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ON SPECIAL CONCIRCULAR *R*-LIE-RECURRENCE IN SPECIAL FINSLER SPACES

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Abstract: In this paper we discuss a special concircular R-Lie-recurrence in special Finsler spaces such as R-recurrent, R-symmetric, R-birecurrent and R-bisymmetric. Apart from other theorems, it is being proved that an R-recurrent Finsler space can not admit a special concircular R-Lie-recurrence while a non-flat R-symmetric Finsler space $F_n(n > 2)$ admitting a special concircular R-Lie-recurrence is necessarily of constant Riemannian curvature.

AMS Subject Classification: 53B40.
Key Words: Finsler space, R-Lie-recurrence, Riemannian curvature, special concircular vector field, R-recurrent Finsler space.

1. Introduction

In 1982, P. N. pandey [5] introduced the concept of Lie-recurrence in a Finsler Space. In 1992, K. L. Duggal [3] studied the Lie-recurrence in a Riemannian space with its application to fluid space time but he used the term curvature inheriting symmetry in place of Lie-recurrence. He also used the theory to the study of fluid space time. Since then both the terms (Lie-recurrence and curvature inheriting symmetry) are in use (see [1] and [11]-[14]). P. N. Pandey and vaishali pandey [12] discussed a K-curvature inheritance, K-projective Lie-recurrence and special concircular K-Lie-recurrence in a Finsler space.

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The authors (P. N. pandey and Vaishali Pandey) [13] studied a K-Lie-recurrence in a Finsler space. C. K. Mishra and Gautam Lodhi [1] discussed curvature inheriting symmetry and Ricci-inheriting symmetry in a Finsler space and obtained some results. In this paper we have discussed a special concircular R-Lie-recurrence in special Finsler spaces such as R-recurrent, R-symmetric, R-birecurrent and R-bisymmetric.

2. Preliminaries

Let F_n be an n-dimensional Finsler space equipped with a metric function F satisfying the requisite conditions [2]. The relation between the metric tensor g_{ij} of the Finsler space F_n and the metric function F are given by

$$\text{(a)}\quad g_{ij} = \frac{1}{2}\dot{\partial}_i\dot{\partial}_j F^2, \ \text{(b)}\quad g_{ij}y^iy^j = F^2, \tag{1}$$

where $\dot{\partial}_i \equiv \frac{\partial}{\partial y^i}$.
The Cartan h-covariant derivative of an arbitrary vector field T^i with respect to connection coefficients Γ^{*i}_{jk} is given by

$$T^i_{|k} = \partial_k T^i - (\dot{\partial}_r T^i)\Gamma^{*r}_{hk}y^h + T^r\Gamma^{*i}_{rk}, \tag{2}$$

where $\partial_k \equiv \frac{\partial}{\partial x^k}$.
The Cartan v-covariant derivative of an arbitrary vector field T^i is given by

$$T^i|_k = \dot{\partial}_k T^i + T^r C^i_{rk}, \tag{3}$$

where $C^i_{rk} = g^{ij}C_{jrk}$. The tensor C_{jrk} is called Cartan tensor and defined as $C_{jrk} = \frac{1}{2}\dot{\partial}_j g_{rk}$.
The Ricci commutation formula for h-covariant derivative is given by

$$T^i_{|h|k} - T^i_{|k|h} = T^r R^i_{rhk} - T^i|_j R^j_{shk}y^s, \tag{4}$$

where $R^i_{rhk} = \partial_k\Gamma^{*i}_{rh} - (\dot{\partial}_t\Gamma^{*i}_{rh})\Gamma^{*t}_{sk}\dot{x}^s + \Gamma^{*t}_{rh}\Gamma^{*i}_{tk} + C^i_{rm}(\partial_k G^m_h + G^t_h G^m_{tk})$
$- \partial_h\Gamma^{*i}_{rk} + (\dot{\partial}_t\Gamma^{*i}_{rk})\Gamma^{*t}_{sh}\dot{x}^s - \Gamma^{*t}_{rk}\Gamma^{*i}_{th} - C^i_{rm}(\partial_h G^m_k + G^t_k G^m_{th})$.
The tensor R^i_{jkh} is called h-curvature tensor. This tensor is skew-symmetric in last two lower indices and positively homogeneous of degree zero in y^i.
Cartan curvature tensor K^i_{jkh}, Cartan h-curvature tensor R^i_{jkh} are related by

$$R^i_{jkh} = K^i_{jkh} + C^i_{jm}H^m_{kh}, \tag{5}$$

where $H^m_{kh} = \partial_h G^m_k - \partial_k G^m_h + G^r_k G^m_{rh} - G^r_h G^m_{rk}$.
Transvecting (5) by y^j and using the fact $C^i_{jm} y^j = 0$, we get

$$R^i_{jkh} y^j = H^i_{kh}. \tag{6}$$

The tensor H^i_{kh} is connected with Berwald deviation tensor H^i_h by

$$\text{(a)}\ \ y^k H^i_{kh} = H^i_h,\ \text{(b)}\ \ \dot\partial_k H^i_h - \dot\partial_h H^i_k = 3H^i_{kh}. \tag{7}$$

Berwald deviation tensor satisfies the following:

$$\text{(a)}\ g_{ik} H^i_h = g_{ih} H^i_k,\ \text{(b)}\ y_i H^i_h = 0,\ \text{(c)}\ \ H^i_i = (n-1)H, \tag{8}$$

where $y_i = g_{ij} y^j$ and H is scalar curvature.
The commutation formula for the operators of partial differentiation with respect to y^k and h-covariant differentiation is given by

$$\dot\partial_k (T^i_{|h}) - (\dot\partial_k T^i)_{|h} = T^r \dot\partial_k \Gamma^{\square i}_{rh} - (\dot\partial_r T^i)(\dot\partial_k \Gamma^{\square r}_{sh}) y^s. \tag{9}$$

Let us consider an infinitesimal transformation

$$\overline{x}^i = x^i + \epsilon v^i(x^j), \tag{10}$$

generated by a contravariant vector field $v^i(x^j)$ which depends on position coordinates only. ϵ appearing in (10) is an infinitesimal constant.
The Lie-derivative of an arbitrary tensor T^i_j with respect to the infinitesimal transformation (10) is given by [4]

$$\pounds T^i_j = T^i_{j|r} v^r - T^r_j v^i_{|r} + T^i_r V^i_{|j} + (\dot\partial_r T^i_j) v^r_{|s} y^s. \tag{11}$$

The commutation formula for the operators $\pounds$ and $\dot\partial_h$ is given by

$$\dot\partial_h \pounds \Omega - \pounds \dot\partial_h \Omega = 0, \tag{12}$$

where Ω is any geometrical object. An infinitesimal transformation (10) is Lie-recurrence or H-Lie-recurrence if the Lie-derivative of Berwald curvature tensor H^i_{jkh} of the Finsler space satisfies

$$\pounds H^i_{jkh} = \phi H^i_{jkh}, \tag{13}$$

where ϕ is a non-zero scalar field [5]. In view of this concept, the infinitesimal transformation (10) is called R-Lie-recurrence if the Lie-derivative of Cartan h-curvature tensor satisfies [13]

$$\pounds R^i_{jkh} = \phi R^i_{jkh},\ \ \phi \neq 0. \tag{14}$$

A vector field v^i in the Finsler space F_n is said to be special concircular if

$$\text{(a)}\quad v^i_{|k} = \rho\delta^i_k, \quad \text{(b)}\quad v^i|_k = 0, \tag{15}$$

where $\rho = \rho(x)$ [13].

3. Special Concircular R-Lie-Recurrence

Theorem 1. *An R-recurrent Finsler space $F_n (n > 2)$ can not admit a special concircular R-Lie-recurrence.*

Proof. Let us consider a Finsler space admitting the infinitesimal transformation (10) generated by a special concircular vector field $v^i(x^j)$. Differentiating (15a) covariantly with respect to x^h,we get

$$v^i_{|k|h} = \rho_h\delta^i_k, \tag{16}$$

where $\rho_h = \rho_{|h}$. Taking skew-symmetric part of (16) and utilizing commutation formula (4) and using (15b), we have

$$v^r R^i_{rkh} = \rho_h\delta^i_k - \rho_k\delta^i_h. \tag{17}$$

Contraction of indices i and h in (17) gives

$$v^r R_{rk} = -(n-1)\rho_k, \tag{18}$$

where R_{rk} is Ricci tensor defined as $R_{rk} = R^h_{rkh}$. From equations (17) and (18), we may write

$$v^r R^i_{rkh} = \frac{1}{n-1} v^r (R_{rk}\delta^i_h - R_{rh}\delta^i_k), \tag{19}$$

which implies

$$\begin{aligned} &\{R^i_{rkh|m} - \frac{1}{n-1}(R_{rk|m}\delta^i_h - R_{rh|m}\delta^i_k)\}v^r \\ &+\rho\{R^i_{mkh} - \frac{1}{n-1}(R_{mk}\delta^i_h - R_{mh}\delta^i_k)\} = 0. \end{aligned} \tag{20}$$

Let the Finsler space F_n be R-recurrent characterized by

$$R^i_{jkh|m} = \lambda_m R^i_{jkh}, \tag{21}$$

where λ_m are components of a non-zero covariant vector field [7]. Contracting the indices i and h in (21), we have

$$R_{jk|m} = \lambda_m R_{jk}. \tag{22}$$

From (19), (20), (21) and (22), we get

$$R^i_{mkh} = \frac{1}{n-1}(R_{mk}\delta^i_h - R_{mh}\delta^i_k). \tag{23}$$

Transvecting (23) by y^m and using (6) and $R_{mk}y^m = H_k$, we have

$$H^i_{kh} = \frac{1}{n-1}(H_k\delta^i_h - H_h\delta^i_k). \tag{24}$$

Transvecting (24) by y_i and using $y_i H^i_{kh} = 0$ [8], we get $H_k y_h = H_h y_k$, which implies

$$H_k = \frac{n-1}{F^2} H y_k, \tag{25}$$

for $H_h y^h = (n-1)H$ and $y_h y^h = F^2$. In view of (25), (24) may be rewritten as

$$H^i_{kh} = R(y_k\delta^i_h - y_h\delta^i_k), \tag{26}$$

where $R = \frac{H}{F^2}$. In view of Berwald theorem [2], equation (26) implies that R is a constant and the space $F_n(n > 2)$ is of constant Riemannian curvature. Differentiating (26) covariantly, we find

$$H^i_{Kh|m} = 0, \tag{27}$$

for $y_{k|m} = 0$.
Transvecting (21) by y^j and using equation (6), we get $H^i_{kh|m} = \lambda_m H^i_{kh}$, which in view of (27), implies $\lambda_m = 0$, a contradiction. Therefore, an R-recurrent Finsler space $F_n(n > 2)$ can not admit a special concircular infinitesimal transformation. □

Definition 2. A Finsler space $F_n(n > 2)$ be R-symmetric characterized by [6]

$$R^i_{jkh|m} = 0. \tag{28}$$

Theorem 3. *A special concircular R-Lie-recurrence in a non-flat R-symmetric Finsler space $F_n(n > 2)$ is an H-Lie-recurrence and the R-symmetric Finsler space is necessarily of constant Riemannian curvature.*

Proof. Then, equation (20) implies equation (23). Adapting the above proceedure, we may show that equation (23) implies that the space $F_n(n > 2)$ is of constant Riemannian curvature if it is non-flat. Suppose that the special concircular transformation (10) is a Lie-recurrence in the R-symmetric Finsler space $F_n(n > 2)$. Then we have equation (14). In view of equation (11), equation (14) may be written as

$$R^i_{jkh|r}v^r + (\dot{\partial}_r R^i_{jkh})v^r_{|s}y^s - R^r_{jkh}v^i_{|r} + R^i_{rkh}v^r_{|j} + R^i_{jrh}v^r_{|k} + R^i_{jkr}v^r_{|h} = \phi R^i_{hjk}.$$

Using equations (16), (28) and the fact that the curvature tensor R^i_{hjk} is positively homogeneous of degree zero in y^j, we get $\phi = 2\rho$ if the space is non-flat. Since ρ is independent of y^i and $\phi = 2\rho$, ϕ is also independent of y^i. Transvecting equation (14) by y^j and using (6), we get

$$\pounds H^i_{kh} = \phi H^i_{kh}. \tag{29}$$

Differentiating (29) partially with respect to y^j and using the fact that ϕ is independent of y^j, we get $\pounds H^i_{jkh} = \phi H^i_{jkh}$, which shows that the special concircular R-Lie-recurrence is an H-Lie-recurrence. □

4. Special Concircular R-Lie-Recurrence in a Birecurrent Finsler Space

Definition 4. A birecurrent Finsler space F_n characterized by

$$R^i_{jkh|l|m} = a_{lm}R^i_{jkh}, \tag{30}$$

where a_{lm} are components of a non-zero covariant tensor of type $(0, 2)$ and $R^i_{jkh} \neq 0$ (see [8] and [10]).

Theorem 5. *A birecurrent Finsler space $F_n(n > 2)$ admitting a special concircular R-Lie-recurrence necessarily satisfies the conditions $\phi = 3\rho$ and $v^r a_{mr} = \rho_m$.*

Proof. Suppose that this space admits a special concircular R-Lie-recurrence characterized by equations (15) and (14). In view of (11), equation (14) may be written as

$$R^i_{jkh|r}v^r = (\phi - 2\rho)R^i_{jkh}. \tag{31}$$

Differentiating equation (31) covariantly with respect to x^m, we get

$$v^r_{|m} R^i_{jkh|r} + v^r R^i_{jkh|r|m} = (\phi_m - 2\rho_m) R^i_{jkh} + (\phi - 2\rho) R^i_{jkh|m}, \tag{32}$$

where $\phi_m = \phi_{|m}$. Using (15a) and equation (30) in equation (32), we have

$$(v^r a_{rm} - \phi_m + 2\rho_m) R^i_{jkh} = (\phi - 3\rho) R^i_{jkh|m}. \tag{33}$$

In view of the definition for a birecurrent Finsler space, $R^i_{jkh} \neq 0$. In equation (33), for $R^i_{jkh|m} \neq 0$, implies $a_{lm} = 0$, a contradiction. Therefore, equation (33) implies either of the follwing conditions:

(i) $\phi - 3\rho = 0, \ \ v^r a_{mr} - \phi_m + 2\rho_m = 0,$

(ii) $\phi - 3\rho \neq 0, \ \ v^r a_{mr} - \phi_m + 2\rho_m \neq 0.$

We can write the condition (i) as $\phi = 3\rho, \ \ v^r a_{mr} = \rho_m$.
Let us consider the condition (ii). In this case equation (33) may be written as

$$R^i_{jkh|m} = \frac{(v^r a_{mr} - \phi_m + 2\rho_m)}{\phi - 3\rho} R^i_{jkh}, \tag{34}$$

which shows that the space is R-recurrent. In view of theorem 1, an R-recurrent Finsler space $F_n (n > 2)$ does not admit a special concircular R-Lie-recurrence. Therefore, the conditions (ii) is not possible. □

5. Special Concircular *R*-Lie-Recurrence in a Bisymmetric Finsler Space

Definition 6. A bisymmetric Finsler space F_n characterized by [9]

$$R^i_{jkh|l|m} = 0, \tag{35}$$

Theorem 7. *An R-bisymmetric Finsler space F_n admitting a special concircular R-Lie-recurrence with condition $\phi = 3\rho$ is flat.*

Proof. Suppose that this space admits a special concircular R-Lie-recurrence characterized by equations (15) and (14). Differentiating equation (31) covariantly with respect to x^m, we have

$$v^r_{|m} R^i_{jkh|r} + v^r R^i_{jkh|r|m} = (\phi_m - 2\rho_m) R^i_{jkh} + (\phi - 2\rho) R^i_{jkh|m}. \tag{36}$$

Using equations (15a) and equation (35) in equation (36), we get

$$(\phi - 3\rho)R^i_{jkh|m} = (2\rho_m - \phi_m)R^i_{jkh}. \tag{37}$$

If $\phi = 3\rho$, equation (37) reduces to $\rho_m R^i_{jkh} = 0$ which implies $R^i_{jkh} = 0$ for $\rho_m \neq 0$. □

Theorem 8. *An R-bisymmetric Finsler space $F_n (n > 2)$ admitting a special concircular R-Lie-recurrence with $\phi = 2\rho$ is a R-symmetric Finsler space.*

Proof. If $\phi = 2\rho$ then $\phi_m = 2\rho_m$. Therefore, equation (37) may be written as

$$R^i_{jkh|m} = 0. \tag{38}$$

This shows that the space is symmetric. Thus, we see that a R-bisymmetric Finsler space admitting a special concircular R-Lie-recurrence with $\phi = 2\rho$ is a symmetric space admitting a special concircular R-Lie-recurrence. □

From Theorem 3 and Theorem 8, we may conclude:

Theorem 9. *A special concircular R-Lie-recurrence in a non-flat R-bisymmetric Finsler space $F_n (n > 2)$ with $\phi = 2\rho$ is an H-Lie-recurrence and the R-bisymmetric Finsler space $F_n (n > 2)$ is necessarily of constant Riemannian curvature.*

Theorem 10. *An R-bisymmetric Finsler space $F_n (n > 2)$ can not admit a special concircular R-Lie-recurrence if ϕ is neither 2ρ nor 3ρ.*

Proof. If $\phi \neq 2\rho$ and $\phi \neq 3\rho$, then equation (37) may be written as

$$R^i_{jkh|m} = \frac{2\rho_m - \phi_m}{\phi - 3\rho} R^i_{jkh}. \tag{39}$$

This shows that the space is recurrent, but in view of Theorem 1, a recurrent space does not admit a special concircular R-Lie-recurrence. □

From Theorems 7, 8, 9 and 10 we may conclude:

Theorem 11. *An R-bisymmetric Finsler space $F_n (n > 2)$ admitting a special concircular R-Lie-recurrence is either flat or a Finsler space of constant Riemannian curvature.*

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