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ON SOLVABILITY OF COMPLETELY GENERALIZED NONLINEAR QUASI-VARIATIONAL INCLUSIONS

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Abstract: The purpose of this paper is to introduce and study a class of completely generalized nonlinear quasi-variational inclusions. A perturbed iterative algorithm for finding the approximate solutions of the completely generalized nonlinear quasi-variational inclusion is suggested. The existence and uniqueness of solution of the completely generalized nonlinear quasi-variational inclusion is proved and the convergence of iterative sequence generated by the perturbed iterative algorithm is established.

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1. Introduction

One of the most interesting and important problems in the variational inequality theory is the development of an efficient and implementable iterative algorithm. Among the most effective numerical techniques, the resolvent operator technique is an important one.

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In 1996, Adly [1] studied a class of general variational inclusions with maximal monotone mapping by using the resolvent operator technique. Recently, Huang [3], Wang et al. [7] and others introduced and studied some classes of variational inequalities, established the existence of solutions and the convergence of iterative algorithms by applying the resolvent operator technique, respectively.

In this paper, we introduce and study a new class of completely generalized nonlinear quasi-variational inclusions involving strongly monotone, generalized pseudocontractive, relaxed monotone mappings, and suggest a perturbed iterative algorithm by applying the resolvent operator technique. The solvability of the completely nonlinear quasi-variational inclusion and the convergence of iterative sequence generated by the perturbed iterative algorithm are established. Our results extend, improve and unify the corresponding results due to Dong et al. [2], Huang [3], Huang et al. [4], Verma [6] and Wang et al. [7].

2. Preliminaries

Let H be a real Hilbert space with a norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$, respectively. Assume that I and 2^H denote the identity mapping on H and the family of nonempty subsets of H, respectively. Let g, m, A, B, C, $D : H \to H$ and $N : H \times H \times H \to H$ be mappings. Suppose that $W : H \times H \to 2^H$ is a mapping such that $W(\cdot, y) : H \to 2^H$ is a maximal monotone operator and $range(g-m) \cap dom\, W(\cdot, y) \neq \emptyset$ for each $y \in H$. For a given $f \in H$, we consider the following problem: Find $x \in H$ such that $(g-m)(x) \in dom(W(\cdot, x))$ and

$$f \in N(A(x), B(x), C(x)) + W((g-m)(x), D(x)), \tag{2.1}$$

which is called a *completely generalized nonlinear quasi-variational inclusion*, where

$$(g-m)(x) = g(x) - m(x), \quad \forall x \in H.$$

Special cases:

(A) If $f = 0$, $D = I$ and $N(A(x), B(x), C(x)) = A(x) - B(x)$ for all $x \in H$, then the problem (2.1) is equivalent to finding $x \in H$ such that $(g-m)(x) \in dom(W(\cdot, x))$ and

$$0 \in A(x) - B(x) + W((g-m)(x), x),$$

which was introduced and studied by Huang [3].

(B) If $f = 0$, $N(A(x), B(x), C(x)) = A(x) - B(x)$ and $W((g-m)(x), D(x)) = W(g(x))$ for all $x \in H$, then the problem (2.1) is equivalent to finding $x \in H$ such that $(g - m)(x) \in dom(W(\cdot, x))$ and

$$0 \in A(x) - B(x) + W(g(x)),$$

which was introduced and studied by Adly [1].

The following definitions and results play crucial roles in this paper.

Let $W : H \to 2^H$ be a maximal monotone mapping. For a given $\rho > 0$, the resolvent operator associated with W is defined by

$$J_\rho^W(u) = (I + \rho W)^{-1}(u), \quad \forall u \in H.$$

It is known that J_ρ^W is single-valued and nonexpansive.

Definition 2.1. Let A, B, $C : H \to H$ be mappings. The mapping $N : H \times H \times H \to H$ is said to be

(1) *strongly monotone* with respect to A in the first argument if there exists a constant $\alpha > 0$ such that

$$\langle N(A(x), u, v) - N(A(y), u, v), x - y\rangle \geq \alpha\|x - y\|^2, \quad \forall x, y, u, v \in H;$$

(2) *relaxed monotone* with respect to B in the second argument if there exists a constant $\beta > 0$ such that

$$\langle N(u, B(x), v) - N(u, B(y), v), x - y\rangle \geq -\beta\|x - y\|^2, \quad \forall x, y, u, v \in H;$$

(3) *relaxed Lipschitz* with respect to C in the third argument if there exists a constant $\gamma > 0$ such that

$$\langle N(u, v, C(x)) - N(u, v, C(y)), x - y\rangle \leq -\gamma\|x - y\|^2, \quad \forall x, y, u, v \in H;$$

(4) *Lipschitz continuous* with respect to the first argument if there exists a constant $a > 0$ such that

$$\|N(x, u, v) - N(y, u, v)\| \leq a\|x - y\|, \quad \forall x, y, u, v \in H.$$

Similarly, we can define the Lipschitz continuity of N with respect to the second and third arguments, respectively.

Definition 2.2. A mapping $g : H \to H$ is said to be

(1) *strongly monotone* if there exists a constant $\eta > 0$ satisfying

$$\langle g(x) - g(y), x - y\rangle \geq \eta\|x - y\|^2, \quad \forall x, y \in H;$$

(2) *Lipschitz continuous* if there exists a constant $b > 0$ satisfying

$$\|g(x) - g(y)\| \leq b\|x - y\|, \quad \forall x, y \in H.$$

Definition 2.3. Let g, $m : H \to H$ be mappings. The mapping m is said to be *relaxed monotone* with respect to g if there exists a constant $\lambda > 0$ satisfying

$$\langle m(x) - m(y), g(x) - g(y)\rangle \geq -\lambda\|x - y\|^2, \quad \forall x, y \in H.$$

Lemma 2.4. ([5]) *Let $\{\alpha_n\}_{n\geq 0}$, $\{\beta_n\}_{n\geq 0}$ and $\{\gamma_n\}_{n\geq 0}$ be three nonnegative real sequences satisfying the inequality*

$$\alpha_{n+1} \leq (1 - w_n)\alpha_n + \beta_n w_n + \gamma_n, \quad \forall n \geq 0,$$

where $\{w_n\}_{n\geq 0} \in [0, 1]$, $\sum_{n=0}^{\infty} w_n = \infty$, $\lim_{n\to\infty} \beta_n = 0$ and $\sum_{n=0}^{\infty} \gamma_n < \infty$. Then $\lim_{n\to\infty} \alpha_n = 0$.

Lemma 2.5. *Let ρ be a positive constant. Then the completely generalized nonlinear quasi-variational inclusion* (2.1) *has a solution $x \in H$ if and only if the mapping $G : H \to H$ defined by*

$$\begin{aligned} G(x) =& x - (g - m)(x) + J_\rho^{W(\cdot,D(x))}[(g - m)(x) \\ & - \rho N(A(x), B(x), C(x)) + \rho f], \quad x \in H \end{aligned} \tag{2.2}$$

has a fixed point $x \in H$.

Proof. The completely generalized nonlinear quasi-variational inclusion (2.1) has a solution $x \in H$ if and only if

$$\begin{aligned} & f \in N(A(x), B(x), C(x)) + W((g - m)(x), D(x)) \\ & \iff \rho f - \rho N(A(x), B(x), C(x)) + (g - m)(x) \\ & \qquad \in (g - m)(x) + \rho W((g - m)(x), D(x)) \\ & \iff x = x - (g - m)(x) \\ & \qquad\qquad + J_\rho^{W(\cdot,D(x))}[(g - m)(x) - \rho N(A(x), B(x), C(x)) + \rho f]. \end{aligned}$$

This completes the proof. □

Lemma 2.5 implies that the completely generalized nonlinear quasi-variational inclusion (2.1) is equivalent to a fixed point problems. On the basic of this observation, we now suggest and analyze the following perturbed iterative algorithm for finding the approximate solutions of the completely generalized nonlinear quasi-variational inclusion (2.1).

Algorithm 2.6. *Let A, B, C, D, g, $m : H \to H$, $N : H \times H \times H \to H$ be mappings. For each $n \geq 0$, $W_n : H \times H \to 2^H$ be a mapping. For given $f \in H$, $x_0 \in H$, compute sequence $\{x_n\}_{n\geq 0}$ by the following scheme*

$$\begin{aligned} y_n &= (1 - \alpha_n' - \beta_n')x_n + \alpha_n'\{x_n - (g - m)(x_n) \\ &\quad + J_\rho^{W_n(\cdot, D(x_n))}[(g - m)(x_n) - \rho N(A(x_n), B(x_n), C(x_n)) + \rho f]\} \\ &\quad + \beta_n' v_n, \\ x_{n+1} &= (1 - \alpha_n - \beta_n)x_n + \alpha_n\{y_n - (g - m)(y_n) \\ &\quad + J_\rho^{W_n(\cdot, D(y_n))}[(g - m)(y_n) - \rho N(A(y_n), B(y_n), C(y_n)) + \rho f]\} \\ &\quad + \beta_n u_n, \quad \forall n \geq 0, \end{aligned}$$

where ρ is a positive constant and $\{u_n\}_{n\geq 0}$, $\{v_n\}_{n\geq 0}$ are any bounded sequences in H and $\{\alpha_n\}_{n\geq 0}$, $\{\beta_n\}_{n\geq 0}$, $\{\alpha_n'\}_{n\geq 0}$ and $\{\beta_n'\}_{n\geq 0}$ are any sequences in $[0, 1]$.

3. Main results

Now we establish the existence of solutions of the completely generalized nonlinear quasi-variational inclusion (2.1) and the convergence of the sequence generalized by Algorithm 2.6.

Theorem 3.1. *Let A, B, C, D, g, $m : H \to H$ be Lipschitz continuous with constants a, b, c, d, p and q, respectively, $g - m$ be strongly monotone with constant λ, and m be relaxed monotone with respect to g with constant δ. Assume that $N : H \times H \times H \to H$ is Lipschitz continuous in the first, second and third arguments with constants t, k, h, respectively, and is strongly monotone with respect to A in the first argument with constant α, relaxed monotone with respect to B in the second argument with constant β, and relaxed Lipschitz with respect to C in the third argument with constant γ, respectively. Suppose that $W : H \times H \to 2^H$ is a mapping such that $W(\cdot, y) : H \to 2^H$ is a maximal mapping, $range(g - m) \cap dom(W(\cdot, y)) \neq \emptyset$ for each $y \in H$ and there exists a constant ξ satisfying*

$$\|J_\rho^{W(\cdot,x)}(z) - J_\rho^{W(\cdot,y)}(z)\| \leq \xi\|x - y\|, \quad \forall x, y, z \in H. \tag{3.1}$$

Let

$$\begin{aligned} L &= 2\sqrt{1 - 2\lambda + q^2 + p^2 + 2\delta} + \xi d, \\ K &= \sqrt{1 + 2\beta + b^2k^2} + \sqrt{1 - 2\gamma + h^2c^2}. \end{aligned}$$

Suppose that there exists a constant $\rho > 0$ *satisfying*

$$L + \rho K < 1 \tag{3.2}$$

and one of the following conditions:

$$\begin{aligned} &ta > K, \quad (K - KL - \alpha)^2 > (t^2a^2 - K^2)(2L - L^2), \\ &\left|\rho - \frac{\alpha - K(1-L)}{t^2a^2 - K^2}\right| \\ &\quad < \frac{\sqrt{(K - KL - \alpha)^2 - (t^2a^2 - K^2)(2L - L^2)}}{t^2a^2 - K^2}; \end{aligned} \tag{3.3}$$

$$\begin{aligned} &ta < K, \\ &\left|\rho - \frac{\alpha - K(1-L)}{t^2a^2 - K^2}\right| \\ &\quad > \frac{\sqrt{(K - KL - \alpha)^2 - (t^2a^2 - K^2)(2L - L^2)}}{K^2 - t^2a^2}. \end{aligned} \tag{3.4}$$

Then for a given $f \in H$, *the completely generalized nonlinear quasi-variational inclusion* (2.1) *has a unique solution* $x^* \in H$.

Proof. Let f, x, y be arbitrary points in H and G be defined by (2.2). In view of the Lipschitz continuity of g, m and strong monotonicity of $g - m$, we infer that

$$\begin{aligned} &\|x - y - ((g-m)(x) - (g-m)(y))\|^2 \\ &= \|x - y\|^2 - 2\langle x - y, (g-m)(x) - (g-m)(y)\rangle \\ &\quad + \|m(x) - m(y)\|^2 + \|g(x) - g(y)\|^2 \\ &\quad - 2\langle m(x) - m(y), g(x) - g(y)\rangle \\ &\le (1 - 2\lambda + q^2 + p^2 + 2\delta)\|x - y\|^2. \end{aligned} \tag{3.5}$$

Since $N : H \times H \times H \to H$ is Lipschitz continuous with respect to the first, second and third arguments, respectively, and is strongly monotone with respect to A in the first argument, relaxed monotone with respect to B in the second argument and relaxed Lipschitz with respect to C in the third argument, A, B and C are Lipschitz continuous, we have

$$\begin{aligned} &\|x - y - \rho(N(A(x), B(x), C(x)) - N(A(y), B(x), C(x)))\|^2 \\ &\le (1 - 2\alpha\rho + \rho^2 t^2 a^2)\|x - y\|^2, \end{aligned} \tag{3.6}$$

$$\begin{aligned} &\|x - y - (N(A(y), B(x), C(x)) - N(A(y), B(y), C(x)))\|^2 \\ &\le (1 + 2\beta + b^2 k^2)\|x - y\|^2 \end{aligned} \tag{3.7}$$

and

$$\begin{aligned}&\|x-y+N(A(y),B(y),C(x))-N(A(y),B(y),C(y))\|^2\\&\le(1-2\gamma+h^2c^2)\|x-y\|^2.\end{aligned}\tag{3.8}$$

Using (3.1), we conclude that

$$\begin{aligned}&\|G(x)-G(y)\|\\&\le\|x-y-((g-m)(x)-(g-m)(y))\|\\&\quad+\|J_\rho^{W(\cdot,D(x))}[(g-m)(x)-\rho N(A(x),B(x),C(x))+\rho f]\\&\qquad-J_\rho^{W(\cdot,D(y))}[(g-m)(x)-\rho N(A(x),B(x),C(x))+\rho f]\|\\&\quad+\|J_\rho^{W(\cdot,D(y))}[(g-m)(x)-\rho N(A(x),B(x),C(x))+\rho f]\\&\qquad-J_\rho^{W(\cdot,D(y))}[(g-m)(y)-\rho N(A(y),B(y),C(y))+\rho f]\|\\&\le 2\|x-y-((g-m)(x)-(g-m)(y))\|+\xi\|D(x)-D(y)\|\\&\quad+\|x-y-\rho(N(A(x),B(x),C(x))-N(A(y),B(x),C(x))\|\\&\quad+\rho\|x-y-(N(A(y),B(x),C(x))-N(A(y),B(y),C(x)))\|\\&\quad+\rho\|x-y+N(A(y),B(y),C(x))-N(A(y),B(y),C(y))\|\\&\le\theta\|x-y\|,\end{aligned}\tag{3.9}$$

where

$$\theta=L+\rho K+\sqrt{1-2\alpha\rho+\rho^2t^2a^2}.\tag{3.10}$$

Obviously, (3.2) and one of (3.3) and (3.4) ensure that $\theta<1$. Hence $G(x)$ has a unique fixed point $x^*\in H$. It follows from Lemma 2.5 that x^* is the unique solution of the completely generalized nonlinear quasi-variational inclusion (2.1). This completes the proof. □

Theorem 3.2. *Let A, B, C, D, g, m, N, W, K and L be as in Theorem 3.1 and (3.1) hold. Suppose that for each $n\ge 0$, $W_n(\cdot,y):H\to 2^H$ is maximal monotone mapping, $range(g-m)\cap dom\,W_n(\cdot,y)\ne\emptyset$ for each $y\in H$ and*

$$\|J_\rho^{W_n(\cdot,x)}(z)-J_\rho^{W_n(\cdot,y)}(z)\|\le\xi\|x-y\|,\quad\forall x,y,z\in H\tag{3.11}$$

and

$$\lim_{n\to\infty}\|J_\rho^{W_n(\cdot,x)}(z)-J_\rho^{W(\cdot,x)}(z)\|=0,\quad\forall x,z\in H,\tag{3.12}$$

where ξ is a positive constant. Assume that $\{\alpha_n\}_{n\ge0}$, $\{\beta_n\}_{n\ge0}$, $\{\alpha'_n\}_{n\ge0}$ and $\{\beta'_n\}_{n\ge0}$ satisfy

$$\alpha_n+\beta_n\le1,\quad\alpha'_n+\beta'_n\le1,\quad\forall n\ge0;\tag{3.13}$$

$$\lim_{n\to\infty}\beta_n'=0,\quad \sum_{n=0}^{\infty}\alpha_n=\infty,\quad \sum_{n=0}^{\infty}\beta_n<\infty. \tag{3.14}$$

If there exists a positive constant ρ satisfying (3.2) *and one of* (3.3) *and* (3.4), *then for a given $f\in H$, the completely generalized nonlinear quasi-variational inclusion* (2.1) *has a unique solution $x^*\in H$ and the sequence $\{x_n\}_{n\geq 0}$ defined by Algorithm 2.6 converges strongly to x^*.*

Proof. Let $f\in H$ be a fixed point. It follows from Theorem 3.1 that the completely generalized nonlinear quasi-variational inclusion (2.1) has a unique solution $x^*\in H$. In light of Lemma 2.5, we arrive at

$$\begin{aligned} x^* &= (1-\alpha_n'-\beta_n')x^*+\alpha_n'\{x^*-(g-m)(x^*)\\ &\quad + J_\rho^{W_n(\cdot,D(x^*))}[(g-m)(x^*)-\rho N(A(x^*),B(x^*),C(x^*))\\ &\quad + \rho M(E(x^*),F(x^*))+\rho f]\}+\beta_n'x^*\\ &= (1-\alpha_n-\beta_n)x^*+\alpha_n\{x^*-(g-m)(x^*)\\ &\quad + J_\rho^{W_n(\cdot,D(x^*))}[(g-m)(x^*)-\rho N(A(x^*),B(x^*),C(x^*))\\ &\quad + \rho M(E(x^*),F(x^*))+\rho f]\}+\beta_n x^*,\quad \forall n\geq 0. \end{aligned} \tag{3.15}$$

According to Algorithm 2.6 and (3.15), we obtain that

$$\begin{aligned} &\|y_n-x^*\|\\ &\leq (1-\alpha_n'-\beta_n')\|x_n-x^*\|\\ &\quad +\alpha_n'\|x_n-x^*-((g-m)(x_n)-(g-m)(x^*))\|\\ &\quad +\alpha_n'\|J_\rho^{W_n(\cdot,D(x_n))}[(g-m)(x_n)-\rho N(A(x_n),B(x_n),C(x_n))+\rho f]\\ &\qquad -J_\rho^{W_n(\cdot,D(x^*))}[(g-m)(x_n)-\rho N(A(x_n),B(x_n),C(x_n))+\rho f]\|\\ &\quad +\alpha_n'\|J_\rho^{W_n(\cdot,D(x^*))}[(g-m)(x_n)-\rho N(A(x_n),B(x_n),C(x_n))+\rho f]\\ &\qquad -J_\rho^{W_n(\cdot,D(x^*))}[(g-m)(x^*)-\rho N(A(x^*),B(x^*),C(x^*))+\rho f]\|\\ &\quad +\alpha_n'\|J_\rho^{W_n(\cdot,D(x^*))}[(g-m)(x^*)-\rho N(A(x^*),B(x^*),C(x^*))+\rho f]\\ &\qquad -J_\rho^{W(\cdot,D(x^*))}[(g-m)(x^*)-\rho N(A(x^*),B(x^*),C(x^*))+\rho f]\|\\ &\quad +\beta_n'\|v_n-x^*\|,\quad \forall n\geq 0. \end{aligned} \tag{3.16}$$

It follows from (3.11), (3.12), (3.16) and the proof of Theorem 3.1 that

$$\begin{aligned} \|y_n-x^*\| &\leq (1-\alpha_n'-\beta_n')\|x_n-x^*\|\\ &\quad +\theta\alpha_n'\|x_n-x^*\|+M_n\alpha_n'+M\beta_n' \end{aligned} \tag{3.17}$$

for all $n \geq 0$, where θ is defined by (3.10) and

$$M := \sup\{\|v_n - x^*\|, \|u_n - x^*\| : n \geq 0\} < \infty$$

and

$$\begin{aligned} M_n = \|J_\rho^{W_n(\cdot, D(x^*))}&[(g-m)(x^*) - \rho N(A(x^*), B(x^*), C(x^*)) + \rho f] \\ - J_\rho^{W(\cdot, D(x^*))}&[(g-m)(x^*) - \rho N(A(x^*), B(x^*), C(x^*)) + \rho f]\| \end{aligned}$$

for each $n \geq 0$.

In a similar way, we derive that

$$\begin{aligned} \|x_{n+1} - x^*\| \leq (1 - \alpha_n - \beta_n)&\|x_n - x^*\| \\ &+ \theta\alpha_n\|y_n - x^*\| + M_n\alpha_n + M\beta_n \end{aligned} \tag{3.18}$$

for all $n \geq 0$. Substituting (3.17) into (3.18), we conclude that

$$\begin{aligned} \|x_{n+1} - x^*\| \leq (1 - (1-\theta)\alpha_n)&\|x_n - x^*\| \\ &+ (2M_n + M\beta'_n)\alpha_n + M\beta_n \end{aligned} \tag{3.19}$$

for all $n \geq 0$. It follows from (3.13), (3.14), (3.19), Lemma 2.4 and $\theta < 1$ that $x_n \to x^*$ as $n \to \infty$. This completes the proof. □

Remark 3.3. Theorems 3.1 and 3.2 are generalizations of Theorem 3.1 in [2], Theorem 4.1 in [3], Theorem 4.1 in [4], Theorem 2.2 in [6] and Theorem 3.1 in [7].

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