left(r - \frac{2rS^*}{k} - \frac{\beta I^*}{1+\alpha I^*} - d\right) u_1 + \left(\gamma - \frac{\beta S^*}{(1+\alpha I^*)^2}\right) u_2\right] u_1 \\ + w_2 \left[\frac{\beta I^*}{1+\alpha I^*} u_1 + \left(\frac{\beta S^*}{(1+\alpha I^*)^2} - (d+\gamma)\right) u_2\right] u_2 + \frac{1}{2} Tr\left[g^T(u) \frac{\partial^2 W(t,u)}{\partial u^2} g(u)\right] \end{aligned} \tag{25}$$

$$\text{with} \quad \frac{1}{2} Tr\left[g^T(u) + \frac{\partial^2 W(t,u)}{\partial u^2} g(u)\right] = \frac{1}{2}\left[w_1 \sigma_1^2 u_1^2 + w_2 \sigma_2^2 u_2^2\right]$$

If in (25) we choose $\left(\frac{\beta S^*}{(1+\alpha I^*)^2} - \gamma\right) w_1 = \frac{\beta I^*}{(1+\alpha I^*)} w_2$, then

$$LW(u) = -\left(\frac{2rS^*}{k} + \frac{\beta I^*}{1+\alpha I^*} + d - r - \frac{1}{2}\sigma_1^2\right) w_1 u_1^2 - \left((d+\gamma) - \frac{\beta S^*}{(1+\alpha I^*)^2} - \frac{1}{2}\sigma_2^2\right) w_2 u_2^2$$

According to Theorem (3), we conclude that the trivial solution of system (20) is globally asymptotically stable.

5. Numerical Simulations

In this section, we substantiate as well as augment our analytical results through numerical simulations considering the following examples.

Example 1. We take the following parameters. $r = 1.9; k = 79.6; d = 0.964; \beta = 0.47; \alpha = 0.102; \gamma = 0.068$.

System (2) has the unique positive equilibrium $E_1(2.7312, 2.3906)$ and $R_0 = 1.8693 > 1$. It follows from result (18), that the critical positive time delay $\tau_0 = 0.5499$ and we know that when $0 \leq \tau < \tau_0, E_1$ is asymptotically stable. From the Theorem (2) when τ passes through the critical value $\tau_0 = 0.5499$, the positive equilibrium loses its stability and a family of periodic solutions bifurcate from E_1. (Fig.1-3)

Example 2. We numerically simulate the dynamics of the deterministic and the stochastic model system around the positive interior steady state for the above set of parameter values with $\sigma_1 = 0.02, \sigma_2 = 0.01$. It is easy to see that, all the conditions of Theorem (4) are satisfied. Fig (4-5) represents deterministic and stochastic trajectories of the system (3).

6. Conclusion

From the above results and numerical examples the following conclusions may be inferred: The disease-free equilibrium exists only if $r > d$ and asymptotically stable when $R_0 < 1$. When $R_0 > 1$, endemic equilibrium is achieved and the population is asymptotically stable. We showed that the local stability of the endemic equilibrium point E_1 depends on the time delay τ. The system changes behaviour from stable to unstable nature around E_1 when τ crosses the critical value τ_0 via a Hopf bifurcation from E_1. The model in question here supplies evidence to the effect that the stochastic model is globally asymptotically stable in probability when the intensities of white noise are less than certain threshold

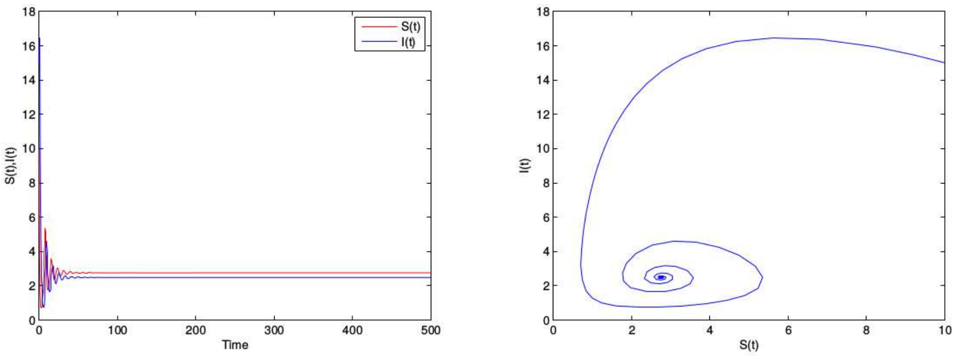

Figure 1: The trajectories and graphs of system (2) with $\tau = 0.1 < \tau_0 = 0.5499$

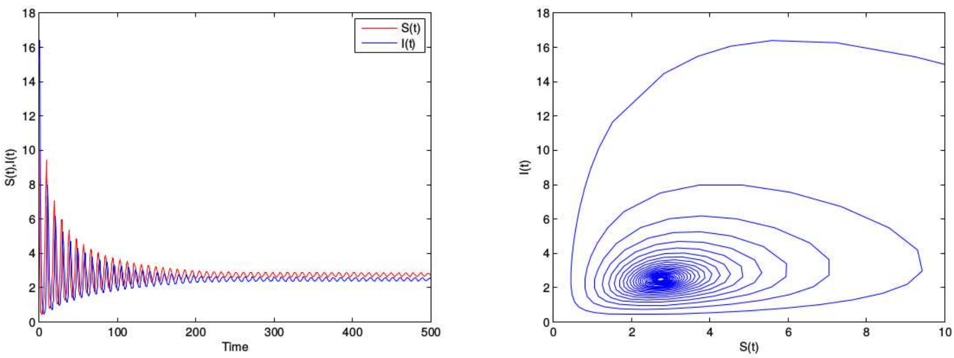

Figure 2: The trajectories and phase graphs of system (2) with $\tau = \tau_0 = 0.5499$

parameters. Finally, numerical simulations have been factored in to verify the analytical results and prove beyond doubt the reliability and validity of the mathematical model assumed for explaining the phenomenon.

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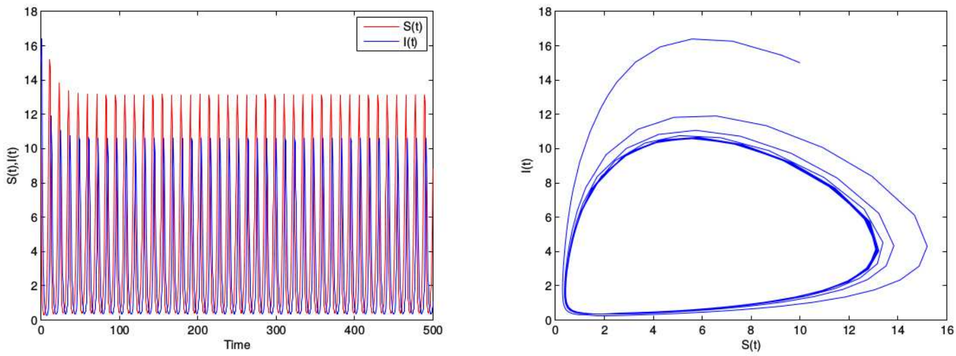

Figure 3: The trajectories and phase graphs of system (2) with $\tau = 1 > \tau_0 = 0.5499$

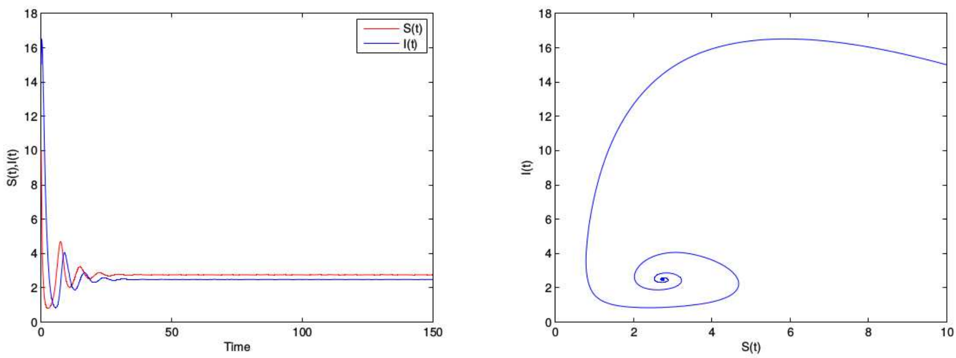

Figure 4: represents deterministic trajectories and phase graphs of model (3)

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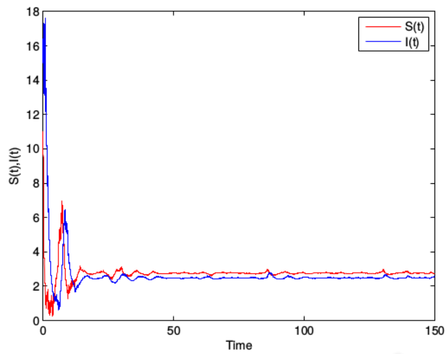

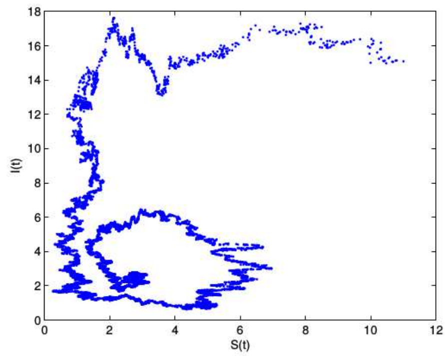

Figure 5: represents stochastic trajectories and phase graphs of model (3)

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