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ON THE MAXIMAL NUMERICAL RANGE OF ELEMENTARY OPERATORS

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Abstract: The notion of the numerical range has been generalized in different directions. One such direction, is the maximal numerical range introduced by Stampfli (1970) to derive an identity for the norm of a derivation on $L(H)$. Unlike the other generalizations, the maximal numerical range has not been largely explored by researchers as many only refer to it in their quest to determine the norm of operators. In this paper we establish how the algebraic maximal numerical range of elementary operators is related to the closed convex hull of the maximal numerical range of the implementing operators $A = (A_1, A_2, ..., A_n)$, $B = (B_1, B_2, ..., B_n)$, on the algebra of bounded linear operators on a Hilbert space H. The results obtained are an extension of the work done by Seddik [2] and Fong [9].

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1. Introduction

For a bounded linear operator T on a Hilbert space H, we denote the set of bounded linear operators on H by $L(H)$ and define the numerical range implemented by the operator T by $W(T) = \{\langle Tx, x\rangle : \|x\| = 1\}$. It is clear $L(H)$ is an algebra when multiplication is pointwise defined. In fact it is a C^*-algebra with the Hilbert adjoint defining an involution on $L(H)$. For more information on numerical ranges we refer the reader to [3], [4], [5] and [7] and for C^*-algebra we refer to [14], [15], [16], and [17].

The concept of maximal numerical range was introduced by Stampfli [8] in proving the norm of a derivation.

Definition 1. The maximal numerical range of $T \in L(H)$, denoted by $W_\circ(T)$ is the set

$$W_\circ(T) = \{\lambda : \langle Tx_n, x_n\rangle \to \lambda, \|x_n\| = 1, \|Tx_n\| \to \|T\|\}.$$

When H is finite dimensional, $W_\circ(T)$ corresponds to the numerical range produced by the maximal vectors (vectors x such that $\|x\| = 1$ and $\|Tx\| = \|T\|$).

The Joint maximal numerical range of $A = (A_1, A_2, ..., A_n)$ is given by

$$W_\circ(A) = \{\{\lambda_i\} \in \mathbb{C}^n : \langle A_i x_n, x_n\rangle \to \lambda_i, \|x_n\| = 1, \|A_i x_n\| \to \|A_i\|\},$$

for $1 \le i \le n$.

If $\mathscr{A}$ is a C*-algebra with identity I, $a \in \mathscr{A}$ and $\mathscr{A}^*$ its dual space, we denote by $S(\mathscr{A}) = \{f \in \mathscr{A}^* : f(I) = 1 = \|f\|\}$, the set of states on $\mathscr{A}$.

The algebraic numerical range of an element $a \in \mathscr{A}$ is the set

$$V(a; \mathscr{A}) = \{f(a) : f \in S(\mathscr{A})\}.$$

$S_\circ(a, \mathscr{A}) = \left\{f : f(I) = 1 = \|f\|, f(a^*a) = \|a\|^2\right\}$ is denote the set of maximal states on $\mathscr{A}$. We introduce the following definition:

Definition 2. The algebraic maximal numerical range, denoted by $V_\circ(a, \mathscr{A})$ is the set $\{f(a) : f \in S_\circ(a, \mathscr{A})\}$

Let $A = (A_1, A_2, ..., A_n), B = (B_1, B_2, ..., B_n)$ be two n-tuples with $A_i, B_i \in L(H)$ for $1 \le i \le n$.

The elementary operator $R_{A,B}$ associated with A and B is the operator on $L(H)$ into itself defined by

$$R_{A,B}(X) = A_1XB_1 + A_2XB_2 + \cdots + A_nXB_n, \forall X \in L(H).$$

For T_1 and T_2 in $L(H)$, we have the following examples of elementary operators:

i) the left multiplication operator L_{T_1} defined by $L_{T_1}(X) = T_1X, \forall X \in L(H)$;

ii) the right multiplication operator R_{T_2} defined by $R_{T_2}(X) = XT_2, \forall X \in L(H)$;

iii) the elementary multiplication operator $M_{T_1,T_2} = L_{T_1}R_{T_2}$ defined by

$$M_{T_1,T_2}(X) = T_1XT_2, \forall X \in L(H),$$

i.e. the elementary operator of length one;

iv) the inner derivation Δ_{T_1} defined by $\Delta_{T_1}(X) = T_1X - XT_1$;

v) the generalized derivation Δ_{T_1,T_2} defined by $\Delta_{T_1,T_2}(X) = T_1X - XT_2$.

Symmetric studies on elementary operators begun in the late 1950's with Lumer and Rosenblum [6] establishing their spectral properties and applications to systems of operator equations. More results on their spectral and structural properties are found in, [10], [11], [12], and [13].

If A and B are n-tuples of commuting operators on $H, W(A), W(B)$ the usual numerical ranges of A and B, $V(R_{A,B})$ the algebraic numerical range of $R_{A,B}$, then:

(i) [1] establishes that the numerical range of an elementary operator acting on the Banach space of the p-Schatten class operators on H,$(\mathscr{C}_p(H), \|.\|_p)$,for $p \geq 1$, satisfies the relation $co(W(A) \circ W(B) \subset V(R_p(A,B))$;

(ii) in [2] it is proved that

$$\overline{co\left\{\sum_{i=1}^{n} \lambda_i\beta_i\right\}} \subset V(R_{A,B}),$$

where $(\lambda_1, ..., \lambda_n) \in W(A), (\beta_1, ..., \beta_n) \in W(B)$.

In this paper we establish results using the maximal numerical range.

Notation. For vectors x and y in a Hilbert space H, define a finite rank operator by $(x \otimes y)z = \langle z, y\rangle\, x, \quad \forall z \in H$.

We take $S \subseteq L(H)$ to be an operator algebra containing finite rank operators. For a set M, we denote by $\overline{M}$ and coM the closure and the convex hull of M respectively.

2. Main Result

Theorem 2.1. *We have*

$$co\overline{\left\{\sum_{i=1}^{n}\lambda_i\beta_i\right\}} \subset V_\circ(R_{A,B} \mid S),$$

where $(\lambda_1, \lambda_2, ..., \lambda_n) \in W_\circ(A), (\beta_1, \beta_2, ..., \beta_n) \in W_\circ(B)$ *and* $V_\circ(R_{A,B} \mid S)$ *the algebraic maximal numerical range of the elementary operator restricted on* S*.*

Proof. et $\lambda \in W_\circ(A), \beta \in W_\circ(B)$.

This implies, there exist sequences $\{x_n\}, \{y_n\} \in H$ such that $\|x_n\| = \|y_n\| = 1$ and

$$\lim_{n\to\infty}\{\langle A_1x_n, x_n\rangle, ..., \langle A_nx_n, x_n\rangle\} = \lim_{n\to\infty}\{\lambda_1, ..., \lambda_n\} = \lambda, \|A_ix_n\| \to \|A_i\|,$$

$$\lim_{n\to\infty}\{\langle B_1x_n, x_n\rangle, ..., \langle B_nx_n, x_n\rangle\} = \lim_{n\to\infty}\{\beta_1, ..., \beta_n\} = \beta, \|B_ix_n\| \to \|B_i\|, \quad 1 \le i \le n.$$

Define f on $L(S)$ by

$$f(\Omega) = \lim_{n\to\infty}\langle \Omega(x_n \otimes y)z_n, (x_n \otimes y_n)z_n\rangle, \quad \forall \Omega \in L(S).$$

f is clearly linear and moreover

$$\begin{aligned}\|x_n \otimes y_n\| &= \sup_n\{\|(x_n \otimes y_n)z_n\| : \|z_n\| = 1\}\\ &= \sup_n\{|\langle z_n, y_n\rangle|\,\|x_n\| : \|z_n\| = 1\}\\ &= \|z_n\|^2\,\|x_n\|\\ &=1.\end{aligned}$$

Also,

$$\begin{aligned}|f(\Omega)| &= |\langle \Omega(x_n \otimes y_n)z_n, (x_n \otimes y_n)z_n\rangle|\\ &= |\langle z_n, y_n\rangle|^2\,|\langle \Omega x_n, x_n\rangle|\\ &\le \|z_n\|^2\,\|y_n\|^2\,\|\Omega x_n\|\,\|x_n\|\\ &\le \|z_n\|^4\,\|\Omega\|\,\|x_n\|^2\\ &= \|\Omega\|, \quad \forall z_n = y_n.\end{aligned}$$

Therefore f is bounded and $\|f\| \leq 1$.

Assume $I \in S$ (S can be unitized in case its non-unital) then,

$$\begin{aligned} f(I) &= \lim \langle (x_n \otimes y_n)z_n, (x_n \otimes y_n)z_n \rangle \\ &= \lim \langle \langle z_n, y_n \rangle x_n, \langle z_n, y_n \rangle x_n \rangle \\ &= \lim |\langle z_n, y_n \rangle|^2 \|x_n\|^2 \\ &= \|z_n\|^4 \|x_n\|^2 \\ &= 1, \quad \forall z_n = y_n \end{aligned}$$

Hence, $f(I) = 1$, so that $\|f(I)\| = 1 \leq \|f\| \Longrightarrow \|f\| \geq 1$.

Thus $\|f\| = 1$.

A linear functional f is said to be positive if $f(\omega\omega^*) \geq 0$ for all $\omega \in S$. Taking a sequence of unit vectors z in H we see that f is a positive linear functional since

$$\begin{aligned} f(\Omega\Omega^*) &= \langle \Omega\Omega^*(x_n \otimes y_n)z_n, (x_n \otimes y_n)z_n \rangle \\ &= \langle \Omega^*(x_n \otimes y_n)z_n, \Omega^*(x_n \otimes y_n)z_n \rangle \\ &= \|\Omega^*(x_n \otimes y_n)z_n\|^2 \\ &= \|\Omega^*\|^2 \geq 0. \end{aligned}$$

Since f is a positive linear functional of unit norm, it follows that f is a state on $L(S)$.Moreover,f is clearly a maximal state.

Recall now that

$$R_{A,B}(X) = \sum_{i=1}^{n} A_i X B_i = A_1 X B_1 + A_2 X B_2 + \cdots + A_n X B_n, \forall X \in L(H)$$

Thus for $X \in S$ we have,

$$\begin{aligned} f\left(\sum_{i=1}^{n} A_i X B_i\right) &= f\left(\sum_{i=1}^{n} A_i (x_n \otimes y_n) B_i z_n\right) \\ &= f\left(\sum_{i=1}^{n} A_i x_n \langle B_i z_n, y_n \rangle\right) \\ &= \sum_{i=1}^{n} \left(f(A_i x_n) \langle B_i z_n, y_n \rangle\right) \\ &= \sum_{i=1}^{n} f(A_i x_n) g(B_i y_n), \quad \forall z_n = y_n \end{aligned}$$

$$= \sum_{i=1}^{n} \lambda_i \beta_i \subset V_\circ(R_{A,B} \mid_S)$$

Since the algebraic maximal numerical range is compact and convex, then

$$\overline{co\left\{\sum_{i=1}^{n} \lambda_i \beta_i\right\}} \subset V_\circ(R_{A,B}). \qquad \square$$

Corollary 2.2. *Let $A \in L(H)$. Then $V_\circ(L_A) = V_\circ(R_A) = V_\circ(A)$.*

Proof. If A is an operator on a Hilbert space H, Fong [9] has showed that $V_\circ(A) = W_\circ(A)$. The inclusion $V_\circ(A) \subseteq V_\circ(L_A)$ follows from this and the above theorem.

Now let $\lambda \in V_\circ(L_A)$. Then there exists f in $(L(L_A))^*$ such that

$$f(L_A) = \lambda, f(I) = 1 = \|f\|$$

and

$$f(L_A^* L_A) = \|L_A\|^2 .$$

Define a functional g on $L(H)$ by $g(A) = f(L_A)$. By simple computation,we see that g is a maximal state on $L(H)$ so that $g(A) = f(L_A) \in V_\circ(A)$. Therefore $V_\circ(L_A) \subseteq V_\circ(A)$. By the same argument, we find also that $V_\circ(R_A) = V_\circ(A)$. $\square$

Corollary 2.3. *For $A, B \in L(H), V_\circ(A) - V_\circ(B) = V_\circ(\Delta_{A,B})$.*

Proof. By the above theorem, we have $W_\circ(A) - W_\circ(B) \subseteq V_\circ(\Delta_{A,B})$ and since $V_\circ(\Delta_{A,B})$ is closed, then we have

$$\overline{(W_\circ(A) - W_\circ(B))} = V_\circ(A) - V_\circ(B) \subseteq V_\circ(\Delta_{A,B}).$$

For the reverse inclusion,

$$\begin{aligned} V_\circ\left(\Delta_{A,B}, \mathscr{A}\right) &= \{f\left(\Delta_{A,B}\right) : f \in S_\circ\left(L\left(\mathscr{A}\right)\right)\} \\ &= \{f\left(L_A - R_B\right) : f \in S_\circ\left(L\left(\mathscr{A}\right)\right)\} \\ &\subseteq \{f\left(L_A\right) : f \in S_\circ\left(L\left(\mathscr{A}\right)\right)\} - \{f\left(R_B\right) : f \in S_\circ\left(L\left(\mathscr{A}\right)\right)\} \\ &= V_\circ\left(L_A\right) - V_\circ\left(R_B\right) \\ &= V_\circ(A) - V_\circ(B). \end{aligned}$$

Therefore $V_\circ\left(\Delta_{A,B}, \mathscr{A}\right) \subseteq V_\circ(A) - V_\circ(B)$. $\square$

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