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EXISTENCE AND GLOBAL ATTRACTIVITY OF POSITIVE PERIODIC SOLUTION TO A HOLLING II TYPE FUNCTIONAL RESPONSE MODEL WITH MUTUAL INTERFERENCE AND GROUP DEFENSE

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Abstract: In this paper, by using the comparison principle of differential equation, Mawhins continuation theorem and Lyapunov functional, a Holling II type functional response model with mutual interference and group defense is studied. Some sufficient conditions are obtained for the existence and global attractivity of a positive periodic solution of the model.

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Key Words: Holling II type functional response model, positive periodic solutions, Lyapunov functional, global attrativity, Mawhin's coincidence theorem

1. Introduction

Many species coexist in the same ecological environment, and they are bound to affect each other, which is a common phenomenon in nature. To describe and analyze the mutual interference among them, various mathematical ecological differentiation models have extensively studied. A Volterra predator-prey model with mutual interference was introduced by Hassell in [1, 2] as follows

$$\begin{cases} \dot{x} = xg(x) - y^m p(x), \\ \dot{y} = y\left(-s + cy^{m-1}p(x) - q(y)\right), \end{cases} \tag{1}$$

where $x(t), y(t)$ stand for the population of the prey and the predator at time t

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respectively, $g(t)$ is the growth rate of the prey population without the predators capturing effects,$0 < m \leq 1$ is a mutual interference constantand $p(x)$ is the so-called predator functional response to prey.

Zhu and Wang [3] considered a Volterra model with mutual interference and a Holling II type functional response:

$$\begin{cases} \dot{x}(t) = x(t)\left(r_1(t) - b_1(t)x(t) - \frac{a_1(t)y(t)}{x(t)+k_1}\right), \\ \dot{y}(t) = y(t)\left(r_2(t) - \frac{a_2(t)x(t)}{x(t)+k_2}\right). \end{cases} \tag{2}$$

Wang and Du et al. [4], Lv and Du [5] discussed a Volterra model with mutual interference and a Holling III type functional response:

$$\begin{cases} \dot{x}(t) = x(t)\left(r_1(t) - b_1(t)x(t)\right) - \frac{c_1(t)x^2(t)}{x^2(t)+k_2}y^m(t), \\ \dot{y}(t) = y(t)\left(-r_2(t) - b_2(t)y(t)\right) + \frac{c_2(t)x^2(t)}{x^2(t)+k_2}y^m(t). \end{cases} \tag{3}$$

And some sufficient conditions which guarantee the existence and global attractivity of a positive periodic solution for the above two kinds of models are obtained.

In this paper, we study a Holling II type functional response model with mutual interference and group defense [6]:

$$\begin{cases} \dot{x}(t) = x(t)\left(r_1(t) - a_1(t)x(t)\right) - \frac{a_2(t)x(t)}{1+d_2(t)x(t)+d_2(t)x^2(t)}y^m(t), \\ \dot{y}(t) = y(t)\left(-r_2(t) - b_1(t)y(t)\right) + \frac{b_2(t)x(t)}{1+d_1(t)x(t)+d_2(t)x^2(t)}y^m(t), \end{cases} \tag{4}$$

with the initial conditions

$$x(0) = x_0 > 0, y(0) = y_0 > 0. \tag{5}$$

where $x(t), y(t)$ stand for the population of the prey and the predator at time t respectively;$a_i(t), b_i(t), r_i(t)$ and $d_i(t)(i = 1, 2)$ are positive ω-periodic continuous functions on $[0, +\infty)$; $0 < m < 1$ is a mutual interference constant. More ecological significance about the model can be seen in [2, 6]. We obtain some sufficient conditions for the ultimate boundedness , the existence and global attractivity of a positive periodic solution of the model. Furthermore our conclusion extends the studied in paper [3, 4, 8]. For convenience and simplicity in the following discussion, we denote

$$f = \inf\{f(t) : t \in R^+\}, F = \sup\{f(t) : t \in R^+\},$$

$$\bar{F} = \frac{1}{\omega}\int_0^{+\infty} f(t)d(t),$$

where $f(t)$ is a continuous function on $[0, +\infty)$.

2. Boundedness of Solutions

Suppose $f(t)$ is continuous on $[0,+\infty)$, it's easy to attain the solution $x(t)>0$ on $[0,+\infty)$ of the initial-value problem

$$\begin{cases} \dot{x}(t)=x(t)f(t), \\ x(0)=x_0>0. \end{cases}$$

Therefore, through the integration of system (4), the solutions $(x(t),y(t))^T$ for system (4) with initial conditions (5) exist and are positive, i.e.,$x(t)>0, y(t)>0$ on $t\in[0,+\infty)$.

Lemma 2.1 Suppose $x(t)$ is the solution of the initial-value problem

$$\begin{cases} \dot{x}(t)\le x(t)\left(p-qx(t)\right), \\ x(0)=x_0>0, \end{cases} \quad \text{or} \quad \begin{cases} \dot{x}(t)\ge x(t)\left(p-qx(t)\right), \\ x(0)=x_0>0, \end{cases}$$

where p,q are positive constants. Then there exist $T_1>0$ such that $x(t)\le\frac{p}{q}$ or $x(t)\ge\frac{p}{q}$ when $t>T_1$.

Proof. It is obvious that $u(t)=\frac{p}{q+(x_0^{-1}p-q)e^{-pt}}$ is the solution of the initial-value problem $\dot{u}(t)=u(t)\left(p-qu(t)\right), u(0)=x_0$.

By the comparison principle we obtain

$$x(t)\le\frac{p}{q+(x_0^{-1}p-q)e^{-pt}}, \qquad \text{or} \qquad x(t)\ge\frac{p}{q+(x_0^{-1}p-q)e^{-pt}},$$

This implies $\lim_{t\to+\infty}\sup x(t)\le\frac{p}{q}$, or $\lim_{t\to+\infty}\inf x(t)\ge\frac{p}{q}$. So there exist $T_1>0$ such that $x(t)\le\frac{p}{q}$ or $x(t)\ge\frac{p}{q}$ when $t>T_1$. □

Lemma 2.2 Suppose p,q are positive constants, and $x(t)$ is the solution of the initial-value problem

$$\begin{cases} \dot{x}(t)\le x(t)\left(-p+qx^{m-1}(t)\right), \\ x(0)=x_0>0, \end{cases} \text{or} \begin{cases} \dot{x}(t)\ge x(t)\left(-p-qx^{m-1}(t)\right), \\ x(0)=x_0>0, \end{cases}$$

where $0<m<1$.Then there exist $T_2>0$ such that $x(t)\le\left(\frac{q}{p}\right)^{\frac{1}{1-m}}$ or $x(t)\ge\left(\frac{q}{p}\right)^{\frac{1}{1-m}}$ when $t>T_2$.

Proof. The result can be obtained by using the same proof method of Lemma 2.1. □

By Lemma 2.1 and Lemma 2.2, we will proof the ultimate boundedness of solutions for system (4).

Theorem 2.1 Suppose $(x(t), y(t))^T$ is the solution of system (4) with initial conditions (5). If $r_1 - L_2^m > 0$, then, there exist positive constants $K_i, L_i (i = 1, 2)$ and $T > 0$ such that $K_1 \leq x(t) \leq L_1$ and $K_2 \leq y(t) \leq L_2$ when $t > T_2$, where $L_i, K_i (i = 1, 2)$ are given in the following proof.

Proof. From the first equation of system (4), we obtain

$$\dot{x} \leq x(t) \left(R_1 - a_1 x(t) \right).$$

By Lemma 2.1, there exist $T_1 > 0$,when $t > T_1$, we have

$$x(t) \leq \frac{R}{a_1} := L_1. \tag{6}$$

From the second equation of system (4),we obtain

$$\dot{y}(t) \leq y(t) \left(-r_2 + \frac{B_2}{d_1} y^{m-1} \right).$$

By Lemma 2.2, there exist $T_2 > T_1$, when $t > T_2$, we have

$$y(t) \leq \left(\frac{B_2}{d_1 r_2} \right)^{\frac{1}{1-m}} := L_2. \tag{7}$$

Then from the first equation of system (4), we have

$$\begin{aligned} \dot{x}(t) &\geq \left(r_1(t) - \frac{y^m(t)}{1 + d_1(t)x(t) + d_2(t)x^2(t)} - a_1 x(t) \right) \\ &\geq x(t) \left((r_1 - L_2^m) - A_2 x(t) \right). \end{aligned}$$

By Lemma 2.1, when $r_1 - L_2^m > 0$, there exist $T_3 > T_2$ such that

$$x(t) \geq \frac{r_1 - L_2^m}{A_1} := K_1, \quad \text{while } t > T_3. \tag{8}$$

Then from the second equation of system (4), we have

$$\dot{y}(t) \geq y(t) \left(-(R_2 + B_1 L_2) + \frac{b_1 K_1}{1 + D_1 L_1 + D_2 L_1^2} y^{m-1}(t) \right)$$

By Lemma 2.2, there exist $T_4 > T_3$, when $t > T_4$, we have

$$y(t) \geq \left[\frac{b_1 K_1}{(R_2 + B_1 L_2)(1 + D_1 L_1 + D_2 L_1^2)} \right]^{\frac{1}{1-m}} := K_2. \tag{9}$$

From formulas (6), (7), (8) and (9), there exist $T > T_4$, when $t > T$, we have

$$K_1 \leq x(t) \leq L_1, K_2 \leq y(t) \leq L_2, \tag{10}$$

where $K_i, Li(i = 1, 2)$ are irrelevant to any solutions of system (4), but only decided by the systems coefficients. The proof is complete. $\square$

3. Existence for Periodic Solutions

Firstly, We introduce the Mawhin's coincidence theorem.

Definition 3.1. (see Gain) Let X and Y be both Banach spaces, $L : \mathrm{Dom}L \subset X \to Y$ be linear mapping. If $\mathrm{Im}L \in Y$ is closed and $\dim \mathrm{Ker}L = \mathrm{codim}\ \mathrm{Im}L < +\infty$, we call operator L a Fredholm operator with index zero.

Let L be a Fredholm operator with index zero, then there exist linear projection operators $P : X \to \mathrm{Dom}L$ and $Q : X \to Y$ such that $\mathrm{Im}P = \mathrm{Ker}L, \mathrm{Ker}Q = \mathrm{Im}L = \mathrm{Im}(I - Q)$ and $X = \mathrm{Ker}L \oplus \mathrm{Ker}P$, $Y = \mathrm{Im}L \oplus \mathrm{Im}Q$. The mapping $L_P = L \mid_{\mathrm{Dom}L\cap \mathrm{Ker}P}$: $\mathrm{Dom}L \cap \mathrm{Ker}P \to \mathrm{Im}L$ is reversible, its inverse mapping is denoted by K_P, so $K_P : \mathrm{Im}L \to \mathrm{Dom}L \cap \mathrm{Ker}P$.

Definition 3.2[7] Let $L : \mathrm{Dom}L \subset X \to Y$ be a Fredholm operator with index zero,$Q \subset X$ be an open set and $N : X \to Y$ be a continuous mapping. If $QN(\bar{\Omega})$ is bounded on Y and $K_P(I - Q)N(\overline{\Omega}) \in X$ is relative compact, we say N is L-compact on $\bar{\Omega}$.

Lemma 3.1. (see Gain) Let X and Y be both Banach spaces,$L : \mathrm{Dom}L \subset X \to Y$ be a Fredholm operator with index zero,$\Omega \subset X$ be an open bounded set, and $N : \bar{\Omega} \to Y$ be L-compact on $\bar{\Omega}$. If all the following conditions hold:
(i)$Lx \neq \lambda Nx, x \in \partial\Omega \cap \mathrm{Dom}L, \lambda \notin (0, 1)$;
(ii)$QNx \neq 0, x \in \partial\Omega \cap \mathrm{Ker}L$;
(iii)$\deg\{JQN, \Omega \cap \mathrm{Ker}L, 0\} \neq 0$,where $J : \mathrm{Im}Q \to \mathrm{Ker}L$ is an isomorphism.
then the equation $Lx = Nx$ has at least one solution on $\bar{\Omega} \cap \mathrm{Dom}L$.

Let $u(t) = \ln x(t), v(t) = \ln y(t)$,then system (4) can be changed to

$$\begin{cases} \dot{u}(t) = F_1 (t, u(t), v(t)), \\ \dot{v}(t) = F_2 (t, u(t), v(t)), \end{cases} \tag{11}$$

where

$$F_1 (t, u(t), v(t)) = r_1(t) - a_1(t)e^{u(t)} - \frac{a_2(t)e^{u(t)}}{1 + d_1(t)e^{u(t)} + d_2(t)e^{2u(t)}} e^{mv(t)},$$

$$F_2\left(t,u(t),v(t)\right)=-r_2(t)-a_2(t)e^{u(t)}+\frac{b_2(t)e^{2u(t)}}{1+d_1(t)e^{u(t)}+d_2(t)e^{2u(t)}}e^{(m-1)v(t)}.$$

Now we consider the system

$$\begin{cases} \dot{u}(t)=\lambda F_1\left(t,u(t),v(t)\right), \\ \dot{v}(t)=\lambda F_2\left(t,u(t),v(t)\right), \end{cases} \lambda\in(0,1). \tag{12}$$

Lemma 3.2. (see Gain) If $(u(t),v(t))^T$ is a positive ω-periodic solution of system (12), then there exist a positive number S_1 such that $|u(t)|+|v(t)|\le S_1$, where S_1 will be given in the following proof.

Proof. Since $(u(t),v(t))^T$ is a periodic solution of system (12), we only need to prove the result on the interval $[0,\omega]$.

Let

$$u(\xi_1)=\max_{t\in[0,\omega]}u(t), u(\eta_1)=\min_{t\in[0,\omega]}u(t),$$

$$v(\xi_2)=\max_{t\in[0,\omega]}v(t), v(\eta_2)=\min_{t\in[0,\omega]}v(t),$$

then $\dot{u}(\xi_1)=\dot{v}(\xi_2)=0, \dot{u}(\eta_1)=\dot{v}(\eta_2)=0$.

It follows from (11) that

$$\begin{cases} r_1(\xi_1)-a_1(\xi_1)e^{u(\xi_1)}-\frac{a_2(\xi_1)}{1+d_1(\xi_1)e^{u(\xi_1)}+d_2(\xi_1)e^{2u(\xi_1)}}e^{mv(\xi_1)}=0, \\ -r_2(\xi_2)-b_1(\xi_2)e^{v(\xi_2)}+\frac{b_2(\xi_2)e^{u(\xi_2)}}{1+d_1(\xi_2)e^{u(\xi_2)}+d_2(\xi_2)e^{2u(\xi_2)}}e^{(m-1)v(\xi_2)}=0. \end{cases} \tag{13}$$

In view of the first formula of system (13), we obtain

$$r_1(\xi_1)-a_1(\xi_1)e^{u(\xi_1)}>0, e^{u(\xi_1)}<\frac{r_1(\xi_1)}{a_1(\xi_1)}, u(\xi_1)<\ln\frac{r_1(\xi_1)}{a_1(\xi_1)}.$$

So

$$u(t)\le u(\xi_1)<\ln\frac{r_1(\xi_1)}{a_1(\xi_1)}\le\ln\frac{R_1}{a_1}:=H_1. \tag{14}$$

From the second formula of system (13), we have

$$\frac{b_2(\xi_2)e^{u(\xi_2)}e^{(m-1)v(\xi_2)}}{1+d_1(\xi_2)e^{u(\xi_2)}+d_2(\xi_2)e^{2u(\xi_2)}}>r_2(\xi_2),$$

$$r_2(\xi_2)e^{(m-1)v(\xi_2)}<\frac{b_2(\xi_2)e^{u(\xi_2)}}{1+d_1(\xi_2)e^{u(\xi_2)}+d_2(\xi_2)e^{2u(\xi_2)}}<b_2(\xi_2)e^{u(\xi_2)},$$

$$v_2(\xi_2) < \frac{1}{m-1}\ln\frac{b_2(\xi_2)e^{u(\xi_2)}}{r_2(\xi_2)} \le \frac{1}{m-1}\ln\frac{B_2e^{H_1}}{r_2} := H_2. \tag{15}$$

Similarly, by substituting $\dot{u}(\eta_1) = \dot{v}(\eta_2) = 0$ into system (11), we have

$$\begin{cases} r_1(\eta_1) - a_1(\eta_1)e^{u(\eta_1)} - \frac{a_2(\eta_1)}{1+d_1(\eta_1)e^{u(\eta_1)}+d_2(\eta_1)e^{2u(\eta_1)}}e^{mv(\eta_1)} = 0, \\ -r_2(\eta_2) - b_1(\eta 2)e^{v(\eta_2)} + \frac{b_2(\eta_2)e^{u(\eta_2)}}{1+d_1(\eta_2)e^{u(\eta_2)}+d_2(\eta_2)e^{2u(\eta_2)}}e^{(m-1)v(\eta_2)} = 0. \end{cases} \tag{16}$$

From the first formula of system (16), we can obtain

$$e^{u(\eta_1)} > \frac{r_1(\eta_1) - a_2(\eta_1)e^{mv(\eta_1)}}{a_1(\eta_1)}.$$

Hence, when $r_1 - A_2e^{mH_2} > 0$, we have

$$u(\eta_1) > \ln\frac{r_1(\eta_1) - a_2(\eta_1)e^{mv(\eta_1)}}{a_1(\eta_1)} \ge \ln\frac{r_1 - A_2e^{mH_2}}{A_1} := G_1. \tag{17}$$

Also from the second formula of system (16), we have

$$\frac{b_2(\eta_2)}{1 + d_1(\eta_2)e^{u(\eta_2)} + d_2(\eta_2)e^{2u(\eta_2)}} = r_2(\eta_2)e^{(1-m)v(\eta_2)} + b_1(\eta_2)e^{(2-m)v(\eta_2)}. \tag{18}$$

If $v(\eta_2) \ge 0$,we have $v(t) \ge 0$. And if $v(\eta_2) < 0$, then we have $e^{(1-m)v(\eta_2)} > e^{(2-m)v(\eta_2)}$. So from formula (18), we obtain

$$\frac{b_2(\eta_2)e^{u(\eta_2)}}{1 + D_1e^{H_1} + D_2e^{2H_1}} \le \left(r_2(\eta_2) + b_1(\eta_2)\right)e^{(1-m)v(\eta_2)},$$

$$v(\eta_2) \ge \frac{1}{1-m}\ln\frac{b_2e^{G_1}}{(1 + D_1e^{H_1} + D_2e^{2H_1})(R_2 + B_1)}.$$

Thus, we have

$$v(t) \ge \min\left\{0, \frac{1}{1-m}\ln\frac{b_2e^{G_1}}{(1 + D_1e^{H_1} + D_2e^{2H_1})(R_2 + B_1)}\right\} := G_2. \tag{19}$$

From formulas (14), (15), (17) and (19), to any $t \in [0, \omega]$, we obtain

$$G_1 \le u(t) \le H_1, G_2 \le v(t) \le H_2. \tag{20}$$

Set $E_1 = \max\{|G_1|, |H_1|\}, E_2 = \max\{|G_2|, |H_2|\}, S_1 = E_1 + E_2$, then we have

$$|u(t)| \le E_1, |v(t)| \le E_2, |u(t)| + |v(t)| \le S_1, \forall t \in [0, \omega]. \tag{21}$$

The proof is complete. □

If $(u,v)^T$ is a constant vector solution of system (12), then we have

$$\begin{cases} r_1(t) - a_1(t)e^u - \frac{a_2(t)e^{mv}}{1+d_1(t)e^u+d_2(t)e^{2u}} = 0, \\ -r_2(t) - b_1(t)e^v + \frac{b_2(t)e^u e^{(m-1)v}}{1+d_1(t)e^u+d_2(t)e^{2u}} = 0. \end{cases}$$

Through the integration of the above two formulas, and by the second integral mean-value theorem, we obtain

$$\begin{cases} \bar{r}_1 - \bar{a}_1 e^u - \frac{\bar{a}_2 e^{mv}}{1+d_1(t_1)e^u+d_2(t_1)e^{2u}} = 0, \\ -\bar{r}_2 - \bar{b}_1 e^v + \frac{\bar{b}_2 e^u e^{(m-1)v}}{1+d_1(t_2)e^u+d_2(t_2)e^{2u}} = 0, \end{cases}$$

where $t_1, t_2 \in [0,\omega]$.

Considering the algebraic equations

$$\begin{cases} \bar{r}_1 - \bar{a}_1 e^u - \mu\frac{\bar{a}_2 e^{mv}}{1+d_1(t_1)e^u+d_2(t_1)e^{2u}} = 0, \\ \frac{\bar{b}_2 e^u e^{(m-1)v}}{1+d_1(t_2)e^u+d_2(t_2)e^{2u}} - \bar{b}_1 e^v - \mu\bar{r}_2 = 0, \end{cases} \tag{22}$$

where $\mu \in [0,1]$ is a parameters, we have the following lemma.
Lemma 3.3 If $(u,v)^T$ is a solution of equations (22), then we have $|u|+|v| \le S_2$, where S_2 will be given in the following proof.

Proof. From the first equation of algebraic equations (22), we can easily obtain

$$u \le \ln\frac{\bar{r}_1}{\bar{a}_1} := H_3. \tag{23}$$

From the second equation of algebraic equations (22), we have

$$\frac{\bar{b}_2 e^u e^{(m-1)v}}{1+d_1(t_2)e^u+d_2(t_2)e^{2u}} - \bar{b}_1 e^v \ge 0,$$

$$\bar{b}_1 e^{(2-m)v} \le \frac{\bar{b}_2 e^u}{1+d_1(t_2)e^u+d_2(t_2)e^{2u}} \le \frac{\bar{b}_2}{d_1(t_2)} \le \frac{\bar{b}_2}{d_1},$$

$$v \le \frac{1}{2-m}\ln\frac{\bar{b}_2}{\bar{b}_1 d_1} := H_4. \tag{24}$$

And from the first equation of algebraic equations (22), we have

$$\bar{r}_1 - \bar{a}_1 e^u = \mu\frac{a_2 e^{mv}}{1+d_1(t_1)e^u+d_2(t_1)e^{2u}} \le \bar{a}_2 e^{mv},$$

$$\bar{a}_1 e^u \ge \bar{r}_1 - \bar{a}_2 e^{mv} \ge \bar{r}_1 - \bar{a}_2 e^{mH_4}.$$

So, when $\bar{r}_1 - \bar{a}_2 e^{mH_4} > 0$, we obtain

$$u \geq \ln \frac{\bar{r}_1 - \bar{a}_2 e^{mH_4}}{\bar{a}_1} := G_3. \tag{25}$$

From the second equation of algebraic equations (22), we have

$$\bar{b}_1 e^{(2-m)v} + \bar{r}_1 e^{(1-m)v} \geq \frac{\bar{b}_2 e^u}{1 + d_1(t_2)e^u + d_2(t_2)e^{2u}} \geq \frac{\bar{b}_2 e^{G^3}}{1 + D_1 e^{H_3} + D_2 e^{2H_3}}.$$

If $v \geq 0$, we set 0 to be a lower bound of v.
If $v < 0$, we have $e^{(2-m)v} < e^{(1-m)v}$. Hence we obtain

$$(\bar{b}_1 + \bar{r}_1)e^{(1-m)v} \geq \frac{\bar{b}_2 e^{G^3}}{1 + D_1 e^{H_3} + D_2 e^{2H_3}},$$

$$v \geq \frac{1}{1-m} \ln \frac{\bar{b}_2 e^{G^3}}{(\bar{b}_1 + \bar{r}_1)(1 + D_1 e^{H_3} + D_2 e^{2H_3})}.$$

Thus,we have

$$v \geq \min \left\{ 0, \frac{1}{1-m} \ln \frac{\bar{b}_2 e^{G_3}}{(\bar{b}_1 + \bar{r}_1)(1 + D_1 e^{H_3} + D_2 e^{2H_3})} \right\} := G_4. \tag{26}$$

From formulas (23), (24), (25) and (26), we have

$$G_3 \leq u \leq H_3, G_4 \leq v \leq H_4. \tag{27}$$

Set $E_3 = \max\{|G_3|, |H_3|\}$,$E_4 = \max\{|G_4|, |H_4|\}$,$S_3 = E_3 + E_4$, we have

$$|u| \leq E_3, |v| \leq E_4, |u| + |v| \leq S_2. \tag{28}$$

The proof is complete. □

Theorem 3.1 System (4) has at least one positive ω-periodic solution satisfying initial condition (5).

Proof. Suppose $(u(t), v(t))^T$ is any positive solution of system (4), and let $x(t) = e^{u(t)}, y(t) = e^{v(t)}$, then system (4) is changed to system (11).

Let $X = Y = \{z(t) = (u(t), v(t))^T \in C(R, R^2) : z(t + \omega) = z(t)\}$, and the norm $\|z(t)\| = \max_{t\in[0,\omega]} |u(t)| + \max_{t\in[0,\omega]} |v(t)|$, then X and Y are two Banach spaces.

To $z(t) \in X \to Y$, we define operators L, P, Q as follows:

$$L : \mathrm{Dom}L \cap X \to Y, Lz = \frac{dz}{dt}; P(z) = z(0); Q(z) = \frac{1}{\omega}\int_0^{\omega} z(t)dt,$$

where $\mathrm{Dom}L = \left\{z(t) \in X, z(t) \in C^1(R, R^2)\right\}$. And we define operator $N : X \to Y$ by the form

$$Nz = \begin{pmatrix} r_1(t) - a_1(t)e^{u(t)} - \frac{a_2(t)}{1+d_1(t)e^{u(t)}+d_2(t)e^{2u(t)}}e^{mv(t)} \\ -r_2(t) - b_1(t)e^{v(t)} + \frac{b_2(t)e^{u(t)}}{1+d_1(t)e^{u(t)}+d_2(t)e^{2u(t)}}e^{(m-1)v(t)} \end{pmatrix}.$$

Then $\mathrm{Ker}L = R^2, \mathrm{Im}Q = R^2, \dim \mathrm{Ker}L = \mathrm{codim}\,\mathrm{Im}L = 2$, and $\mathrm{Im}L = \left\{z \in Y : \int_0^{\omega} z(t)dt = 0\right\}$. From Lebesgue dominated convergence theorem, we can proof $\mathrm{Im}L$ is close on Y . Thus L is a Fredholm operator with index zero.

Obviously,P, Q are two continuous projection operators which satisfying $\mathrm{Im}P = \mathrm{Ker}L$ and $\mathrm{Im}P = \mathrm{Ker}L, \mathrm{Im}L = \mathrm{Ker}Q = \mathrm{Im}(I - Q)$. This means that L is reversible on $\mathrm{Dom}L \cap \mathrm{Ker}P$. We define the inverse of L by $K_P : \mathrm{Im}L \to \mathrm{Dom}L \cap \mathrm{Ker}P$, and we can easily obtain $K_P(z) = \int_0^1 z(s)ds - \frac{1}{\omega}\int_0^{\omega} dt \int_0^t z(s)ds$. To $\forall z(t) \in X$, we have

$$\begin{aligned} QN(z) &= Q\left(F_1(t, u(t), v(t)), F_2(t, u(t), v(t))\right)^T \\ &= \left(\frac{1}{\omega}\int_0^{\omega} F_1(t, u(t), v(t))dt, \frac{1}{\omega}\int_0^{\omega} F_2(t, u(t), v(t))dt\right)^T \\ &= \left(\bar{F}_1, \bar{F}_2\right)^T. \end{aligned} \tag{29}$$

Let $w_i(t) = \int_0^t F_i(s, u(s), v(s))ds + \left(\frac{2}{\omega} - t\right)\bar{F}_i$
$-\frac{1}{\omega}\int_0^{\omega} dt \int_0^t F_i(s, u(s), v(s))ds, (i = 1, 2),$
then

$$K_P(I - Q)N(z) = (w_1(t), w_2(t))^T. \tag{30}$$

Therefore, it is easy to check by the Lebesgue convergence theorem that QN and $K_P(I - Q)N$ are both continuous. Suppose any bounded open set $\Omega \in X$, since $F_i(t, u(t), v(t))$ is bounded on $[0, \omega] \times \bar{\Omega}$,so $QN(\bar{\Omega})$ and $K_P(I-Q)N(\bar{\Omega})$ are both uniform bounded and equicontinuous on $[0, \omega]$. By using the Arzela-Ascoli theorem, we know that $QN(\bar{\Omega})$ and $K_P(I-Q)N(\bar{\Omega})$ are both compact. So N is L-compact on $\bar{\Omega}$. Specially, we take $\Omega = \left\{z(t) = (u(t), v(t))^T \in X : \|z(t)\| \le S\right\}$, where $S = S_1 + S_2 + \varepsilon(\varepsilon > 0)$, and S_1, S_2 are defined as in Lemmas 3.2 and 3.3,N is L-compact on $\bar{\Omega}$. Now we check all conditions of Lemma 3.1(i.e., Mawhin's coincidence theorem).

(i) For each $z(t) \in \partial\Omega \cap \mathrm{Dom}L, \lambda \in (0,1), Lz \neq \lambda Nz$, otherwise that $z(t)$ is a positive ω-periodic solution of system (12). Due to Lemma 3.2, we know $\|z(t)\| \leq S_1$. But $z(t) \in \partial\Omega \cap \mathrm{Dom}L$, so we have $\|z(t)\| = S > S_1$. This is conflict.

(ii) For each $z(t) \in \partial\Omega \cap \mathrm{Ker}L$, we have $\frac{dz(t)}{dt} = 0$. So $z(t)$ is a constant vector $z = (u,v)^T$, and $\|(u,v)^T\| = S$. Suppose $QN(u,v)^T = 0$, then by formula (29), we can obtain $z = (u,v)^T$ is a solution of equation (22) with $\mu = 1$. By Lemma 3.3 we have $\|(u,v)\|^T \leq S_2$ which is contradictory to $\|(u,v)\|^T = S > S_2$. This implies that for each $z(t) \in \partial\Omega \cap \mathrm{Dom}L, QNz \neq 0$.

(iii) Take $J : \mathrm{Im}Q \to \mathrm{Ker}L, Jz = z, \forall z(t) \in \mathrm{Im}Q$. So to each $z(t) \in \Omega \cap \mathrm{Ker}L, z = (u,v)^T$ is a constant vector and

$$\begin{aligned} JQN(u,v)^T &= JQ\left(F_1(t,u,v), F_2(t,u,v)\right)^T \\ &= \left(\frac{1}{\omega}\int_0^\omega F_1(t,u,v)dt, \frac{1}{\omega}\int_0^\omega F_2(t,u,v)dt\right)^T \\ &= \begin{pmatrix} \bar{r}_1 - \bar{a}_1 e^u - \frac{\bar{a}_2 e^{mv}}{1+d_1(t_1)e^u+d_2(t_1)e^{2u}} \\ -\bar{r}_2 - \bar{b}_1 e^v + \frac{\bar{b}_2 e^u e^{(m-1)v}}{1+d_1(t_2)e^u+d_2(t_2)e^{2u}} \end{pmatrix}. \end{aligned}$$

We define $\phi : \mathrm{Dom}L \cap \mathrm{Ker}L \times [0,1] \to X$ by

$$\phi(u,v,\mu) = \begin{pmatrix} \bar{r}_1 - \bar{a}_1 e^u \\ \frac{\bar{b}_2 e^u e^{(m-1)v}}{1+d_1(t_2)e^u+d_2(t_2)e^{2u}} - \bar{b}_1 e^v \end{pmatrix} + \mu \begin{pmatrix} \frac{-\bar{a}_2 e^{mv}}{1+d_1(t_1)e^u+d_2(t_1)e^{2u}} \\ -\bar{r}_2 \end{pmatrix}.$$

Then we have $JQN(u,v)^T = \phi(u,v,1), (u,v)^T \in \partial\Omega \cap \mathrm{Ker}L$ and $\phi(u,v,\mu) \neq (0,0)^T$. Otherwise, $(u,v)^T$ is a solution of equations (22). From Lemma 3.3, $\|(u,v)^T\| \leq S_2$ is conflict with $(u,v)^T \in \partial\Omega \cap \mathrm{Ker}L$. So, by the homotopy invariance theorem of topological degree we have

$$\begin{aligned} & \deg\left\{JQN(u,v)^T, \mathrm{Ker}L \cap \Omega, (0,0)^T\right\} \\ = & \deg\left\{\phi(u,v,1), \mathrm{Ker}L \cap \Omega, (0,0)^T\right\} \\ = & \deg\left\{\phi(u,v,0), \mathrm{Ker}L \cap \Omega, (0,0)^T\right\} \\ = & \deg\Bigg\{\left(\bar{r}_1 - \bar{a}_1 e^u, \frac{\bar{b}_2 e^u e^{(m-1)v}}{1+d_1(t_2)e^u+d_2(t_2)e^{2u}} - \bar{b}_1 e^v\right)^T, \\ & \mathrm{Ker}L \cap \Omega, (0,0)^T\Bigg\}. \end{aligned}$$

To consider the following algebraic equations:

$$\begin{cases} \bar{r}_1 - \bar{a}_1 e^u = 0, \\ \frac{\bar{b}_2 e^u e^{(m-1)v}}{1+d_1(t_2)e^u+d_2(t_2)e^{2u}} - \bar{b}_1 e^v = 0. \end{cases} \tag{31}$$

From the first equation,$u = \ln \frac{\bar{r}_1}{\bar{a}_1} := u^*$. Substituting it into the second equation, we have

$$v = \frac{1}{2-m} \ln \frac{\bar{b}_2 e^{u^*}}{\bar{b}_1 \left(1 + d_1(t_2)e^{u^*} + d_2(t)e^{2u^*}\right)} := v^*.$$

Thus, equations (31) have a unique solution $(u^*, v^*)^T \in \mathrm{Ker}L \cap \Omega$.

For convenience, we denote

$$\psi_1(u,v) = \bar{r}_1 - \bar{a}_1 e^u, \psi_2(u,v) = \frac{\bar{b}_2 e^u e^{(m-1)v}}{1 + d_1(t_2)e^u + d_2(t_2)e^{2u}} - \bar{b}_1 e^v.$$

Then

$$\frac{\partial \psi_1(u,v)}{\partial u} = -\bar{a}_1 e^u, \quad \frac{\partial \psi_1(u,v)}{\partial v} = 0,$$

$$\frac{\partial \psi_2(u,v)}{\partial v} = -\frac{(1-m)\bar{b}_2 e^u e^{(m-1)v}}{1 + d_1(t_2)e^u + d_2(t_2)e^{2u}} - \bar{b}_1 e^v,$$

$$\begin{aligned}
& \deg\left\{JQN(u,v)^T, \mathrm{Ker}L \cap \Omega, (0,0)^T\right\} \\
= \ & \mathrm{sgn} \begin{vmatrix} \frac{\partial \psi_1(u,v)}{\partial u} & \frac{\partial \psi_1(u,v)}{\partial v} \\ \frac{\partial \psi_2(u,v)}{\partial u} & \frac{\partial \psi_2(u,v)}{\partial v} \end{vmatrix}_{(u^*,v^*)} \\
= \ & \mathrm{sgn} \begin{vmatrix} -\bar{a}_1 e^{u^*} & 0 \\ \frac{\partial \psi_2(u,v)}{\partial u}\Big|_{(u^*,v^*)} & \frac{-\bar{b}_2 e^{u^*}(1-m)e^{(m-1)v^*}}{1+d_1(t_2)e^{u^*}+d_2(t_2)e2u^*} - \bar{b}_1 e^{v^*} \end{vmatrix} \\
= \ & \mathrm{sgn}\bar{a}_1 e^{u^*} \left\{ \frac{(1-m)\bar{b}_2 e^{u^*} e^{(m-1)v^*}}{1 + d_1(t_2)e^{u^*} + d_2(t_2)e^{2u^*}} + \bar{b}_1 e^{v^*} \right) = 1 \neq 0.
\end{aligned}$$

So far we have verified the conditions of Mawhins coincidence theorem are satisfied. Thus system (11) has at least one positive ω-periodic solution. This show system (4) also has at least one positive ω-periodic solution. The proof is now finished. □

4. Global Attractivity

Definition 4.1 Suppose $(\bar{x}(t), \bar{y}(t))^T$ is a positive ω-periodic solution of system (4), and $(x(t), y(t))^T$ is any positive solution of system (4). If they satisfy $\lim_{t\to+\infty}|x(t)-\bar{x}(t)| = 0$ and $\lim_{t\to+\infty}|y(t)-\bar{y}(t)| = 0$, we call $(\bar{x}(t), \bar{y}(t))^T$ globally attractive.

Lemma 4.1. (see Gopula) If function $f(t)$ is nonnegative, integrable and uniformly continuous on $[0,+\infty)$, then $\lim_{t\to+\infty} f(x) = 0$.

To the positive ω-periodic solution $(\bar{x}(t), \bar{y}(t))^T$ and any positive solution $(x(t), y(t))^T$ of system (4), from (20) and (21), we have

$$e^{G_1} \le \bar{x}(t) \le e^{H_1}, e^{G_2} \le \bar{y}(t) \le e^{H_2}, \tag{32}$$

$$K_1 \le x(t) \le L_1, K_2 \le y(t) \le L_2, (t > T > 0). \tag{33}$$

Let $E_1 = \min\left\{K_1, e^{G_1}\right\}, E_2 = \min\left\{K_2, e^{G_2}\right\}$,$M_1 = \max\left\{L_1, e^{H_1}\right\}$ and $M_2 = \max\left\{L_2, e^{H_2}\right\}$, then

$$E_1 \le x(t), \bar{x}(t) \le M_1, E_2 \le y(t), \bar{y}(t) \le M_2. \tag{34}$$

For convenience, we denote $P(t, x(t)) = 1 + d_1(t) + x(t) + d_2x^2(t)$. So we have the following global attractivity theorem.

Theorem 4.1 Assume the following two conditions hold:

(i) $\Sigma_1 \min_{t\in[0,\omega]}\Big\{a_1(t) - \frac{1}{P(t,K_1)P(t,e^{G_1})}\left(K_2^m a_2(t)d_1(t) - L_2^m\right.$ $\left.(L_1 + e^{H_1})a_2(t)d_2(t) - (1 + L_1e^{H_1}d_2(t))e^{(m-1)H_2}b_2(t)\right)\Big\} > 0;$

(ii)$\Sigma_2 \min_{t\in[0,\omega]}\left\{b_1(t) - \frac{(1-m)L_1M_2^{m-2}b_2(t)}{P(t,K_1)} - \frac{e^{H_1}a_2(t)}{P(t,e^{G_1})}\right\} > 0.$

Then system (4) has one and only one positive ω-periodic solution which is globally attractive.

Proof. From Theorem 3.1 we know that system (4) has at least one positive ω-periodic solution $(\bar{u}(t), \bar{v}(t))^T$, which also satisfy (31) by Lemma 3.2. Suppose $(x(t), y(t))^T$ is any positive periodic solution of system (4). Then from Lemma 2.2, we know it satisfy (33). Now we define the Lyapunov function $v(t) =$

$v_1(t)+v_2(t)$, where $v_1(t)=|\ln x(t)-\ln \bar{x}(t)|, v_2(t)=|\ln y(t)-\ln \bar{y}(t)|$,

$$\begin{aligned} &D^+v_1(t)|_{(4)} \\ &= \operatorname{sgn}(x(t)-\bar{x}(t))\left(\frac{\dot{x}(t)}{x(t)}-\frac{\dot{\bar{x}}(t)}{\bar{x}(t)}\right) \\ &= \operatorname{sgn}(x(t)-\bar{x}(t))\Bigg[-a_1(t)\left(x(t)-\bar{x}(t)\right) \\ &-\left(\frac{a_2(t)y^m(t)}{P(t,x(t))}-\frac{a_2(t)\bar{y}^m(t)}{P(t,\bar{x}(t))}\right)\Bigg]. \end{aligned} \tag{35}$$

Since

$$\begin{aligned} &\frac{a_2(t)y^m(t)}{P(t,x(t))}-\frac{a_2(t)\bar{y}^m(t)}{P(t,\bar{x}(t))} \\ &=\frac{a_2(t)y^m(t)}{P(t,x(t))}-\frac{a_2(t)y^m(t)}{P(t,\bar{x}(t))}-\frac{a_2(t)y^m(t)}{P(t,\bar{x}(t))}-\frac{a_2(t)\bar{y}^m(t)}{P(t,\bar{x}(t))} \\ &=-\frac{a_2(t)y^m(t)}{P(t,x(t))P(t,\bar{x}(t))}\left[d_1(t)+d_2(t)(x(t)+\bar{x}(t))\right] \\ &\quad (x(t)-\bar{x}(t))+\frac{a_2(t)}{P(t,\bar{x}(t))}\left(y^m(t)-\bar{y}^m(t)\right), \end{aligned} \tag{36}$$

by substituting (36) into (35) we get

$$\begin{aligned} &D^+v_1(t)|_{(4)} \\ &= \operatorname{sgn}(x(t)-\bar{x}(t))\Bigg[\Bigg(\frac{a_2(t)y^m(t)\left(d_1(t)+d_2(t)\left(x(t)+\bar{x}(t)\right)\right)}{P(t,x(t))P(t,\bar{x}(t))} \\ &\quad -a_1(t)\Bigg)\times(x(t)-\bar{x}(t))-\frac{a_2(t)\bar{x}(t)}{P(t,\bar{x}(t))}\left(y^m(t)-\bar{y}^m(t)\right)\Bigg]. \\ &\le\left(-a_1(t)+\frac{a_2(t)y^m(t)d_1(t)}{P(t,x(t))P(t,\bar{x}(t))}+\frac{a_2(t)y^m(t)d_2(t)(x(t)+\bar{x}(t))}{P(t,x(t))P(t,\bar{x}(t))}\right) \\ &\quad |x(t)-\bar{x}(t)|+\frac{a_2(t)\bar{x}(t)}{P(t,\bar{x}(t))}|y^m(t)-\bar{y}^m(t)|, \end{aligned} \tag{37}$$

$$\begin{aligned} D^+ v_2(t)|_{(4)} &= \operatorname{sgn}(y(t)-\bar{y}(t))\left(\frac{\dot{y}(t)}{y(t)}-\frac{\dot{\bar{y}}(t)}{\bar{y}(t)}\right) \\ &= \operatorname{sgn}(y(t)-\bar{y}(t))\left[-b_1(t)(y(t)-\bar{y}(t))+\frac{b_2(t)x(t)y^{(m-1)}(t)}{P(t,x(t))}\right. \\ &\quad \left. -\frac{b_2(t)\bar{x}(t)\bar{y}^{(m-1)}(t)}{P(t,\bar{x}(t))}\right]. \end{aligned} \tag{38}$$

Since

$$\begin{aligned} &\frac{b_2(t)x(t)y^{m-1}(t)}{P(t,x(t))}-\frac{b_2(t)\bar{x}^{(}t)\bar{y}^{m-1}(t)}{P(t,\bar{x}(t))} \\ &= \frac{b_2(t)x(t)y^{m-1}(t)-b_2(t)x(t)\bar{y}^{m-1}(t)}{P(t,x(t))}+\frac{b_2(t)x(t)\bar{y}^{m-1}(t)}{P(t,x(t))} \\ &\quad -\frac{b_2(t)\bar{x}(t)\bar{y}^{m-1}(t)}{P(t,\bar{x}(t))} \\ &= \frac{b_2(t)x(t)}{P(t,x(t))}\left(y^{m-1}(t)-\bar{y}^{m-1}(t)\right)+\frac{b_2(t)\bar{y}^{m-1}(t)}{P(t,x(t))P(t,\bar{x}(t))} \\ &\quad \times\left(x(t)P(t,\bar{x}(t))-\bar{x}(t)P(t,\bar{x}(t))\right) \\ &= \frac{b_2(t)x(t)}{P(t,x(t))}\left(y^{m-1}(t)-\bar{y}^{m-1}(t)\right)+\frac{b_2(t)\bar{y}^{m-1}(t)}{P(t,x(t))P(t,\bar{x}(t))} \\ &\quad \times\left[x(t)-\bar{x}(t)+d_2(t)x(t)\bar{x}(t)\left(x(t)-\bar{x}(t)\right)\right], \end{aligned} \tag{39}$$

thus substituting (39) into (38), from $0 < m < 1$, we have

$$\begin{aligned} &D^+ v_2(t)|_{(4)} \\ &= \operatorname{sgn}(y(t)-\bar{y}(t))\left[-b_1(t)(y(t)-\bar{y}(t))+\frac{b_2(t)x(t)}{P(t,x(t))}\left(y^{m-1}(t)-\bar{y}^{m-1}(t)\right)\right. \\ &\quad \left.+\frac{b_2(t)\bar{y}^{m-1}(t)}{P(t,x(t))P(t,\bar{x}(t))}\left((1+d_2(t)x(t)\bar{x}(t))(x(t)-\bar{x}(t))\right)\right] \\ &\le -b_1(t)|y(t)-\bar{y}(t)|+\frac{b_2(t)x(t)}{P(t,x(t))}|y^{m-1}(t)-\bar{y}^{m-1}(t)| \\ &\quad +\frac{b_2(t)\bar{y}^{m-1}(t)}{P(t,x(t))P(t,\bar{x}(t))}\left(1+d_2(t)x(t)\bar{x}(t)\right)|x(t)-\bar{x}(t)|. \end{aligned} \tag{40}$$

To consider function $h(x) = x^\alpha, x \in (0,+\infty), \forall x_1, x_2 \in [a,b] \subseteq (0,+\infty)$, from the differential mean value theorem, we have

$$|x_1^\alpha - x_2^\alpha| = |\alpha|\xi^{\alpha-1}|x_1 - x_2| \quad (\xi \text{ is in between } x_1 \text{ and } x_2).$$

Therefore, when $\alpha < 1$,

$$|\alpha|b^{\alpha-1}|x_1 - x_2| \leq |x_1^\alpha - x_2^\alpha| \leq |\alpha|a^{\alpha-1}|x_1 - x_2|. \tag{41}$$

And when $\alpha > 1$,

$$|\alpha|a^{\alpha-1}|x_1 - x_2| \leq |x_1^\alpha - x_2^\alpha| \leq |\alpha|b^{\alpha-1}|x_1 - x_2|. \tag{42}$$

So by (37) and (40), we have

$$\begin{aligned} & D^+v_1(t)|_{(4)} \\ \leq & \left(-a_1(t) + \frac{K_2^m a_2(t)d_1(t)}{P(t,K_1)P(t,e^{G_1})} + \frac{L_2^m a_2(t)d_2(t)(L_1 + e^{H_1})}{P(t,K_1)P(t,e^{G_1})}\right) \\ & \times |(x(t) - \bar{x}(t)| + \frac{e^{H_1}a_2(t)}{P(t,e^{G_1})}|y(t) - \bar{y}(t)|, \end{aligned} \tag{43}$$

$$\begin{aligned} & D^+v_2(t)|_{(4)} \\ \leq & \left(-b_1(t) + \frac{(1-m)L_1M_2^{m-2}b_2(t)}{P(t,K_1)}\right)|y(t) - \bar{y}(t)| \\ & + \frac{e^{(m-1)H_2}b_2(t)}{P(t,K_1)P(t,e^{G_1})}\left(1 + L_1e^{H_1}d_2(t)\right)|x(t) - \bar{x}(t)|. \end{aligned} \tag{44}$$

From (43) and (44), we have

$$\begin{aligned} & D^+v(t)|_{(4)} = D^+v_1(t)|_{(4)} + D^+v_2(t)|_{(4)} \\ \leq & -\left\{a_1(t) - \frac{1}{P(t,K_1)P(t,e^{G_1})}\left(K_2^m a_2(t)d_1(t) - L_2^m(L_1 + e^{H_1})\right.\right. \\ & \left.\left.\times a_2(t)d_2(t) - \left(1 + L_1e^{H_1}d_2(t)\right)e^{(m-1)H_2}b_2(t)\right)\right\}|x(t) - \bar{x}(t)| \\ & -\left\{b_1(t) - \frac{(1-m)L_1M_2^{m-2}b_2(t)}{P(t,K_1)} - \frac{e^{H_1}a_2(t)}{P(t,e^{G_1})}\right\}|y(t) - \bar{y}(t)| \\ \leq & -\Sigma_1|x(t) - \bar{x}(t)| - \Sigma_2|y(t) - \bar{y}(t)|, \\ & (t > T > 0, \text{where} \quad T \quad \text{is as same as it in Theorem 2.1 .}) \end{aligned}$$

Integrating the above inequality from T to t, we obtain $v(t) + \Sigma_1 \int_T^t |x(s) - \bar{x}(s)|ds + \Sigma_2 \int_T^t |y(s) - \bar{y}(s)|ds \leq v(T) < +\infty$. Because the solution of system (4) is ultimately bounded, so it's derivative is bounded, $|x(t) - \bar{x}(t)|$ and $|y(t) - \bar{y}(t)|$ are uniform continuity. From Lemma 4.1, we have

$$\lim_{t\to+\infty} |x(t) - \bar{x}(t)| = 0, \lim_{t\to+\infty} |y(t) - \bar{y}(t)| = 0.$$

So far we proved the positive periodic solution $(\bar{x}(t), \bar{y}(t))^T$ of system (4) is globally attractive. The proof is now finished. $\square$

Note: Literature [4] and [5] only proved the positive periodic solution of the system is global attractivity. But they dont prove any positive solution is attractivity. In this article We prove the result.

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