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ULAM'S STABILITIES OF FRACTIONAL INTEGRO-DIFFERENTIAL EQUATIONS

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Abstract: In this paper we establish Hyers-Ulam and Hyers-Ulam-Rassias stability for fractional integrodifferential equations

$$x^{(\alpha)}(t) = g(t, x(t)) + \int_{t_0}^{t} G(t, s, x(s))ds, \quad \alpha \in \mathbb{R},\ 0 < \alpha \leq 1,$$

with the initial condition $x^{(\alpha-1)}(t_0) = x_0$.

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1. Introduction

The study of stability problems for various functional equations originated from a famous talk given by Ulam In 1940 (see [1]). In the talk, Ulam discussed a problem concerning the stability of homomorphisms. A significant breakthrough came in 1941, when Hyers [2] gave a partial solution to Ulam's problem.. In 1978, Rassias [3] provided a remarkable generalization of the Ulam-Hyers stability of mappings by considering variables. During the last two

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decades very important contirbutions to the stability problems of functional equations were given by many mathematicians (see [4-11]). A generalization of Ulam's problem was recently proposed by replacing functional equations with differential equations: The differential equation $F(t, y(t), y'(t), ..., y^{(n)}(t)) = 0$ has the Hyers-Ulam stability if for given $\varepsilon > 0$ and a function y such that

$$\left|F(t, y(t), y'(t), ..., y^{(n)}(t))\right| \leq \varepsilon$$

there exists a solution y_0 of the differential equation such that

$$|y(t) - y_0(t)| \leq K(\varepsilon)$$

and $\lim_{\varepsilon \to 0} K(\varepsilon) = 0$.

Obloza seems to be the first author who has investigated the Hyers-Ulam stability of linear differential equations (see [12, 13]). Thereafter, Alsina and Ger published their paper [14], which handles the Hyers-Ulam stability of the linear differential equation $y'(t) = y(t)$: If a differentiable function y(t) is a solution of the inequality $|y'(t) - y(t)| \leq \varepsilon$ for any $t \in (a, \infty)$, then there exists a constant c such that $|y(t) - ce^t| \leq 3\varepsilon$ for all $t \in (a, \infty)$.Recently, the Hyers-Ulam stability problems of linear differential equations of first order and second order with constant coefficients were studied by using the method of integral factors (see [15, 16]). The results given in [17-19] have been generalized by Popa and Rus [20, 21] for the linear differential equations of nth order with constant coefficients. For more details on Hyers-Ulam stability and the generalized Hyers-Ulam stability, we refer the reader to the papers [22-28].

Fractional calculus is a generalization of ordinary differentiation and integration to arbitrary order (non-integer). In recent years, fractional differential equations arise naturally in various fields such as rheology, fractals, chaotic dynamics, modeling and control theory, signal processing, bioengineering and biomedical applications, etc; Fractional derivatives provide an excellent instrument for the description of memory and hereditary properties of various materials and processes,[29-30]. Some researchers have used the fixed point approach to investigate the Hyers-Ulam stability for fractional differential equations [e.g. 31,33].The objective of this article is to investigate the Hyers-Ulam-Rassias Stability for the stability and Hyers-Ulam Criteria for fractional integrodifferential equations

$$x^{(\alpha)}(t) = g(t, x(t)) + \int_{t_0}^{t} G(t, s, x(s))ds, \quad \alpha \in \mathbb{R},\ 0 < \alpha \leq 1 \qquad (1.1)$$

with the initial condition

$$x^{(\alpha-1)}(t_0) = x_0. \tag{1.2}$$

where $\mathbb{R}$denotes the set of real numbers,$J = [t_0, t+a], a > 0, g \in C[J\times\mathbb{R}, \mathbb{R}], G \in C[J \times J \times \mathbb{R}, \mathbb{R}]$, and x_0 is a real constant.

2. Preliminaries

In this section, we give some basic definitions and Lemmas which we used to prove the main results.

Definition 1. Let f be a function which is defined on $[a, b]$**.** For $\alpha > 0$, define

$$\underset{a}{I}{}^{b\alpha} g(t) = \frac{1}{\Gamma(\alpha)} \int_a^b (b-s)^{\alpha-1} g(s) ds \tag{2.1}$$

provided that this integral exists, where Γ is the Gamma function.

Definition 2. Assume that for continuously differentiable functions $g : J \to \mathbb{R}$, $G : J \times J \to \mathbb{R}$ and satisfying fractional differential inequality

$$\left| x^{(\alpha)}(t) - g(t, x(t)) - \int_{t_0}^{t} G(t, s, x(s)) ds \right| \leq \varepsilon, \tag{2.2}$$

for all $t \in J$ and for each $\varepsilon > 0$,where $x^{(\alpha)}(t)$ denotes the fractional drivative of order α.there exists a solution $y_0 : J \to Y$

of the fractional initial value problem (1.1)and (1.2) such that $|x(t) - x_0(t)| \leq K\varepsilon$, for all $t \in J$ .Then we say that the above fractional initial value problem (1.1)and (1.2) has the Hyers-Ulam stability.

Definition 3. We say that equation (1.1) with initial condition (1.2) has the Hyers-Ulam-Rassias stability with respect to φ if there exists a positive constant $K > 0$ with the following property:

For each $x(t)$ satisfying

$$\left| x^{(\alpha)}(t) - g(t, x(t)) - \int_{t_0}^{t} G(t, s, x(s)) ds \right| \leq \varphi(t) \tag{2.3}$$

then there exists some solution $x_0(t)$ of the equation (1.1) with (1.2) such that $|x(t) - x_0(t)| \leq K\varphi(t)$.

Lemma 1. **[34]** The IVP (1.1) and (1.2) is equivalent to the nonlinear integral equation

$$x(t) = \frac{x_0}{\Gamma(\alpha)}(t-t_0)^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}g(s,x(s))ds + \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}\int_{s}^{t}G(\tau,s,x(s))d\tau ds, \quad (2.4)$$

where $0 < t_0 \leq t \leq t_0 + a$. In otherwords, every solution of the integral (2.3) is also a solution of the origina lIVP (1.1) and (1.2), and viceversa.

Lemma 2. (Gronwall's lemma). Let u(t) and v(t) be nonnegative continuous functions on some interval $0 < t_0 \leq t \leq t_0 + a$. Also, let the function $f(t)$be positive, continuous, and monotonically nondecreasing on $[t_{0,}t_0 + a]$ and satisfy the inequality

$$u(t) \leq f(t) + \int_{t_0}^{t} u(s)v(s)ds \quad (2.5)$$

then, there holds the inequality

$$u(t) \leq f(t)\exp\left(\int_{t_0}^{t} u(s)ds\right), \text{ for } t_0 \leq t \leq t_0 + a \quad (2.6)$$

Proof. For the proof of Lemma 1.2, see [35].

3. Main Results On Hyers-Ulam and Hyers-Ulam-Rassias Stability

In this section, we will prove our main results, and establish the HU stability of solution of (1.1) satisfying (1.2).

Theorem 4. *let the function g satisfy the inequality*

$$|g(t,x(t)) - g(t,y(t))| \leq \beta(t)\,|x-y|\,, \quad (3.1)$$

and let G satisfy the inequality

$$\left|\int\limits_s^t [G(\tau,s,x(s)) - G(\tau,s,y(s))]\,d\tau\right| \le \gamma(t)\,|x-y|\,, \qquad s\in[t_0,t], \tag{3.2}$$

where $\beta(t)$ and $\gamma(t)$ are continuous and nonnegative functions such that

$$\sup\int\limits_{t_0}^t (t-s)^{\alpha-1}[(\beta(s)+\gamma(s)]ds < \infty \tag{3.3}$$

Then the problem (1.1),(1.2) is stable in the sense of Hyers and Ulam.

Proof. Let $x(t) \in C[J,\mathbb{R}]$ be a solution of inequation (2.2) i.e.

$$\left|x^{(\alpha)}(t) - g(t,x(t)) - \int\limits_{t_0}^t G(t,s,x(s))ds\right| \le \varepsilon, \tag{3.4}$$

Applying the integral operator (2.1) to the inequality (3.9) we obtain

$$| \, x(t) - \frac{x_0}{\Gamma(\alpha)}(t-t_0)^{\alpha-1} - \frac{1}{\Gamma(\alpha)}\int\limits_{t_0}^t (t-s)^{\alpha-1}g(s,x(s))ds$$

$$-\frac{1}{\Gamma(\alpha)}\int\limits_{t_0}^t (t-s)^{\alpha-1}\int\limits_s^t G(\tau,s,x(s))d\tau ds \,| \le \frac{\varepsilon}{\Gamma(\alpha)}(t-t_0)^{\alpha-1}$$

On can easily show that $z(t) \in C[J,\mathbb{R}]$ defined by

$$\begin{aligned} z(t) &= \frac{x_0}{\Gamma(\alpha)}(t-t_0)^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\int\limits_{t_0}^t (t-s)^{\alpha-1}g(s,z(s))ds \\ &\quad + \frac{1}{\Gamma(\alpha)}\int\limits_{t_0}^t (t-s)^{\alpha-1}\int\limits_s^t G(\tau,s,z(s))d\tau ds \end{aligned}$$

satisfies the IVP (1.1),(1.2).

Now we estimate the difference

$$| x(t) - z(t) | \leq | x(t) - \frac{x_0}{\Gamma(\alpha)}(t-t_0)^{\alpha-1} - \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}g(s,x(s))ds$$

$$-\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}\int_{s}^{t}G(\tau,s,x(s))d\tau ds |$$

$$+ | z(t) - \frac{x_0}{\Gamma(\alpha)}(t-t_0)^{\alpha-1} - \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}g(s,x(s))ds$$

$$-\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}\int_{s}^{t}G(\tau,s,x(s))d\tau ds |$$

Or equivalently,

$$| x(t) - z(t) | \leq \frac{\varepsilon}{\Gamma(\alpha)}(t-t_0)^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1} | g(s,x(s)) - g(s,z(s))ds$$

$$+\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}\int_{s}^{t} | G(\tau,s,x(s))-G(\tau,s,z(s)) | d\tau ds$$

$$\leq \frac{\varepsilon}{\Gamma(\alpha)}(t-t_0)^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1} | x(s))-z(s)) | [\beta(t)+\gamma(t)]ds$$

By Gronwell's lemma, we obtain

$$| x(t) - z(t) | \leq \frac{\varepsilon}{\Gamma(\alpha)}(t-t_0)^{\alpha-1}\exp\left(\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}[\beta(t)+\gamma(t)]ds\right)$$

In view of (3.3) there exists a positive constant K such that

$$| x(t) - z(t) | \leq K\varepsilon,$$

which completes the proof. □

Now we will prove the Hyers-Ulam-Rassias stability (HUR) of problem (1.1), (1.2).

Theorem 5. *Let the function g satisfy the inequality*

$$|g(t,x(t))-g(t,y(t))| \le \beta(t)\,|x-y|\,, \tag{3.5}$$

and let G satisfy the inequality

$$\left|\int_s^t [G(\tau,s,x(s))-G(\tau,s,y(s))]\,d\tau\right| \le \gamma(t)\,|x-y|\,, \qquad s\in[t_0,t], \tag{3.6}$$

where $\gamma(t)$ and $\beta(t)$ are continuous and nonnegative functions such that

$$\sup\int_{t_0}^t (t-s)^{\alpha-1}[(\beta(s)+\gamma(s)]ds<\infty \tag{3.7}$$

If $\varphi(t):[0,\infty)\to(0,\infty)$ is a continuous function such that

$$\int_{t_0}^t (t-s)^{\alpha-1}\varphi(s)ds \le C\varphi(t) \tag{3.8}$$

then the problem (1.1),(1.2) is stable in the sense of HUR.

Proof. Let $x(t)\in C[J,\mathbb{R}]$ be a solution of inequation (2.2) i.e.

$$\left|x^{(\alpha)}(t)-g(t,x(t))-\int_{t_0}^t G(t,s,x(s))ds\right| \le \varphi(t), \tag{3.9}$$

Applying the integral operator (2.1) to the inequality (3.9) we obtain

$$|\,x(t)-\frac{x_0}{\Gamma(\alpha)}(t-t_0)^{\alpha-1}-\frac{1}{\Gamma(\alpha)}\int_{t_0}^t (t-s)^{\alpha-1}g(s,x(s))ds$$

$$-\frac{1}{\Gamma(\alpha)}\int_{t_0}^t (t-s)^{\alpha-1}\int_s^t G(\tau,s,x(s))d\tau ds\,|\le \frac{1}{\Gamma(\alpha)}\int_{t_0}^t (t-s)^{\alpha-1}\varphi(s)ds$$

On can easily show that $z(t) \in C[J, \mathbb{R}]$ defined by

$$z(t) = \frac{x_0}{\Gamma(\alpha)}(t-t_0)^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}g(s,z(s))ds + \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}\int_{s}^{t}G(\tau,s,z(s))d\tau ds$$

satisfies the IVP (1.1),(1.2).
Now we estimate the difference

$$| x(t)-z(t) | \leq | x(t) - \frac{x_0}{\Gamma(\alpha)}(t-t_0)^{\alpha-1} - \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}g(s,x(s))ds - \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}\int_{s}^{t}G(\tau,s,x(s))d\tau ds |$$

$$+ | z(t) - \frac{x_0}{\Gamma(\alpha)}(t-t_0)^{\alpha-1} - \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}g(s,x(s))ds - \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}\int_{s}^{t}G(\tau,s,x(s))d\tau ds |$$

Using the conditions (3.5), (3.6) and (3.8) we infer that

$$| x(t)-z(t) | \leq \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}\varphi(s)ds + \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1} | g(s,x(s))-g(s,z(s))ds + \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}\int_{s}^{t} | G(\tau,s,x(s))-G(\tau,s,z(s)) | d\tau ds$$

$$\leq \frac{C\varphi(t)}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1} | x(s))-z(s)) | [\beta(t)+\gamma(t)]ds$$

By Gronwell's lemma, we obtain

$$| x(t) - z(t) |\leq \frac{C\varphi(t)}{\Gamma(\alpha)}(t-t_0)^{\alpha-1}\exp\left(\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}[\beta(t)+\gamma(t)]ds\right)$$

According to (3.7) there exists a positive constant K such that

$$| x(t) - z(t) |\leq K\varepsilon,$$

which completes the proof. □

In the following theorem we establish the Hyers-Ulam-Rassias stability for (1.1),
(1.2) in the interval $0 \leq t_0 \leq t \leq \infty$.

Theorem 6. *Let the function g satisfy the inequality*

$$|g((t,x(t)) - g((t,y(t))| \leq \beta(t)\,|x-y|\,,$$

and let G satisfy the inequality

$$\left|\int_{s}^{t}[G(\tau,s,x(s)) - G(\tau,s,y(s))]\,d\tau\right| \leq \gamma(t)\,|x-y|\,, \qquad t_0 \leq s \leq t \leq \infty, \quad (3.10)$$

where $\beta(t)$ and $\gamma(t)$ are continuous and nonnegative functions such that

$$\sup_{t_0\leq t\leq\infty}\int_{t_0}^{t}(t-s)^{\alpha-1}[(\beta(s)+\gamma(s)]ds < \infty \quad (3.11)$$

If $\varphi(t) : [0,\infty) \to (0,\infty)$ is a continuous function such that

$$\int_{t_0}^{\infty}(t-s)^{\alpha-1}\varphi(s)ds \leq C\varphi(t) \quad (3.12)$$

then the problem (1.1),(1.2) is stable in the sense of HUR as $t\to\infty$.

Proof. Applying the same approach used in the Theorem 2, we can get the proof of theorem . □

4. Conclusion

In this work, the problem of the Hyers-Ulam and Hyers-Ulam-Rassias Stability of solution of Ulam's Stabilities of Fractional Integro-Differential Equations has been investigated and solved using the direct method.

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