International Journal of Pure and Applied Mathematics
Volume 113 No. 3 2017, 511-522
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: 10.12732/ijpam.v113i3.13

ijpam.eu

SOME PROPERTIES OF L-FUZZY PREUNIFORMITIES, L-FUZZY NEIGHBORHOOD SYSTEMS AND L-FUZZY TOPOLOGIES

Ju-Mok Oh[1], A.A.Ramadan[2], Yong Chan Kim[3 §]

[1,3] Department of Mathematics
Gangneung-Wonju University
Gangneung, Gangwondo 25457, KOREA

[2]Mathematics Department
Faculty of Science
Beni-Suef University
Beni-Suef, Egypt

Abstract: This article gives results on L-fuzzy preuniformities, L-fuzzy neighborhood systems and L-fuzzy topologies in complete residuated lattices, and includes some properties. The notion of their continuity property is investigated

AMS Subject Classification: 03E72, 06A15, 06F07, 54A05.

Key Words: Complete residuated lattice, L-fuzzy neighborhood space, L-fuzzy preuniform space, L-fuzzy topologies.

1. Introduction

Hájek [7] introduced a complete residuated lattice which is an algebraic structure for many valued logic. Bělohlávek [2] investigated information systems and decision rules in complete residuated lattices. Höhle and S.E. Rodabaugh [8] introduced L-fuzzy topologies with algebraic structures L (stsc-quantales, MValgebras) as an extension for a completely distributive lattice L or the unit

Received: January 20, 2017
Revised: March 3, 2017
Published: April 3, 2017

§Correspondence author

interval or the two-point lattice $2 = \{0,1\}$. Hutton [9] introduced the notion of fuzzy uniformities in a completely distributive lattice.

Fuzzy uniformities have the following different approaches as follows the entourage approach of Lowen [15,16] , the uniform covering approach of Kotzé [13] and the unification approach of Hutton [9] based on the powersets of the form $(L^X)^{(L^X)}$, the unification approach of Gutiérrez García[6]. Recently, Gutiérrez García introduced L-valued Hutton uniformity where a quadruple $(L, \leq, \otimes, *)$ is defined by a GL-monoid $(L, *)$ as an extension of a completely distributive lattice L. Many researchers studied the different approach as powerset [9] or the uniform covering [13]. Using the Lowen neighborhood system [15], Katsaras [10] proved that every linear fuzzy neighborhood space is uniformizable in the sense of Lowen uniformity. Kim [11] introduced the notion of L-fuzzy uniformities as an extension of Lowen in a strictly two-sided, commutative quantale.

This article gives results on L-fuzzy preuniformities, stratified L-neighborhood systems and L-fuzzy topologies in complete residuated lattices, and includes some properties. The notion of their continuity property is investigated

2. Preliminaries

Definition 2.1. [2,4] An algebra $(L, \wedge, \vee, \odot, \rightarrow, \bot, \top)$ is called a complete residuated lattice if it satisfies the following conditions:

(C1) $L = (L, \leq, \vee, \wedge, \bot, \top)$ is a complete lattice with the greatest element $\top$ and the least element $\bot$;

(C2) $(L, \odot, \top)$ is a commutative monoid;

(C3) $x \odot y \leq z$ iff $x \leq y \rightarrow z$ for $x, y, z \in L$.

An L-subset on a set X is a mapping from X to L, and the family of all L-subsets on X will be denoted by L^X; in particular, the L-subsets $\top_X$ and $\bot_X$ defined by $\top_X(x) = \top$ and $\bot_X(x) = \bot$, $\forall x \in X$, are respectively the universal upper and lower bound in L^X. We denote the characteristic function of a subset $\{x\}$ of X by $\top_x$. We do not distinguish between an element $\alpha \in L$ and the constant function $\alpha : X \rightarrow L$ such that $\alpha(x) = \alpha$ for all $x \in X$.

All algebraic operations on L can be extended pointwise to the power set L^X. That is, for all $\lambda, \mu \in L^X$, $\alpha \in L$ and $x \in X$,

(1) $\lambda \leq \mu$ if and only if $\lambda(x) \leq \mu(x)$,

(2) $(\lambda \odot \mu)(x) = \lambda(x) \odot \mu(x)$,

(3) $(\alpha \odot \lambda)(x) = \alpha \odot \lambda(x)$,

(4) $(\alpha \to \lambda)(x) = \alpha \to \lambda(x),\ (\alpha \odot \lambda)(x) = \alpha \odot \lambda(x)$.

In this paper, we assume that $(L, \vee, \wedge, \odot, \to, \bot, \top)$ is a complete residuated lattice.

Lemma 2.2. [2,7,25] For each $x, y, z, w, x_i, y_i \in L$, the following properties hold.

(1) If $y \le z$, then $x \odot y \le x \odot z$.

(2) If $y \le z$, then $x \to y \le x \to z$ and $z \to x \le y \to x$.

(3) $x \to y = \top$ iff $x \le y$.

(4) $x \to \top = \top$ and $\top \to x = x$.

(5) $x \odot y \le x \wedge y$.

(6) $x \odot (\bigvee_{i\in\Gamma} y_i) = \bigvee_{i\in\Gamma}(x \odot y_i)$ and $(\bigvee_{i\in\Gamma} x_i) \odot y = \bigvee_{i\in\Gamma}(x_i \odot y)$.

(7) $x \to (\bigwedge_{i\in\Gamma} y_i) = \bigwedge_{i\in\Gamma}(x \to y_i)$ and $(\bigvee_{i\in\Gamma} x_i) \to y = \bigwedge_{i\in\Gamma}(x_i \to y)$.

(8) $\bigvee_{i\in\Gamma} x_i \to \bigvee_{i\in\Gamma} y_i \ge \bigwedge_{i\in\Gamma}(x_i \to y_i)$ and $\bigwedge_{i\in\Gamma} x_i \to \bigwedge_{i\in\Gamma} y_i \ge \bigwedge_{i\in\Gamma}(x_i \to y_i)$.

(9) $(x \to y) \odot x \le y$ and $(x \to y) \odot (y \to z) \le (x \to z)$.

(10) $x \to y \le (y \to z) \to (x \to z)$ and $x \to y \le (z \to x) \to (z \to y)$.

(11) $(x \odot y) \to z = x \to (y \to z) = y \to (x \to z)$.

(12) $y \to z \le x \odot y \to x \odot z$ and $(x \to z) \odot (y \to w) \le x \odot y \to z \odot w$.

Lemma 2.3. [3,4] For a given set X, define a binary mapping $S : L^X \times L^X \to L$ by

$$S(\lambda, \mu) = \bigwedge_{x\in X} (\lambda(x) \to \mu(x)).$$

Then, for each $\lambda, \mu, \rho, \nu \in L^X$, and $\alpha \in L$, the following properties hold.

(1) S is an L-partial order on L^X.

(2) $\lambda \le \mu$ iff $S(\lambda, \mu) \ge \top$,

(3) If $\lambda \le \mu$, then $S(\rho, \lambda) \le S(\rho, \mu)$ and $S(\lambda, \rho) \ge S(\mu, \rho)$,

(4) $S(\lambda, \mu) \odot S(\nu, \rho) \le S(\lambda \odot \nu, \mu \odot \rho)$,

(5) Let $\phi : X \to Y$ be an ordinary mapping. Define $\phi^{\to} : L^X \to L^Y$ and $\phi^{\leftarrow} : L^Y \to L^X$ by

$$\phi^{\to}(\lambda)(y) = \bigvee_{\phi(x)=y} \lambda(x), \quad \forall \lambda \in L^X,\ y \in Y,$$

$$\phi^{\leftarrow}(\mu)(x) = \mu(\phi(x)) = \mu \circ \phi(x), \ \ \forall \mu \in L^Y.$$

Then for $\lambda, \mu \in L^X$ and $\rho, \nu \in L^Y$,

$$S(\lambda, \mu) \le S(\phi^{\to}(\lambda), \phi^{\to}(\mu)),$$

$$S(\rho,\nu)\le S(\phi^{\leftarrow}(\rho),\phi^{\leftarrow}(\nu)),$$

and the equalities hold if ϕ is bijective.

Definition 2.4. [8] A map $\mathcal{T}: L^X \to L$ is called an L-fuzzy topology on X if it satisfies the following conditions.

(O1) $\mathcal{T}(\bot_X)=\mathcal{T}(\top_X)=\top$,

(O2) $\mathcal{T}(\lambda\odot\mu)\ge\mathcal{T}(\lambda)\odot\mathcal{T}(\mu),\ \ \forall\ \ \lambda,\mu\in L^X$,

(O3) $\mathcal{T}(\bigvee_i\lambda_i)\ge\bigwedge_i\mathcal{T}(\lambda_i),\ \ \forall\ \ \{\lambda_i\}_{i\in\Gamma}\subseteq L^X$.

An L-fuzzy topology is called enriched if

(R) $\mathcal{T}(\alpha\odot\lambda)\ge\mathcal{T}(\lambda)$ for all $\lambda\in L^X$ and $\alpha\in L$.

The pair $(X,\mathcal{T})$ is called an L-fuzzy topological space.

Let $(X,\mathcal{T}_1)$ and $(Y,\mathcal{T}_2)$ be two L-fuzzy topological spaces. A mapping $\phi: X\to Y$ is said to be fuzzy continuous iff for each $\lambda\in L^Y$,

$$\mathcal{T}_2(\lambda)\le\mathcal{T}_1(\phi^{\leftarrow}(\lambda)).$$

Definition 2.5. [8] An L-fuzzy neighborhood system on X refers to a collection of mappings $\mathcal{P}=\{p_x: L^X\to L\mid x\in X\}$ such that for any $\lambda,\mu\in L^X$ and $x\in X$

(P1) $p_x(\top_X)=\top$,

(P2) $p_x(\lambda\odot\mu)\ge p_x(\lambda)\odot p_x(\mu)$,

(P3) If $\lambda\le\mu$, then $p_x(\lambda)\le p_x(\mu)$,

(P4) $p_x(\lambda)\le\lambda(x)$ for all $\lambda\in L^X$.

An L-fuzzy neighborhood system is called stratified if

(R) $p_x(\alpha\odot\lambda)\ge\alpha\odot p_x(\lambda)$ for all $\lambda\in L^X, x\in X$ and $\alpha\in L$.

The pair $(X,\mathcal{P})$ is called an L-fuzzy neighborhood space.

Let $(X,\mathcal{P})$ and $(Y,\mathcal{Q})$ be two L-fuzzy neighborhood spaces. A mapping $\phi: X\to Y$ is said to be fuzzy continuous at $x\in X$ iff $q_{\phi(x)}(\lambda)\le p_x(\phi^{\leftarrow}(\lambda))$ for each $\lambda\in L^Y, \phi$ is continuous if it is fuzzy continuous at every $x\in X$, where $p_x\in\mathcal{P}$ and $q_{\phi(x)}\in\mathcal{Q}$.

Theorem 2.6. [12] Let $(X,\mathcal{P})$ be an L-fuzzy neighborhood space. Define a map $\mathcal{T}_{\mathcal{P}}: L^X\to L$ by:

$$\mathcal{T}_{\mathcal{P}}(\lambda)=\bigwedge_{x\in X}(\lambda(x)\to p_x(\lambda))=S(\lambda,p_x(\lambda)).$$

Then, $\mathcal{T}_{\mathcal{P}}$ is an L-fuzzy topology on X. If $\mathcal{P}$ is stratified, then $\mathcal{T}_{\mathcal{P}}$ is an enriched L-fuzzy topology.

Theorem 2.7. [12] If a mapping $\phi : (X, \mathcal{P}) \to (Y, \mathcal{Q})$ is fuzzy continuous, then $\phi : (X, \mathcal{T}_{\mathcal{P}}) \to (Y, \mathcal{T}_{\mathcal{Q}})$ is fuzzy continuous.

Theorem 2.8. [12] Let $(X, \mathcal{T})$ be an L-fuzzy topological space. Define a mapping $\mathcal{P}^{\mathcal{T}} : X \to L^{L^X}$ as follows:

$$(\mathcal{P}^{\mathcal{T}}(x) = p_x^{\mathcal{T}})(\lambda) = \bigvee_{\mu \in L^X} (\mathcal{T}(\mu) \odot S(\mu, \lambda) \odot \mu(x)).$$

Then $(X, \mathcal{P}^{\mathcal{T}})$ is a stratified L-fuzzy neighborhood space.

Theorem 2.9. [12] Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be L-fuzzy topological spaces. If a mapping $\phi : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ is L-fuzzy continuous, then $\phi : (X, \mathcal{P}^{\mathcal{T}_X}) \to (Y, \mathcal{P}^{\mathcal{T}_Y})$ is fuzzy continuous.

3. L-Fuzzy Preuniformities Induced by L-Fuzzy Neighborhood Systems and L-Fuzzy Topologies

Definition 3.1. [6, 8] A map $\mathcal{U} : L^{X \times X} \to L$ is called an L-fuzzy preuniformity on X iff the following conditions hold.

(U1) There exists $u \in L^{X \times X}$ such that $\mathcal{U}(u) = \top$.

(U2) If $v \leq u$, then $\mathcal{U}(v) \leq \mathcal{U}(u)$.

(U3) For every $u, v \in L^{X \times X}, \mathcal{U}(u \odot v) \geq \mathcal{U}(u) \odot \mathcal{U}(v)$.

(U4) $\mathcal{U}(u) \leq S(\top_\triangle, u) = \bigwedge_{x \in X} u(x, x)$, where

$$\top_\triangle(x, y) = \begin{cases} \top, & \text{if } x = y \\ \bot, & \text{if } x \neq y, \end{cases}$$

(U5) $\mathcal{U}(u) \leq \mathcal{U}(u^{-1})$, where $u^{-1}(x, y) = u(y, x)$.

An L-fuzzy preuniformity $\mathcal{U}$ on X is said to be stratified if

(R) $\mathcal{U}(\alpha \odot u) \geq \alpha \odot \mathcal{U}(u), \quad \forall\, u \in L^{X \times X}, \alpha \in L$.

The pair $(X, \mathcal{U})$ is called an L-fuzzy preuniform space.

Let $(X, \mathcal{U})$ and $(Y, \mathcal{V})$ be L-fuzzy preuniform spaces, and $\phi : X \to Y$ ba a mapping. Then ϕ is said to be fuzzy preuniformly continuous if $\mathcal{V}(v) \leq \mathcal{U}((\phi \times \phi)^{\leftarrow}(v))$, for every $v \in L^{Y \times Y}$.

Remark 3.2. Let $(X, \mathcal{U})$ be an L-fuzzy preuniform space.

(1) By (U1) and (U2), we have $\mathcal{U}(\top_{X\times X}) = \top$ because $u \le \top_{X\times X}$ for all $u \in L^{X\times X}$.

(2) Since $\mathcal{U}(u) \le \mathcal{U}(u^{-1}) \le \mathcal{U}((u^{-1})^{-1}) = \mathcal{U}(u)$, then $\mathcal{U}(u) = \mathcal{U}(u^{-1})$.

Lemma 3.3. Let $\mathcal{U} : L^{X\times X} \to L$ be a map. The following statement are equivalent

(1) For all $u, v \in L^{X\times X}$, $S(u,v) \le \mathcal{U}(u) \to \mathcal{U}(v)$.

(2) If $u \le v$, then $\mathcal{U}(u) \le \mathcal{U}(v)$ and $\mathcal{U}(\alpha \odot u) \ge \alpha \odot \mathcal{U}(u)$, $\forall\ u \in L^{X\times X}$ and $\alpha \in L$.

(3) If $u \le v$, then $\mathcal{U}(u) \le \mathcal{U}(v)$ and $\mathcal{U}(\alpha \to u) \le \alpha \to \mathcal{U}(u)$, $\forall\ u \in L^{X\times X}$ and $\alpha \in L$.

Proof. (1)$\Rightarrow$ (2). If $u \le v$, $\top = S(u,v) \le \mathcal{U}(u) \to \mathcal{U}(v)$. Hence, $\mathcal{U}(u) \le \mathcal{U}(v)$. Put $v = \alpha \odot u$, then $\alpha \le S(u, \alpha \odot u) \le \mathcal{U}(u) \to \mathcal{U}(\alpha \odot u)$. Hence, $\alpha \odot \mathcal{U}(u) \le \mathcal{U}(\alpha \odot u)$.

(2)$\Rightarrow$ (3). Since $\alpha \odot \mathcal{U}(\alpha \to u) \le \mathcal{U}(\alpha \odot (\alpha \to u)) \le \mathcal{U}(u)$, by Lemma (2,2)(9).

Hence, $\mathcal{U}(\alpha \to u) \le \alpha \to \mathcal{U}(u)$.

(3)$\Rightarrow$ (1). Since $S(u,v) \odot u \le v$ if and only if $u \le S(u,v) \to v$,

$$\mathcal{U}(u) \le \mathcal{U}(S(u,v) \to v) \le S(u,v) \to \mathcal{U}(v).$$

Hence, $S(u,v) \le \mathcal{U}(u) \to \mathcal{U}(v)$.

Definition 3.4. For $\lambda \in L^X$, we define $u_\lambda \in L^{X\times X}$ associated with λ by

$$u_\lambda(x,y) = \begin{cases} \top, & \text{if } x = y \\ \lambda(x) \odot \lambda(y), & \text{if } x \ne y. \end{cases}$$

Theorem 3.5. Let $(X, \mathcal{P})$ be an L-fuzzy neighborhood space. Define a map $\mathcal{U}_{\mathcal{P}} : L^{X\times X} \to L$ by

$$\mathcal{U}_{\mathcal{P}}(u) = \bigvee_{\lambda \in L^X} \Big(\bigvee_{x\in X} p_x(\lambda) \odot S(u_\lambda, u)\Big), \quad \forall\ u \in L^{X\times X}.$$

Then $(X, \mathcal{U}_{\mathcal{P}})$ is an L-fuzzy preuniform space. If $(X, \mathcal{P})$ is stratified, then $(X, \mathcal{U}_{\mathcal{P}})$ is stratified.

Proof. (U1) Since $u_{\top_X} = \top_{X\times X}$, $S(u_{\top_X}, \top_{X\times X}) = \top$, we have

$$\mathcal{U}_{\mathcal{P}}(\top_{X\times X}) \ge \bigvee_{x\in X} p_x(\top_X) \odot \top = \top \odot \top = \top.$$

(U2) If $u_1 \leq u_2, u_1, u_2 \in L^{X\times X}$, then for any $\lambda \in L^X$, we have by Lemma 2.3(3)

$$p_x(\lambda) \odot S(u_\lambda, u_1) \leq p_x(\lambda) \odot S(u_\lambda, u_2).$$

Taking the supremum we get, $\mathcal{U}_{\mathcal{P}}(u_1) \leq \mathcal{U}_{\mathcal{P}}(u_2)$.

(U3) If $\lambda, \rho \in L^X$, $u, v \in L^{X\times X}$, since $u_\lambda \odot u_\rho = u_{\lambda\odot\rho}$, we have

$$\begin{aligned}
&\mathcal{U}_{\mathcal{P}}(u) \odot \mathcal{U}_{\mathcal{P}}(v) = \bigvee_{\lambda\in L^X} \Big(\bigvee_{x\in X} p_x(\lambda) \odot S(u_\lambda, u)\Big)\\
&\odot \bigvee_{\rho\in L^X} \Big(\bigvee_{x\in X} p_x(\rho) \odot S(u_\rho, v)\Big)\\
&\leq \bigvee_{\lambda,\rho\in L^X} \Big(\bigvee_{x\in X} p_x(\lambda) \odot p_x(\rho) \odot S(u_\lambda \odot u_\rho, u \odot v)\Big)\\
&\qquad \text{(by Lemma 2.3(4))}\\
&\leq \bigvee_{\lambda,\rho\in L^X} \Big(\bigvee_{x\in X} p_x(\lambda \odot \rho) \odot S(u_{\lambda\odot\rho}, u \odot v)\Big)\\
&\leq \bigvee_{\nu\in L^X} \Big(\bigvee_{x\in X} p_x(\nu) \odot S(u_\nu, u \odot v)\Big)\\
&= \mathcal{U}_{\mathcal{P}}(u \odot v).
\end{aligned}$$

(U4) If $u \in L^{X\times X}$, then

$$\begin{aligned}
\mathcal{U}_{\mathcal{P}}(u) &= \bigvee_{\lambda\in L^X} \Big(\bigvee_{x\in X} p_x(\lambda) \odot \bigwedge_{(x,y)\in X\times X} \big(u_\lambda(x,y) \to u(x,y)\big)\Big)\\
&\leq \bigvee_{\lambda\in L^X} \Big(\bigvee_{x\in X} p_x(\lambda) \odot \bigwedge_{y\in X} \big(u_\lambda(y,y) \to u(y,y)\big)\Big)\\
&\leq \bigvee_{\lambda\in L^X} (\bigvee_{x\in X} \lambda(x) \odot \bigwedge_{y\in X} u(y,y))\\
&\leq \bigwedge_{y\in X} u(y,y).
\end{aligned}$$

(U5) Let $u \in L^{X\times X}$, then

$$\begin{aligned}
\mathcal{U}_{\mathcal{P}}(u) &= \bigvee_{\lambda\in L^X} \Big(\bigvee_{x\in X} p_x(\lambda) \odot \bigwedge_{(x,y)\in X\times X} (u_\lambda(x,y) \to u(x,y)\Big)\\
&= \bigvee_{\lambda\in L^X} \Big(\bigvee_{x\in X} p_x(\lambda) \odot \bigwedge_{(x,y)\in X\times X} (u_\lambda^{-1}(y,x) \to u^{-1}(y,x)\Big)\\
&= \mathcal{U}_{\mathcal{P}}(u^{-1}).
\end{aligned}$$

Let $(X, \mathcal{P})$ be stratified, $u \in L^{X\times X}$, $\lambda \in L^X$ and $\alpha \in L$, we have

$$\begin{aligned}
\mathcal{U}_{\mathcal{P}}(\alpha \odot u) &= \bigvee \Big(\bigvee_{x\in X} (p_x(\rho) \odot S(u_\rho, \alpha \odot u))\Big)\\
&\geq \bigvee \Big(\bigvee_{x\in X} (p_x(\alpha \odot \lambda) \odot S(u_{\alpha\odot\lambda}, \alpha \odot u))\Big)\\
&\geq \bigvee \Big(\alpha \odot \bigvee_{x\in X} (p_x(\lambda) \odot S(\alpha \odot u_\lambda, \alpha \odot u))\Big)\\
&\geq \alpha \odot \bigvee \Big(\bigvee_{x\in X} (p_x(\lambda) \odot S(u_\lambda, u))\Big)\\
&= \alpha \odot \mathcal{U}_{\mathcal{P}}(u).
\end{aligned}$$

Hence $(X,\mathcal{U}_{\mathcal{P}})$ is stratified.

Theorem 3.6. Let $(X,\mathcal{P})$ and $(Y,\mathcal{Q})$ be L-fuzzy neighborhood spaces. Let $\phi:(X,\mathcal{P})\to(Y,\mathcal{Q})$ be fuzzy continuous. Then $\phi:(X,\mathcal{U}_{\mathcal{P}})\to(Y,\mathcal{V}_{\mathcal{Q}})$ is fuzzy preuniformly continuous.

Proof. Since $(\phi\times\phi)^{\leftarrow}(v_\lambda)\geq v_{\phi^{\leftarrow}(\lambda)}$ and $v(\phi(x),\phi(y))=(\phi\times\phi)^{\leftarrow}(v)(x,y)$, we have,

$$\begin{aligned}\mathcal{V}_{\mathcal{Q}}(v)&=\bigvee\Big(\bigvee_{y\in Y}q_y(\lambda)\odot S(v_\lambda,v)\Big)\\&\leq\bigvee\Big(\bigvee_{x\in X}q_{\phi(x)}(\lambda)\odot S((\phi\times\phi)^{\leftarrow}(v_\lambda),(\phi\times\phi)^{\leftarrow}(v))\Big)\\&\leq\bigvee\Big(\bigvee_{x\in X}p_x(\phi^{\leftarrow}(\lambda))\odot S(v_{\phi^{\leftarrow}(\lambda)},(\phi\times\phi)^{\leftarrow}(v))\Big)\\&\leq\mathcal{U}_{\mathcal{P}}((\phi\times\phi)^{\leftarrow}(v)).\end{aligned}$$

Theorem 3.7. Let $(X,\mathcal{T})$ be an L-fuzzy topological space. Define a map $\mathcal{U}_{\mathcal{T}}:L^{X\times X}\to L$ by

$$\mathcal{U}_{\mathcal{T}}(u)=\bigvee_{\lambda\in L^X}\bigvee_{x\in X}\Big(\lambda(x)\odot\mathcal{T}(\lambda)\odot S(u_\lambda,u)\Big).$$

Then (1) $(X,\mathcal{U}_{\mathcal{T}})$ is an L-fuzzy preuniform space,

(2) If $(X,\mathcal{T})$ is enriched, then $\mathcal{U}_{\mathcal{T}}(\alpha\odot u)\geq\mathcal{U}_{\mathcal{T}}(u)$ for all $\alpha\in L$ and $u\in L^{X\times X}$.

Proof. (U1) Since $u_{\top_X}=\top_{X\times X}$, we have

$$\mathcal{U}_{\mathcal{T}}(\top_{X\times X})\geq\top_X(x)\odot\mathcal{T}(\top_X)=\top.$$

(U2) If $u_1\leq u_2$, $u_1,u_2\in L^{X\times X}$, then

$$\begin{aligned}\mathcal{U}_{\mathcal{T}}(u_1)&=\bigvee_{\lambda\in L^X}\bigvee_{x\in X}\Big(\lambda(x)\odot\mathcal{T}(\lambda)\odot S(u_\lambda,u_1)\Big)\\&\leq\bigvee_{\lambda\in L^X}\bigvee_{x\in X}\Big(\lambda(x)\odot\mathcal{T}(\lambda)\odot S(u_\lambda,u_2)\Big)=\mathcal{U}_{\mathcal{T}}(u_2).\end{aligned}$$

(U3) Let $\lambda,\rho\in L^X$, then $u_\lambda\odot u_\rho=u_{\lambda\odot\rho}$ and

$$\begin{aligned}&\mathcal{U}_{\mathcal{T}}(u)\odot\mathcal{U}_{\mathcal{T}}(w)=\bigvee_{\lambda\in L^X}\bigvee_{x\in X}\Big(\lambda(x)\odot\mathcal{T}(\lambda)\odot S(u_\lambda,u)\Big)\\&\odot\bigvee_{\lambda\in L^X}\bigvee_{x\in X}\Big(\rho(x)\odot\mathcal{T}(\rho)\odot S(u_\rho,w)\Big)\\&\leq\bigvee_{\lambda\in L^X}\bigvee_{x\in X}\Big(\lambda(x)\odot\rho(x)\odot\mathcal{T}(\lambda)\odot\mathcal{T}(\rho)\odot S(u_\lambda\odot u_\rho,u\odot w)\Big)\\&\leq\bigvee_{\lambda\in L^X}\bigvee_{x\in X}\Big((\lambda\odot\rho)(x)\odot\mathcal{T}(\lambda\odot\rho)\odot S(u_{\lambda\odot\rho},u\odot w)\Big)\\&\leq\mathcal{U}_{\mathcal{T}}(u\odot w).\end{aligned}$$

(U4)

$$\begin{aligned}
\mathcal{U}_{\mathcal{T}}(u) &= \bigvee_{\lambda\in L^X}\bigvee_{x\in X}\Big(\lambda(x)\odot\mathcal{T}(\lambda)\odot S(u_\lambda,u)\Big)\\
&= \bigvee_{\lambda\in L^X}\bigvee_{x\in X}\Big(\lambda(x)\odot\mathcal{T}(\lambda)\odot\bigwedge_{(x,y)\in X\times X}\big(u_\lambda(x,y)\to u(x,y)\big)\Big)\\
&\le \bigvee_{\lambda\in L^X}\bigvee_{x\in X}\Big(\lambda(x)\odot\mathcal{T}(\lambda)\odot\bigwedge_{y\in X}\big(u_\lambda(y,y)\to u(y,y)\big)\Big)\\
&\le \bigvee_{\lambda\in L^X}\Big(\mathcal{T}(\lambda)\odot\bigwedge_{y\in X}u(y,y)\big)\\
&= \bigwedge_{y\in X}u(y,y).
\end{aligned}$$

(U5) Let $u\in L^{X\times X}$, then

$$\begin{aligned}
\mathcal{U}_{\mathcal{T}}(u) &= \bigvee_{\lambda\in L^X}\bigvee_{x\in X}\Big(\lambda(x)\odot\mathcal{T}(\lambda)\odot S(u_\lambda,u)\Big)\\
&= \bigvee_{\lambda\in L^X}\bigvee_{x\in X}\Big(\lambda(x)\odot\mathcal{T}(\lambda)\odot S(u_\lambda^{-1},u^{-1})\Big)=\mathcal{U}_{\mathcal{T}}(u^{-1}).
\end{aligned}$$

Finally, let $u\in L^{X\times X}$, $\lambda\in L^X$ and $\alpha\in L$, we have by Lemma 2.3(4),

$$\begin{aligned}
\mathcal{U}_{\mathcal{T}}(u) &= \bigvee_{\lambda\in L^X}\bigvee_{x\in X}\Big(\lambda(x)\odot\mathcal{T}(\lambda)\odot S(u_\lambda,u)\Big)\\
&\le \bigvee_{\lambda\in L^X}\bigvee_{x\in X}\Big(\lambda(x)\odot\mathcal{T}(\lambda)\odot S(\alpha\odot u_\lambda,\alpha\odot u)\Big)\\
&= \bigvee_{\lambda\in L^X}\bigvee_{x\in X}\Big(\lambda(x)\odot\mathcal{T}(\lambda)\odot S(u_{\alpha\odot\lambda},\alpha\odot u)\Big)\\
&\le \bigvee_{\lambda\in L^X}\bigvee_{x\in X}\Big(\lambda(x)\odot\mathcal{T}(\alpha\odot\lambda)\odot S(u_{\alpha\odot\lambda},\alpha\odot u)\Big)\\
&\le \bigvee_{\lambda\in L^X}\bigvee_{x\in X}\Big(\lambda(x)\odot\mathcal{T}(\mu)\odot S(u_\mu,\alpha\odot u)\Big)\\
&= \mathcal{U}_{\mathcal{T}}(\alpha\odot u).
\end{aligned}$$

Theorem 3.8. Let $(X,\mathcal{T}_X)$ and $(Y,\mathcal{T}_Y)$ be L-fuzzy topological spaces. Let $\phi:(X,\mathcal{T}_X)\to(Y,\mathcal{T}_Y)$ be surjective fuzzy continuous. Then $\phi:(X,\mathcal{U}_{\mathcal{T}_X})\to(Y,\mathcal{V}_{\mathcal{T}_Y})$ is fuzzy preuniformly continuous.

Proof. Since $(\phi\times\phi)^{\leftarrow}(v_\lambda)(x_1,x_2)=v_\lambda(\phi(x_1),\phi(x_2))\ge v_{\phi^{\leftarrow}(\lambda)}(x_1,x_2)$ and $v(\phi(x),\phi(y))=(\phi\times\phi)^{\leftarrow}(v)(x,y)$, we have

$$\begin{aligned}
&\mathcal{V}_{\mathcal{T}_Y}(v)=\bigvee_{\lambda\in L^Y}\bigvee_{y\in Y}\Big(\lambda(y)\odot\mathcal{T}_Y(\lambda)\odot S(v_\lambda,v)\Big)\\
&\le \bigvee_{\lambda\in L^Y}\bigvee_{y\in Y}\Big(\lambda(y)\odot\mathcal{T}_Y(\lambda)\odot S((\phi\times\phi)^{\leftarrow}(v_\lambda),(\phi\times\phi)^{\leftarrow}(v))\Big)\\
&\le \bigvee_{\lambda\in L^Y}\bigvee_{x\in X}\Big(\lambda(\phi(x))\odot\mathcal{T}_X(\phi^{\leftarrow}(\lambda))\odot S(v_{\phi^{\leftarrow}(\lambda)},(\phi\times\phi)^{\leftarrow}(v))\Big)\\
&\le \mathcal{U}_{\mathcal{T}_X}((\phi\times\phi)^{\leftarrow}(v)).
\end{aligned}$$

Example 3.9. Let $(L = [0,1], \odot, \rightarrow)$ be a complete residuated lattice defined by

$$x \odot y = (x + y - 1) \vee 0, \ \ x \rightarrow y = (1 - x + y) \wedge 1.$$

Let $X = \{x, y, z\}$ be a set and $\rho \in L^X$ with $\rho(x) = 0.6, \rho(y) = 0.6, \rho(z) = 0.5$. Define $\mathcal{T} : L^X \rightarrow L$ as follows

$$\mathcal{T}(\lambda) = \begin{cases} 1, & \text{if } \lambda \in \{1_X, 0_X\}, \\ 0.6, & \text{if } \lambda = \rho, \\ 0.3, & \text{if } \ \lambda = \rho \odot \rho, \\ 0, & \text{otherwise.} \end{cases}$$

We obtain $u_{\rho \otimes \rho} = 1_{\triangle}$ and

$$u_\rho = \begin{pmatrix} 1 & 0.2 & 0.1 \\ 0.2 & 1 & 0.1 \\ 0.1 & 0.1 & 1 \end{pmatrix}.$$

By Theorem 3.7, we can construct $\mathcal{U}_{\mathcal{T}} : L^{X \times X} \rightarrow L$ as follows

$$\mathcal{U}_{\mathcal{T}}(u) = (\bigwedge_{(x,y) \in X \times X} u(x,y)) \vee (0.6 \odot S(u_\rho, u)) \vee (0.3 \odot \bigwedge_{x \in X} u(x,x)).$$

By Theorem 2.8, we can obtain $\mathcal{P}^{\mathcal{T}} : X \rightarrow L^{L^X}$ as follows

$$p_x^{\mathcal{T}}(\lambda) = (\bigwedge_{x \in X} \lambda(x)) \vee (0.6 \odot S(\rho, \lambda) \odot 0.6),$$

$$p_y^{\mathcal{T}}(\lambda) = (\bigwedge_{x \in X} \lambda(x)) \vee (0.6 \odot S(\rho, \lambda) \odot 0.6),$$

$$p_z^{\mathcal{T}}(\lambda) = (\bigwedge_{x \in X} \lambda(x)) \vee (0.6 \odot S(\rho, \lambda) \odot 0.5).$$

By Theorem 3.5, we can construct $\mathcal{U}_{\mathcal{P}_{\mathcal{T}}} : L^{X \times X} \rightarrow L$ as follows

$$\mathcal{U}_{\mathcal{P}_{\mathcal{T}}}(u) = \bigvee_{\lambda \in L^X} ((\bigwedge_{x \in X} \lambda(x) \vee (0.2 \odot S(\rho, \lambda))) \odot S(u_\lambda, u)).$$

Acknowledgments

This work was supported by Research Fund of Gangneung-Wonju National University in 2016.

References

[1] R. Badard, A.A. Ramadan, A.S. Mashhour, Smooth preuniform and proximity spaces, *Fuzzy Sets and Systems*, **59** (1993), 95-107.

[2] R. Bělohlávek,*Fuzzy Relational Systems*, Kluwer Academic Publishers, New York (2002), **doi:** 10.1007/978-1-4615-0633-1.

[3] Fang Jinming, I-fuzzy Alexandrov topologies and specialization orders, *Fuzzy Sets and Systems,* **158** (2007), 2359-2374.

[4] J. Fang, The relationship between L-ordered convergence structures and strong L-topologies, *Fuzzy Sets and Systems*, **161** (2010), 2923-2944.

[5] J.Gutiérrez García, I. Mardones Peerez, M.H. Burton, The relationship between various filter notions on a GL-monoid, *J. Math. Anal. Appl.*, **230** (1999), 291- 302.

[6] J.Gutiérrez García, M. A. de Prade Vicente, A.P. Šostak, A unified approach to the concept of fuzzy Luniform spaces, Chapter 3, 81-114 in [23].

[7] P. Hájek, Metamathematices of Fuzzy Logic, Kluwer Academic Publishers, Dordrecht, 1998, **doi:** 10.1007/978-94-011-5300-3.

[8] U. Höhle, S.E. Rodabaugh, Mathematics of Fuzzy Sets: Logic, Topology, and Measure Theory, The Handbooks of Fuzzy Sets Series 3, Kluwer Academic Publishers, Boston (1999), **doi:** 10.1007/978-1-4615-5079-2.

[9] B. Hutton, Uniformities in fuzzy topological spaces, *J. Math. Anal. Appl.*, **58** (1977), 74-79, **doi:** 10.1016/0022-247x(77)90192-5.

[10] A.K. Katsaras, On fuzzy uniform spaces, *J. Math. Anal. Appl.*, **101** (1984), 97-113, **doi:** 10.1016/0022-247x(84)90060-x.

[11] Y.C. Kim, A.A. Ramadan, M. A. Usama, L-fuzzy Uniform Spaces, *The Journal of Fuzzy Mathematics*, **14** (2006), 821-850.

[12] J.M. Ko, J.M. Oh, Relationships between L-neighborhood systems and L-fuzzy topologies, *Int.J. Pure and Applied Math.*, **112**(3), (2017) 631-643, **doi:** 10.12732/ijpam.v112i3.14.

[13] W. Kotzé, *Uniform spaces*, Chapter 8, 553-580 in [8].

[14] T. Kubiak, *On fuzzy topologies*, Ph.D. Thesis, Adam Mickiewicz Uniformity, Poznan, Poland, (1985)

[15] R. Lowen, Fuzzy uniform spaces, *J. Math. Anal. Appl.*, **82** (1981), 370-385.

[16] R. Lowen, Fuzzy neighborhood spaces, *Fuzzy Sets and Systems*, **7** (1982), 165-189.

[17] H. Lai, D. Zhang, Fuzzy preorder and fuzzy topology, *Fuzzy Sets and Systems,* **157** (2006), 1865-1885.

[18] A.A. Ramadan,Y.C. Kim, M.K. El-Gayyar, On fuzzy uniform spaces, *The Journal of Fuzzy Mathematics*, **11** (2003), 279-299.

[19] A.A. Ramadan, E.H.Elkordy, Yong Chan Kim, Relationships between L-fuzzy quasi-uniform stuctures and L-fuzzy topologies, *Journal of Intelligent and Fuzzy Systems*, **28** (2015), 2319-2327.

[20] A.A. Ramadan, On L-fuzzy interior operators and L-fuzzy quasi-uniform spaces, *Journal of Intelligent and Fuzzy Systems*, in press.

[21] S.E. Rodabaugh, A theory of fuzzy uniformities with applications to the fuzzy real lines, *J. Math. Anal. Appl.*, **129** (1988), 37-70.

[22] S.E. Rodabaugh, E.P. Klement, *Topological and Algebraic Structures In Fuzzy Sets*, The Handbook of Recent Developments in the Mathematics of Fuzzy Sets, *Kluwer Academic Publishers, Boston, Dordrecht, London*, (2003)

[23] A.P. Šostak, *On a fuzzy topological structure, Suppl.*, Rend Circ. Matem. Palermo, Ser. II [11] (1985), 125- 186.

[24] A.P. Šostak,*Towards the theory of M-approximate systems*, Fuzzy Sets and Systems, **161** (2010), 2440- 2461.

[25] E. Turunen,*Mathematics Behind Fuzzy Logic*, *A Springer-Verlag Co., Heidelberg* (1999).