International Journal of Pure and Applied Mathematics
Volume 113 No. 2 2017, 273-282
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: 10.12732/ijpam.v113i2.7
ijpam.eu

ON UPPER AND LOWER ω-IRRESOLUTE MULTIFUNCTIONS

G. Balaji[1], G. Ganesh[2], N. Rajesh[3 §]

[1]Department of Mathematics
Thangavelu Engineering College
Chennai, Tamilnadu, INDIA

[2,3]Department of Mathematics
Rajah Serfoji Govt. College
Thanjavur, 613005, Tamilnadu, INDIA

Abstract: In this paper we define upper (lower) ω-irresolute multifunction and obtain some characterizations of such a multifunction.

AMS Subject Classification: 54C05, 54C601, 54C08, 54C10
Key Words: Topological spaces, ω-open set, ω-irresolute multifunctions.

1. Introduction

It is well known that various types of functions play a significant role in the theory of classical point set topology. A great number of papers dealing with such functions have appeared, and a good number of them have been extended to the setting of multifunctions. This implies that both, functions and multifunctions are important tools for studying other properties of spaces and for constructing new spaces from previously existing ones. The purpose of this paper is to define upper (respectively lower) ω-irresolute multifunctions and to obtain several characterizations of such a multifunction.

Received: November 1, 2016
Revised: January 19, 2017
Published: March 19, 2017

[§]Correspondence author

2. Preliminaries

Throughout this paper, (X, τ) and (Y, σ) (or simply X and Y) always mean topological spaces in which no separation axioms are assumed unless explicitly stated. Let A be a subset of a space X. For a subset A of (X, τ), $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$ denote the closure of A with respect to τ and the interior of A with respect to τ, respectively. A subset A is said to be semiopen [7] if $A \subset \mathrm{Cl}(\mathrm{Int}(A))$. A subset A of a space (X, τ) is called an ω-closed set [12] if $\mathrm{Cl}(A) \subseteq U$ whenever $A \subseteq U$ and U is semiopen in (X, τ). The complement of an ω-closed set is said to be ω-open. The complement of an ω-closed set is said to be ω-open. The family of all ω-open subsets of a topological space (X, τ), denoted by $\omega O(X, \tau)$, forms a topology on X finer than τ. The ω-closure and the ω-interior, that can be defined in the same way as $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively, will be denoted by $\omega\,\mathrm{Cl}(A)$ and $\omega\,\mathrm{Int}(A)$, respectively. We set $\omega O(X, x) = \{A : A \in \omega O(X) \text{ and } x \in A\}$. A subset U of X is called an ω-neighborhood of a point $x \in X$ if there exists $V \in \omega O(X, x)$ such that $V \subset U$. By a multifunction $F : (X, \tau) \to (Y, \sigma)$, following [3], we shall denote the upper and lower inverse of a set B of Y by $F^+(B)$ and $F^-(B)$, respectively, that is, $F^+(B) = \{x \in X : F(x) \subset B\}$ and $F^-(B) = \{x \in X : F(x) \cap B \neq \emptyset\}$. In particular, $F^-(Y) = \{x \in X : y \in F(x)\}$ for each point $y \in Y$ and for each $A \subset X$, $F(A) = \bigcup_{x \in A} F(x)$. Then F is said to be surjection if $F(x) = y$.

Definition 1. A multifunction $F : (X, \tau) \to (Y, \sigma)$ is said to be:

(i) upper ω-continuous [4] if for each point $x \in X$ and each open set V containing $F(x)$, there exists $U \in \omega O(X, x)$ such that $F(U) \subset V$;

(ii) lower ω-continuous [4] if for each point $x \in X$ and each open set V such that $F(x) \cap V \neq \emptyset$, there exists $U \in \omega O(X, x)$ such that $U \subset F^-(V)$.

3. On Upper and Lower ω-Irresolute Multifunctions

Definition 2. A multifunction $F : (X, \tau) \to (Y, \sigma)$ is said to be:

(i) upper ω-irresolute if for each point $x \in X$ and each ω-open set V containing $F(x)$, there exists $U \in \omega O(X, x)$ such that $F(U) \subset V$;

(ii) lower ω-irresolute if for each point $x \in X$ and each ω-open set V such that $F(x) \cap V \neq \emptyset$, there exists $U \in \omega O(X, x)$ such that $U \subset F^-(V)$.

It is clear that every upper (lower) ω-irresolute multifunction is upper (lower) ω-continuous. But the converse is not true as shown by the following example.

Example 3. Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, \{a\}, \{b, c\}, X\}$. Then the multifunction $F : (X, \tau) \to (X, \tau)$ defined by $F(x) = \{x\}$ is upper ω-continuous but is not upper ω-irresolute In a similar form, we can find a multifunction G that is lower ω-continuous but is not lower ω-irresolute.

Theorem 4. *A multifunction $F : (X, \tau) \to (Y, \sigma)$ is lower ω-irresolute if and only if $F : (X, \omega O(X)) \to (Y, \omega O(Y))$ is lower ω-continuous*

Proof. The proof is clear. □

Theorem 5. *The following statements are equivalent for a multifunction $F : (X, \tau) \to (Y, \sigma)$:*

(i) F is upper ω-irresolute;

(ii) for each point x of X and each ω-neighborhood V of $F(x)$, $F^+(V)$ is an ω-neighborhood of x;

(iii) for each point x of X and each ω-neighborhood V of $F(x)$, there exists an ω-neighborhood U of x such that $F(U) \subset V$;

(iv) $F^+(V) \in \omega O(X)$ for every $V \in \omega O(Y)$;

(v) $F^-(V) \in \omega C(X)$ for every $V \in \omega C(Y)$;

(vi) $\omega\,\mathrm{Cl}(F^-(B)) \subset F^-(\omega\,\mathrm{Cl}(B))$ for every subset B of Y.

Proof. *(i)* ⇒ *(ii)*: Let $x \in X$ and W be an ω-neighborhood of $F(x)$. There exists $V \in \omega O(Y)$ such that $F(x) \subset V \subset W$. Since F is upper ω-irresolute, there exists $U \in \omega O(X, x)$ such that $F(U) \subset V$. Therefore, we have $x \in U \subset F^+(V) \subset F^+(W)$; hence $F^+(W)$ is an ω-neighborhood of x.
(ii) ⇒ *(iii)*: Let $x \in X$ and V be an ω-neighborhood of $F(x)$. Put $U = F^+(V)$. Then, by *(ii)*, U is an ω-neighborhood of x and $F(U) \subset V$.
(iii) ⇒ *(iv)*: Let $V \in \omega O(Y)$ and $x \in F^+(V)$. There exists an ω-neighborhood G of x such that $F(G) \subset V$. Therefore, for some $U \in \omega O(X, x)$ such that $U \subset G$ and $F(U) \subset V$. Therefore, we obtain $x \in U \subset F^+(V)$; hence $F^+(V) \in \omega O(Y)$.
(iv) ⇒ *(v)*: Let K be an ω-closed set of Y. We have $X \backslash F^-(K) = F^+(Y \backslash K) \in \omega O(X)$; hence $F^-(K) \in \omega C(X)$.
(v) ⇒ *(vi)*: Let B be any subset of Y. Since $\omega\,\mathrm{Cl}(B)$ is ω-closed in Y,

$F^{-}(\omega\,\mathrm{Cl}(B))$ is ω-closed in X and $F^{-}(B) \subset F^{-}(\omega\,\mathrm{Cl}(B))$. Therefore, we obtain $\omega\,\mathrm{Cl}(F^{-}(B)) \subset F^{-}(\omega\,\mathrm{Cl}(B))$.
$(vi) \Rightarrow (i)$: Let $x \in X$ and $V \in \omega O(Y)$ with $F(x) \subset V$. Then we have $F(x) \cap (Y \backslash V) = \emptyset$; hence $x \notin F^{-}(Y \backslash V)$. By (vi), $x \in \omega\,\mathrm{Cl}(F^{-}(Y \backslash V))$ and there exists $U \in \omega O(X,x)$ such that $U \cap F^{-}(Y \backslash V) = \emptyset$. Therefore, we obtain $F(U) \subset V$ and hence F is upper ω-irresolute □

Theorem 6. *The following statements are equivalent for a multifunction* $F : (X, \tau) \to (Y, \sigma)$*:*

(i) F *is lower* ω*-irresolute;*

(ii) For each $V \in \omega O(Y)$ *and each* $x \in F^{-}(V)$*, there exists* $U \in \omega O(X,x)$ *such that* $U \subset F^{-}(V)$*;*

(iii) $F^{-}(V) \in \omega O(X)$ *for every* $V \in \omega O(Y)$*;*

(iv) $F^{+}(K) \in \omega C(X)$ *for every* $K \in \omega C(Y)$*;*

(v) $F(\omega\,\mathrm{Cl}(A)) \subset \omega\,\mathrm{Cl}(F(A))$ *for every subset* A *of* X*;*

(vi) $\omega\,\mathrm{Cl}(F^{+}(B)) \subset F^{+}(\omega\,\mathrm{Cl}(B))$ *for every subset* B *of* Y*.*

Proof. $(i) \Rightarrow (ii)$: This is obvious.
$(ii) \Rightarrow (iii)$: Let $V \in \omega O(Y)$ and $x \in F^{-}(V)$. There exists $U \in \omega O(X,x)$ such that $U \subset F^{-}(V)$. Therefore, we have $x \in U \subset \mathrm{Cl}(\mathrm{Int}(U)) \cup \mathrm{Int}(\mathrm{Cl}(U)) \subset \mathrm{Cl}(\mathrm{Int}(F^{-}(V))) \cup \mathrm{Int}(\mathrm{Cl}(F^{-}(V)))$; hence $F^{-}(V) \in \omega O(X)$.
$(iii) \Rightarrow (iv)$: Let K be an ω-closed set of Y. We have $X \backslash F^{+}(K) = F^{-}(Y \backslash K) \in \omega O(X)$; hence $F^{+}(K) \in \omega C(X)$.
$(iv) \Rightarrow (v)$ and $(v) \Rightarrow (vi)$: Straightforward.
$(vi) \Rightarrow (i)$: Let $x \in X$ and $V \in \omega O(Y)$ with $F(x) \cap V \neq \emptyset$. Then $F(x)$ is not a subset of $Y \backslash V$ and $x \notin F^{+}(Y \backslash V)$. Since $Y \backslash V$ is ω-closed in Y, by (vi), $x \notin \omega\,\mathrm{Cl}(F^{+}(Y \backslash V))$ and there exists $U \in \omega O(X,x)$ such that $\emptyset = U \cap F^{-}(Y \backslash V) = U \cap (X \backslash F^{-}(V))$. Therefore, we obtain $U \subset F^{-}(V)$; hence F is lower ω-irresolute. □

Lemma 7. *If* $F : (X, \tau) \to (Y, \sigma)$ *is a multifunction, then* $(\omega\,\mathrm{Cl}\,F)^{-}(V) = F^{-}(V)$ *for each* $V \in \omega O(Y)$*.*

Proof. Let $V \in \omega O(Y)$ and $x \in (\omega\,\mathrm{Cl}\,F)^{-}(V)$. Then $V \cap (\omega\,\mathrm{Cl}\,F)(x) \neq \emptyset$. Since $V \in \omega O(Y)$, we have $V \cap F(x) \neq \emptyset$ and hence $x \in F^{-}(V)$. Conversely, let $x \in F^{-}(V)$. Then $\emptyset \neq F(x) \cap V \subset (\omega\,\mathrm{Cl}\,F)(x) \cap V$ and hence $x \in (\omega\,\mathrm{Cl}\,F)^{-}(V)$. Therefore, we obtain $(\omega\,\mathrm{Cl}\,F)^{-}(V) = F^{-}(V)$. □

Theorem 8. *A multifunction $F : (X, \tau) \to (Y, \sigma)$ is lower ω-irresolute if and only if $\omega \operatorname{Cl} F : (X, \tau) \to (Y, \sigma)$ is lower ω-irresolute*

Proof. The proof is an immediate consequence of Lemma 7 and Theorem 6 (iii). □

Definition 9. A subset A of a topological space (X, τ) is said to be:

(i) α-regular [5] (resp. α-ω-regular) if for each $a \in A$ and any open (resp. ω-open) set U containing a, there exists an open set G of X such that $a \in G \subset \operatorname{Cl}(G) \subset U$;

(ii) α-paracompact [5] if every X-open cover A has an X-open refinement which covers A and is locally finite for each point of X.

Lemma 10. *If A is an α-ω-regular, α-paracompact subset of a space X and U is ω- neighborhood of A, then there exists an open set G of X such that $A \subset G \subset \operatorname{Cl}(G) \subset U$.*

Proof. The proof is similar to that [[5], Theorem 2.5]. □

Definition 11. A multifunction $F : (X, \tau) \to (Y, \sigma)$ is said to be punctually α-paracompact (resp. punctually α-ω-regular, punctually α-regular) if for each $x \in X$, $F(x)$ is α-paracompact (resp. α-ω-regular, α-regular).

Lemma 12. *If $F : (X, \tau) \to (Y, \sigma)$ is punctually α-paracompact and punctually α-ω-regular, $(\omega \operatorname{Cl} F)^+(V) = F^+(V)$ for each $V \in \omega O(Y)$.*

Proof. Let $V \in \omega O(Y)$. Suppose that $x \in (\omega \operatorname{Cl} F)^+(V)$. Then, we have $F(x) \subset \omega \operatorname{Cl}(F(x)) \subset V$ and hence $x \in F^+(V)$. Therefore, we obtain $(\omega \operatorname{Cl} F)^+(V) \subset F^+(V)$. Conversely, suppose that $x \in F^+(V)$. Then $F(x) \subset V$ and by Lemma 10 we have $F(x) \subset G \subset \operatorname{Cl}(G) \subset V$ for some open set G of Y. Therefore, $(\omega \operatorname{Cl} F)(x) \subset V$ and hence $x \in (\omega \operatorname{Cl} F)^+(V)$. Thus, we obtain $F^+(V) \subset (\omega \operatorname{Cl} F)^+(V)$; hence $(\omega \operatorname{Cl} F)^+(V) = F^+(V)$. □

Theorem 13. *Let $F : (X, \tau) \to (Y, \sigma)$ be punctually α-paracompact and punctually α-ω-regular multifunction. Then F is upper ω-irresolute if and only if $\omega \operatorname{Cl} F : (X, \tau) \to (Y, \sigma)$ is upper ω-irresolute.*

Proof. The proof follows from Lemma 12. □

Definition 14. A subset K of a space X is said to be ω-compact relative to X [10] (resp. ω-Lindelöf relative to X) if every cover of K by ω-open sets of X has a finite (resp. countable) subcover. A space X is said to be ω-compact [10] (resp. ω-Lindelöf) if X is ω-compact (resp. ω-Lindelöf) relative to X.

Theorem 15. *Let $F : (X, \tau) \to (Y, \sigma)$ be an upper ω-irresolute multifunction and $F(x)$ is ω-compact relative to Y for each $x \in X$. If A is ω-compact relative to X, then $F(A)$ is ω-compact relative to Y.*

Proof. Let $\{V_i : i \in \Delta\}$ be any cover of $F(A)$ by ω-open sets of Y. For each $x \in A$, there exists a finite subset $\Delta(x)$ of Δ such that $F(x) \subset \cup\{V_i : i \in \Delta(x)\}$. Put $V(x) = \cup\{V_i : i \in \Delta(x)\}$. Then $F(x) \subset V(x) \in \omega O(Y)$ and there exists $U(x) \in \omega O(X, x)$ such that $F(U(x)) \subset V(x)$. Since $\{U(x) : x \in A\}$ is an ω-open cover of A, there exists a finite number of points of A, say, x_1, x_2,...x_n such that $A \subset \cup\{U(x_i) : i = 1, 2, ...n\}$. Therefore, we obtain $F(A) \subset F(\overset{n}{\underset{i=1}{\cup}} U(x_i)) \subset \overset{n}{\underset{i=1}{\cup}} F(U(x_i)) \subset \overset{n}{\underset{i=1}{\cup}} V(x_i) \subset \overset{n}{\underset{i=1}{\cup}} \underset{i\in\Delta(x_i)}{\cup} V_i$. This shows that $F(A)$ is is ω-compact relative to Y. □

Corollary 16. *Let $F : (X, \tau) \to (Y, \sigma)$ be an upper ω-irresolute surjective multifunction and $F(x)$ is ω-compact relative to Y for each $x \in X$. If X is ω-compact, then Y is ω-compact.*

Theorem 17. *Let $F : (X, \tau) \to (Y, \sigma)$ be an upper ω-irresolute multifunction and $F(x)$ is ω-Lindelöf relative to Y for each $x \in X$. If A is ω-Lindelöf relative to X, then $F(A)$ is ω-Lindelöf relative to Y.*

Proof. The proof is similar to that of Theorem 15 and is thus omitted. □

Corollary 18. *Let $F : (X, \tau) \to (Y, \sigma)$ be an upper ω-irresolute surjective multifunction and $F(x)$ is ω-Lindelöf relative to Y for each $x \in X$. If X is ω-Lindelöf, then Y is ω-Lindelöf.*

Definition 19. [8] Let A be a subset of a topological space X. The ω-frontier of A denoted by $\omega Fr(A)$, is defined as follows: $\omega Fr(A) = \omega\,\mathrm{Cl}(A) \cap \omega\,\mathrm{Cl}(X\backslash A)$.

Theorem 20. *The set of a point x of X at which a multifunction $F : (X, \tau) \to (Y, \sigma)$ is not upper ω-irresolute (resp. lower ω-irresolute) is identical with the union of the ω-frontiers of the upper (resp. lower) inverse images of ω-open sets containing (resp. meeting) $F(x)$.*

Proof. Let x be a point of X at which F is not upper ω-irresolute. Then there exists $V \in \omega O(Y)$ containing $F(x)$ such that $U \cap (X \backslash F^+(V)) \neq \emptyset$ for each $U \in \omega O(X, x)$. Then $x \in \omega\,\mathrm{Cl}(X \backslash F^+(V))$. Since $x \in F^+(V)$, we have $x \in \omega\,\mathrm{Cl}(F^+(Y)$ and hence $x \in \omega Fr(F^+(A))$. Conversely, let $V \in \omega O(Y)$ containing $F(x)$ and $x \in \omega Fr(F^+(V))$. Now, assume that F is upper ω-irresolute at x, then there exists $U \in \omega O(X, x)$ such that $F(U) \subset V$. Therefore, we obtain $x \in U \subset \omega\,\mathrm{Int}(F^+(V)$. This contradicts that $x \in \omega Fr(F^+(V))$. Thus, F is not upper ω-irresolute. The proof of the second case is similar. □

Lemma 21. *For a multifunction $F : (X, \tau) \to (Y, \sigma)$, the following holds:*

(i) $G_F^+(A \times B) = A \cap F^+(B)$;

(ii) $G_F^-(A \times B) = A \cap F^-(B)$

for any subset A of X and B of Y.

Theorem 22. *Let $F : (X, \tau) \to (Y, \sigma)$ be a multifunction and X be a connected space. If the graph multifunction of F is upper ω-irresolute (respectively lower ω-irresolute), then F is upper ω-irresolute (respectively. lower ω-irresolute).*

Proof. Let $x \in X$ and V be any ω-open subset of Y containing $F(x)$. Since $X \times V$ is an ω-open set of $X \times Y$ and $G_F(x) \subset X \times V$, there exists an ω-open set U containing x such that $G_F(U) \subset X \times V$. By Lemma 21, we have $U \subset G_F^+(X \times V) = F^+(V)$ and $F(U) \subset V$. Thus, F is upper ω-irresolute. The proof of the lower ω-irresolute of F can be done by the simillar manner. □

Definition 23. A topological space (X, τ) is said to be ω-T_2 [8] if for each pair of distinct points x and y in X, there exist disjoint ω-open sets U and V in X such that $x \in U$ and $y \in V$.

Theorem 24. *If $F : (X, \tau) \to (Y, \sigma)$ is an upper ω-irresolute injective multifunction and point closed from a topological space X to an ω-normal space Y, then X is an ω-T_2 space.*

Proof. Let x and y be any two distinct points in X. Then we have $F(x) \cap F(y) = \emptyset$ since F is injective. Since Y is ω-normal, it follows that there exist disjoint open sets U and V containing $F(x)$ and $F(y)$, respectively. Thus, there exist disjoint ω-open sets $F^+(U)$ and $F^+(V)$ containing x and y, respectively such $G \subset F^+(U)$ and $W \subset F^+(V)$. Therefore, we obtain $G \cap W = \emptyset$; hence X is ω-T_2. □

Definition 25. A multifunction $F : (X,\tau) \to (Y,\sigma)$ is said have an ω-closed graph if for each pair $(x,y) \notin G(F)$ there exist $U \in \omega O(X,x)$ and $V \in \omega O(Y,y)$ such that $(U \times V) \cap G(F) = \emptyset$.

Our next several results concern the relationship between upper ω-continuity and ω-closed graphs.

Theorem 26. *Let $F : (X,\tau) \to (Y,\sigma)$ be an upper ω-continuous multifunction from a space X into a T_2 space Y. If $F(x)$ is α-paracompact for each $x \in X$, then $G(F)$ is ω-closed.*

Proof. Suppose that $(x_0,y_0) \notin G(F)$. Then $y_0 \notin F(x_0)$. Since Y is a T_2 space, for each $y \in F(x_0)$ there exist disjoint open sets $V(y)$ and $W(y)$ containing y and y_0, respectively. The family $\{V(y) : y \in F(x_0)\}$ is an open cover of $F(x_0)$. Thus, by α-paracompactness of $F(x_0)$, there is a locally finite open cover $\Delta = \{U_\beta : \beta \in I\}$ which refines $\{V(y) : y \in F(x_0)\}$. Therefore, there exists an open neighborhood W_0 of y_0 such that W_0 intersects only finitely many members U_{β_1}, U_{β_2},...U_{β_n} of Δ. Choose y_1, y_2,...y_n in $F(x_0)$ such that $U_{\beta_i} \subset V(y_i)$ for each $1 \leq i \leq n$, and set $W = W_0 \cap (\bigcap_{i=1}^{n} W(y_i))$. Then W is an open neighborhood of y_0 such that $W \cap (\bigcup_{\beta\in I} V_\beta) = \emptyset$. By the upper ω-continuity of F, there is a $U \in \omega O(X,x_0)$ such that $U \subset F^+(\bigcup_{\beta\in I} V_\beta)$. It follows that $(U \times W) \cap G(F) = \emptyset$. Therefore, $G(F)$ is ω-closed. □

Theorem 27. *Let $F : (X,\tau) \to (Y,\sigma)$ be a multifunction from a space X into an ω-compact space Y. If $G(F)$ is ω-closed, then F is upper ω-continuous.*

Proof. Suppose that F is not upper ω-continuous. Then there exists a nonempty closed subset C of Y such that $F^-(C)$ is not ω-closed in X. We may assume that $F^-(C) \neq \emptyset$. Then there exists a point $x_0 \in \omega\,\mathrm{Cl}(F^-(C))\backslash F^-(C)$. Hence for each point $y \in C$, we have $(x_0,y) \notin G(F)$. Since F has an ω-closed graph, there are ω-open subsets $U(y)$ and $V(y)$ containing x_0 and y, respectively such that $(U(y)\times V(y))\cap G(F) = \emptyset$. Then $\{Y\backslash C\}\cup\{V(y) : y \in C\}$ is an ω-open cover of Y, and thus it has a subcover $\{Y\backslash C\} \cup \{V(y_i) : y_i \in C, 1 \leq i \leq n\}$. Let $U = \bigcap_{i=1}^{n} U(y_i)$ and $V = \bigcup_{i=1}^{n} V(y_i)$. It is easy to verify that $C \subset V$ and $(U \times V) \cap G(F) = \emptyset$. Since U is an ω-neighborhood of x_0, $U \cap F^-(C) \neq \emptyset$. It follows that $\emptyset \neq (U \times C) \cap G(F) \subset (U \times V) \cap G(F)$. This is a contradiction. Hence the proof is completed. □

Corollary 28. *Let $F : (X,\tau) \to (Y,\sigma)$ be a multifunction into an ω-compact T_2 space Y such that $F(x)$ is ω-closed for each $x \in X$. Then F is* *upper ω-continuous if and only if it has an ω-closed graph.*

Theorem 29. *Let $F : (X, \tau) \to (Y, \sigma)$ be an upper ω-irresolute multifunction into an ω-T_2 space Y. If $F(x)$ is α-paracompact for each $x \in X$, then $G(F)$ is ω-closed.*

Proof. The proof is clear. □

Definition 30. Let A be a subset of X. Then $F : X \to A$ is called a retracting multifunction [13] if $x \in F(x)$ for each $x \in A$.

Theorem 31. *Let $F : X \to X$ be an upper ω-irresolute multifunction of a T_2 space X into itself. If $F(x)$ is α-paracompact for each $x \in X$, then the set $A = \{x : x \in F(x)\}$ is ω-closed.*

Proof. Let $x_0 \in \omega\,\mathrm{Cl}(A)\backslash A$. Then $x_0 \notin F(x_0)$. Since X is T_2, for each $x \in F(x_0)$ there exist disjoint open sets $U(x)$ and $V(x)$ containing x_0 and x respectively. Then $\{V(x) : x \in F(x_0)\}$ is an open cover of $F(x_0)$. By the α-paracompactness of $F(x_0)$, $\{V(x) : x \in F(x_0)\}$ has a locally finite open refinement $\mathcal{W} = \{W_\beta : \beta \in I\}$ which covers $F(x_0)$. Therefore, we can choose an open neighborhood U_0 of x_0 such that U_0 intersects only finitely many members W_{β_1}, W_{β_2},...W_{β_n} of $\mathcal{W}$. Choose x_1, x_2,...x_n in $F(x_0)$ such that $W_{\beta_i} \subset V(x_i)$ for each $1 \leq i \leq n$, and set $U = U_0 \cap (\bigcap_{i=1}^{n} U(x_i))$. Then U is an open neighborhood of x_0 such that $U \cap (\bigcup_{\beta \in I} W_\beta) = \emptyset$. Since F is upper ω-irresolute, there is a $G \in \omega O(X, x_0)$ such that $G \subset F^+(\bigcup_{\beta \in I} W_\beta)$. It follows that $G \cap U$ is an ω-neighborhood of x_0 and satisfies $(G \cap U) \cap A = \emptyset$. This contradicts the fact that $x_0 \in \omega\,\mathrm{Cl}(A)$. □

Corollary 32. *Let A be a subset of X and $F : X \to A$ an upper ω-irresolute retracting multifunction such that $F(x)$ is α-paracompact for each $x \in A$. If X is T_2, then A is is ω-closed.*

References

[1] K. Al-Zoubi and B. Al-Nashef, The topology of ω-open subsets, *Al-Manarah* **(9) (2003)**, 169-179.

[2] A. Al-omari ans M. S. M. Noorani, Contra-ω-continuous and almost ω-continuous functions, *Int. J. Math. Math. Sci.* **(9) (2007)**, 169-179.

[3] T. Banzaru, Multifunctions and M-product spaces, *Bull. Stin. Tech. Inst. Politech. Timisoara, Ser. Mat. Fiz. Mer. Teor. Apl.*, 17(31)(1972), 17-23.

[4] G. Balaji, G. Ganesh and N. Rajesh, On upper and lower ω-continuous multifunctions, Asian Journal of Research in Social Sciences and Humanities, 6(10) 2016, 1920-1928.

[5] I. Kovacevic, Subsets and paracompactness, *Univ. u. Novom Sadu, Zb. Rad. Prirod. Mat. Fac. Ser. Mat.*, 14(1984), 79-87.

[6] H. Z. Hdeib, ω-closed mappings, *Revista Colombiana Mat.*, **16(1982)**, 65-78.

[7] N. Levine, Semiopen sets and semicontinuity in topological spaces, *Amer. Math. Monthly*, 70(1963), 36-41.

[8] T. Noiri and V. Popa, A generalization of ω-continuity, *Fasciculi Mathematici*, 45(2010), 71-86.

[9] N. Rajesh, On Total ω-continuity, Strong ω-continuity and Contran ω-continuity, *Soochow J. Math.*, 33(4)(2007),679-690.

[10] M. Sheik John, A study on generalizations of closed sets and continuous maps in topological spaces, *Ph. D. Thesis*, Bharathiyar University, Coimbatore, India (2002).

[11] M. Sheik John and P. Sundaram, On decomposition of continuity, *Bull. Allahabad Math. Soc.*, 22(2007), 1-9.

[12] P. Sundaram and M. Sheik John, Weakly closed and weakly continuous maps in topological spaces, Proc. 82^{nd} session of the Indian Science Congress, Calcutta, 1995 (P-49).

[13] G. T. Whyburn, Retracting multifunctions, *Proc. Nat. Acad. Sci.*, 59(1968), 343-348.