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PROJECTIVE CURVATURE TENSOR ON $(k,\ \mu)$-CONTACT SPACE FORMS

M. Jawarneh[1 §], S. Samui[2], U. De[3]

[1]Department of Mathematics
AL-Kamil Faculty of Science and Arts
University of Jeddah
P.O. Box 110, Alkamil 21937, SAUDI ARABIA
[2]Umeschandra College
Surya Sen Street, Kolkata, 700012, West Bengal, INDIA
[3]Department of Pure Mathematics
Calcutta University
35, Ballygunge Circular Road, Kol 700019, West Bengal, INDIA

Abstract: We characterize a $(k,\ \mu)$-contact space form satisfying certain curvature conditions on the projective curvature tensor.

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1. Introduction

In [4], Blair, Koufogiorgos and Papantoniou introduced $(k,\ \mu)$-contact metric manifolds. A class of contact metric manifolds with contact metric structure $(\phi,\ \xi,\ \eta,\ g)$ in which the curvature tensor R satisfies the condition

$$R(X,Y)\xi = (kI + \mu h)(\eta(Y)X - \eta(X)Y),$$

for all $X, Y \in$ TM is called $(k,\ \mu)$-contact metric manifolds.

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§Correspondence author

The sectional curvature $K(X, \phi X)$ of a plane section spanned by a unit vector X orthogonal to ξ is called a ϕ-sectional curvature. If the $(k,\ \mu)$-contact metric manifold M has constant ϕ-sectional curvature c, then it is called a $(k,\ \mu)$-contact space form and is denoted by $M(c)$. $(k,\ \mu)$-contact space forms have been studied by K. Arslan, R. Ezentas, I. Mihai, C. Murthan and Özgür, C. [2] and A. Akbar and A. Sarkar [1] and many others.

Apart from conformal curvature tensor, the projective curvature tensor is another important tensor from the differential geometric point of view. Let M be a (2n+1)-dimensional Riemannian manifold. M is said to be locally projectively flat for $n \geq 1$, if and only if the well-known projective curvature tensor P vanishes. P is defined by

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{2n}[S(Y,Z)X - S(X,Z)Y], \tag{1.1}$$

for all $X, Y, Z \in TM$, where R is the curvature tensor and S is the Ricci tensor. In fact M is projectively flat if and only if it is of constant curvature [13]. Thus the projective curvature tensor is the measure of the failure of a Riemannian manifold to be of constant curvature.

Let M be an almost contact metric manifold equipped with an almost contact metric structure $(\phi,\ \xi,\ \eta,\ g)$. Since at each point $p \in M$ the tangent space T_pM can be decomposed into direct sum $T_pM = \phi(T_pM) \oplus \{\xi_p\}$, where $\{\xi_p\}$ is the 1-dimensional linear subspace of T_pM generated by $\{\xi_p\}$, the conformal curvature tensor C is a map

$$C : T_pM \times T_pM \times T_pM \longrightarrow \phi(T_P) \oplus \{\xi_p\} \ \ p \in M.$$

It may be natural to consider the following particular cases: (1) the projection of the image of C in $\phi(T_pM)$ is zero; (2) the projection of the image of C in $\{\xi_p\}$ is zero; (3) the projection of image of $C|_{\phi(T_\mathsf{p}M)\times\phi(T_\mathsf{p}M)\times\phi(T_\mathsf{p}M)})$ in $\phi(T_pM)$ is zero. An almost contact metric manifold satisfying the case (1), (2) and (3) is said to be conformally symmetric [14], ξ-conformally flat [15] and ϕ-conformally flat [6] respectively. In an analogous way, we define ξ-projectively flat $(k,\ \mu)$-contact space forms.

Definition. A contact metric manifold is called ξ-projectively flat if the manifold satisfies $P(X,Y)\xi = 0$ for all vector fields X, Y.

As a generalization of symmetric manifolds Cartan in 1946 introduced the notion of semisymmetric manifolds. A Riemannian manifold is called semisymmetric if the curvature tensor satisfies

$$R(X,Y) \cdot R = 0,$$

where $R(X,Y)Z$ is considered as a field of linear operators acting on R.

An example of curvature condition of semisymmetry type is the following:

$$Q \cdot R = 0,$$

where Q is the Ricci operator of type $(1,1)$ and $S(X,Y) = g(QX,Y)$.

A natural extension of such curvature conditions form curvature conditions of pseudosymmetry type. The condion $Q \cdot R = 0$ have been studied by Verstraelen et al. in [11] .

In this paper, we characterize $(k,\ \mu)$-contact space forms satisfying $Q \cdot P = 0$.

In [7], U. C. De and Avik De studied Projective curvature tensor in K-contact manifolds. Again in [8], some properties of projective curvature tensor in $(k,\ \mu)$-contact metric manifolds was studied by Sujit Ghosh.

Motivated by the above studies, in this paper We characterize a $(k,\ \mu)$-contact space form satisfying certain curvature conditions on the projective curvature tensor. The paper is organized as follows:

In Section 2, we give necessary details about $(k,\ \mu)$-contact space forms. In Section 3, we study ξ-projectively flat $(k,\ \mu)$-contact space forms. Section 4 deals with the study of ϕ-projectively semisymmetric $(k,\ \mu)$-contact space forms. In Section 5, $(k,\ \mu)$-contact space forms satisfying $Q \cdot P = 0$ have been studied. Finally, we construct an example of a $(k,\ \mu)$-contact space form which verifies Theorem 4.1.

2. Preliminaries

A $(2n+1)$-dimensional differentiable manifold M is called an almost contact manifold [3] if there is an almost contact structure $(\phi,\ \xi,\ \eta)$ consisting of a $(1,1)$ tensor field ϕ, a vector field ξ, a 1-form η satisfying

$$\phi^2(X) = -X + \eta(X)\xi,\ \eta(\xi) = 1,\ \phi\xi = 0,\ \eta \circ \phi = 0. \tag{2.1}$$

An almost contact structure is said to be normal if the induced almost complex structure J on the product manifold $\mathrm{M} \times \mathbb{R}$ defined by

$$\mathrm{J}(X, f\frac{d}{dt}) = (\phi X - f\xi, \eta(X)\frac{d}{dt})$$

is integrable where X is tangent to M, t is the coordinate of $\mathbb{R}$ and f is a smooth function on $\mathrm{M} \times \mathbb{R}$.

The condition for being normal is equivalent to vanishing of the torsion tensor $[\phi, \phi] + 2d\eta \otimes \xi$ where $[\phi, \phi]$ is the Nijenhuis tensor of ϕ.

Let g be a compatible Riemannian metric with (ϕ,ξ,η), that is,

$$g(X, Y) = g(\phi X, \phi Y) + \eta(X)\eta(Y), \tag{2.2}$$

or equivalently,

$$g(X, \xi) = \eta(X), g(\phi X, Y) = -g(X, \phi Y), \tag{2.3}$$

for all $X, Y \in$ TM.

An almost contact metric structure becomes a contact metric structure if

$$g(X, \phi Y) = d\eta(X, Y),\ for\ all\ X, Y \in \mathrm{TM}. \tag{2.4}$$

Given a contact metric manifold $M(\phi, \xi, \eta, g)$, we define a $(1, 1)$ tensor field h by $h = \frac{1}{2}\mathrm{L}_\xi\phi$ where L denotes the Lie differentiation. Then h is symmetric and satisfies

$$h\xi = 0,\ h\phi + \phi h = 0, \tag{2.5}$$

$$\nabla\xi = -\phi - \phi h,\ trace(h) = trace(\phi h) = 0, \tag{2.6}$$

where ∇ is the Levi-Civita connection.

A contact metric manifold is said to be an η-Einstein manifold if

$$S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y), \tag{2.7}$$

where a, b are smooth functions and $X, Y \in$ TM, S is the Ricci tensor.

A normal contact metric manifold is called a Sasakian manifold. An almost contact metric manifold is Sasakian if and only if

$$(\nabla_X\phi)Y = g(X, Y)\xi - \eta(Y)X. \tag{2.8}$$

On a Sasakian manifold the following relation holds

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y, \tag{2.9}$$

for all $X, Y \in$ TM.

Blair, Koufogiorgos and Papantoniou [4] considered the $(k,\ \mu)$-nullity condition and gave several reasons for studying it. The $(k,\ \mu)$-nullity distribution $\mathrm{N}(k,\ \mu)$ ([4]) of a contact metric manifold M is defined by

$$N(k, \mu) : p \mapsto N_p(k, \mu) \quad = \quad [U \in T_pM \mid R(X, Y)U$$

$$= (kI + \mu h)(g(Y,U)X - g(X,U)Y)],$$

for all $X, Y \in$ TM, where $(k,\mu) \in \mathbb{R}^2$.

A contact metric manifold M with $\xi \in N(k,\mu)$ is called a $(k,\ \mu)$- contact metric manifold. Then I have

$$\begin{aligned} R(X,Y)\xi &= k[\eta(Y)X - \eta(X)Y] + \\ &\quad \mu[\eta(Y)hX - \eta(X)hY], \end{aligned} \tag{2.10}$$

for all $X, Y \in$ TM. For $(k,\ \mu)$-contact metric manifolds, it follows that $h^2 = (k-1)\phi^2$. This class contains Sasakian manifolds for $k = 1$ and $h = 0$. In fact, for a $(k,\ \mu)$-contact metric manifold, the condition of being Sasakian manifold, K-contact manifold, $k = 1$ and $h = 0$ are equivalent. If $\mu = 0$, then the $(k,\ \mu)$-nullity distribution N$(k,\ \mu)$ is reduced to k-nullity distribution N(k) [9]. If $\xi \in N(k)$, then we call a contact metric manifold M an N(k)- contact metric manifold.

The sectional curvature $K(X, \phi X)$ of a plane section spanned by a unit vector X orthogonal to ξ is called a ϕ-sectional curvature. If the $(k,\ \mu)$-contact metric manifold M has constant ϕ-sectional curvature c, then it is called a $(k,\ \mu)$-contact space form and is denoted by $M(c)$. The curvature tensor of M(c) is given by [10]

$$\begin{aligned} R(X,Y)Z &= \frac{c+3}{4}[g(Y,Z)X - g(X,Z)Y] + \\ &\quad \frac{c-1}{4}[2g(X,\phi Y)\phi Z + g(X,\phi Z)\phi Y - \\ &\quad g(Y,\phi Z)\phi X] + \frac{c+3-4k}{4}[\eta(X)\eta(Z)Y - \\ &\quad \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi] + \\ &\quad \frac{1}{2}[g(hY,Z)hX - g(hX,Z)hY + g(\phi hX,Z) \\ &\quad \phi hY - g(\phi hY,Z)\phi hX + \\ &\quad g(\phi Y,\phi Z)hX - g(\phi X,\phi Z)hY + g(hX,Z) \\ &\quad \phi^2 Y - g(hY,Z)\phi^2 X] + \mu[\eta(Y)\eta(Z)hX - \eta(X) \\ &\quad \eta(Z)hY + g(hY,Z)\eta(X)\xi - g(hX,Z)\eta(Y)\xi], \end{aligned} \tag{2.11}$$

for all $X, Y, Z \in T(M)$, where $c + 2k = -1 = k - \mu$ if $k < 1$.

From (2.11), we obtain the for $(k,\ \mu)$-contact space forms:

$$R(X,Y)\phi Z = \frac{c+3}{4}[g(Y,\phi Z)X - g(X,\phi Z)Y] + \frac{c-1}{4} \tag{2.12}$$

$$[-2g(X,\phi Y)Z + 2g(X,\phi Y)\eta(Z)\xi - g(X,Z)\phi Y$$
$$+\eta(Z)\eta(X)\phi Y + g(Y,Z)\phi X - \eta(Y)\eta(Z)\phi X] +$$
$$\frac{c+3-4k}{4}[g(X,\phi Z)\eta(Y)\xi - g(Y,\phi Z)\eta(X)\xi]$$
$$+\frac{1}{2}[g(hY,\phi Z)hX - g(hX,\phi Z)hY + g(hX,Z)$$
$$\phi hY - g(hY,Z)\phi hX - g(\phi Y,Z)hX +$$
$$g(\phi X,Z)hY - g(hX,\phi Z)Y + g(hX,\phi Z)$$
$$\eta(Y)\xi - g(hY,\phi Z)X - g(hY,\phi Z)\eta(X)\xi] +$$
$$\mu[g(hY,\phi Z)\eta(X)\xi - g(hX,\phi Z)\eta(Y)\xi],$$

$$\begin{aligned}\phi R(X,Y)Z \quad = \quad & \frac{c+3}{4}[g(Y,Z)\phi X - g(X,Z)\phi Y] + \\ & \frac{c-1}{4}[-2g(X,\phi Y)Z + 2g(X,\phi Y)\eta(Z)\xi \\ & -g(X,\phi Z)Y + g(X,\phi Z)\eta(Y)\xi + g(Y,\phi Z)X \\ & -g(Y,\phi Z)\eta(X)\xi] + \frac{c+3-4k}{4} \\ & [\eta(X)\eta(Z)\phi Y - \eta(Y)\eta(Z)\phi X] + \\ & \frac{1}{2}[g(hY,Z)\phi hX - g(hX,Z)\phi hY - g(\phi hX,Z) \\ & hY + g(\phi hY,Z)hX + \\ & g(\phi Y,\phi Z)\phi hX - g(\phi X,\phi Z)\phi hY - g(hX,Z) \\ & \phi Y + g(hY,Z)\phi X] + \mu[\eta(Y)\eta(Z)\phi hX - \eta(X) \\ & \eta(Z)\phi hY],\end{aligned} \tag{2.13}$$

$$R(X,Y)\xi = k[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)hX - \eta(X)hY], \tag{2.14}$$

$$R(X,\xi)\xi = k[X - \eta(X)\xi] + \mu hX, \tag{2.15}$$

$$R(\xi,Y)Z = k[g(Y,Z)\xi - \eta(z)Y] + \mu[g(hY,Z)\xi - \eta(Z)hY], \tag{2.16}$$

$$\begin{aligned}S(Y,Z) \quad = \quad & \frac{1}{2}[c(n+1) + 3(n-1) + 2k]g(Y,Z) + \\ & \frac{1}{2}[-c(n+1) - 3(n-1) + 2k(2n-1)] \\ & \eta(Y)\eta(Z) + [2n-2+\mu]g(hY,Z),\end{aligned} \tag{2.17}$$

$$S(Y,hZ) \quad = \quad \frac{1}{2}[c(n+1) + 3(n-1) + 2k]g(Y,hZ) + \tag{2.18}$$

$$(k-1)[2n-2+\mu]g(Y,Z)-$$
$$(k-1)[2n-2+\mu]\eta(Y)\eta(Z),$$

$$S(Y,\xi)=2nk\eta(Y), \tag{2.19}$$

$$S(\xi,\xi)=2nk, \tag{2.20}$$

$$\begin{aligned} QY &= \frac{1}{2}[c(n+1)+3(n-1)+2k]Y+ \\ &\quad \frac{1}{2}[-c(n+1)-3(n-1)+2k(2n-1)] \\ &\quad \eta(Y)\xi+[2n-2+\mu]hY, \end{aligned} \tag{2.21}$$

$$Q\xi=2nk\xi. \tag{2.22}$$

From (1.1) and using the above relations, we obtain

$$P(X,Y)\xi=\mu[\eta(Y)hX-\eta(X)hY], \tag{2.23}$$

$$P(\xi,Y)\xi=-\mu hY, \tag{2.24}$$

$$P(\xi,Y)Z=k[g(Y,Z)\xi+\mu[g(hY,Z)\xi-\eta(Z)hY]-\frac{1}{2n}S(Y,V)\xi, \tag{2.25}$$

Let us recall the following result:

Lemma 2.1. [5] A contact metric manifold M^{2n+1} satisfying $R(X,Y)\xi=0$ is locally isometric to the Riemannian product $E^{n+1}\times S^n(4)$ for $n>1$.

From (2.17), we obtain the followings:

Proposition 1. A (2n+1)-dimensional (k, μ)-contact space form is an η-Einstein manifold provided the space form is a Sasakian space form or, $\mu=2-2n$.

and

Proposition 2. A 3-dimensional (k, μ)-contact space form is an η-Einstein manifold provided the space form is a Sasakian space form or, a $N(k)$-contact space form.

3. ξ-Projectively Flat (k, μ)-Contact Space Forms

Assume that $M(c)$ is a ξ-projectively flat (k, μ)-contact space form. Then

$$P(X,Y)\xi = 0. \tag{3.1}$$

Putting $Z = \xi$ in (1.1), we obtain

$$P(X,Y)\xi = R(X,Y)\xi - \frac{1}{2n}[S(Y,\xi)X - S(X,\xi)Y]. \tag{3.2}$$

Using (2.14) and (2.19), we get

$$\mu[\eta(Y)hX - \eta(X)hY] = 0. \tag{3.3}$$

From (3.3), we may conclude that either $\mu = 0$ or,

$$\eta(Y)hX = \eta(X)hY. \tag{3.4}$$

Putting $Y = \xi$ in (3.4), we have

$$hX = 0. \tag{3.5}$$

If $\mu = 0$, then $M(c)$ is an $N(k)$-contact space form.

If $h = 0$, then $M(c)$ is a Sasakian space form.

Hence we can state the following:

Theorem 3.1. Let $M(c)$ be a ξ-projectively flat (k, μ)-contact space form. Then $M(c)$ is either an $N(k)$-contact space form or, a Sasakian space form.

4. ϕ-Projectively Semisymmetric (k, μ)-Contact Space Forms

Definition 4.1. A (k, μ)-contact space form is said to be ϕ-projectively semisymmetric if $P(X,Y) \cdot \phi = 0$ for all $X, Y \in$ TM.

Suppose M(c) be a ϕ-projectively semisymmetric (k, μ)-contact space form. Then

$$P(X,Y)\phi Z - \phi(P(X,Y)Z) = 0, \tag{4.1}$$

From (1.1), it follows that

$$P(X,Y)\phi Z = R(X,Y)\phi Z - \frac{1}{2n}[S(Y,\phi Z)X - S(X,\phi Z)Y]. \tag{4.2}$$

Using (2.17) in (4.2) yields

$$\begin{aligned} P(X,Y)\phi Z &= R(X,Y)\phi Z - \frac{1}{2n}\{\frac{1}{2} \\ &[c(n+1)+3(n-1)+2k][g(Y,\phi Z)X - \\ &g(X,\phi Z)Y] + [2n-2+\mu] \\ &[g(hY,\phi Z)X - g(hX,\phi Z)Y]\}. \end{aligned} \tag{4.3}$$

Again,

$$\begin{aligned} \phi P(X,Y)Z &= \phi R(X,Y)Z - \frac{1}{2n}\{\frac{1}{2}[c(n+1) \\ &+3(n-1)+2k][g(Y,Z)\phi X - g(X,Z)\phi Y] \\ &+\frac{1}{2}[-c(n+1)-3(n-1)+2k(2n-1)] \\ &[\eta(Y)\eta(Z)\phi X - \eta(X)\eta(Z)\phi Y] + \\ &[2n-2+\mu][g(hY,Z)\phi X - g(hX,Z)\phi Y]\}. \end{aligned} \tag{4.4}$$

Using (4.3) and (4.4) in (4.1), we have

$$\begin{aligned} (P(X,Y)\cdot\phi)Z &= R(X,Y)\phi Z - \phi R(X,Y)Z \\ &-\frac{1}{2n}\{\frac{1}{2}[c(n+1)+3(n-1)+2k] \\ &[g(Y,\phi Z)X - g(X,\phi Z)Y - \\ &g(Y,Z)\phi X + g(X,Z)\phi Y] + \\ &\frac{1}{2}[-c(n+1)-3(n-1)+2k(2n-1)] \\ &[\eta(Y)\eta(Z)\phi X - \eta(X)\eta(Z)\phi Y] + \\ &[2n-2+\mu][g(hY,\phi Z)X - g(hX,\phi Z)Y \\ &-g(hY,Z)\phi X + g(hX,Z)\phi Y]\} = 0. \end{aligned} \tag{4.5}$$

Putting the value of $R(X,Y)\phi Z$ and $\phi R(X,Y)Z$ in (4.5) and taking inner product with W of (4.5) and contacting Y and W, we obtain

$$\begin{aligned} &\{\frac{c+3}{4}(1-2n) + \frac{c-1}{4}(2n-2) + \\ &\frac{c+3-4k}{4} + 2n - 14n[c(n+1)+3(n-1)+2k]\} \\ &g(\phi Z, X) + \frac{2-2n-\mu}{2n} g(\phi Z, hX) = 0. \end{aligned} \tag{4.6}$$

Putting $X = hX$ in the above equation yields

$$\{\frac{c+3}{4}(1-2n) + \frac{c-1}{4}(2n-2) + \frac{c+3-4k}{4} + 2n - 14n[c(n+1) + 3(n-1) + 2k]\} g(\phi Z, hX) + \frac{2-2n-\mu}{2n} g(\phi Z, h^2 X) = 0. \tag{4.7}$$

Taking trace in both sides of (4.7) and using $traceh = 0$, we get

$$\mu = 2 - 2n. \tag{4.8}$$

From (4.8), we may conclude the following:

Proposition 3. Let M(c) be a ϕ-projectively semisymmetric (k, μ)-contact space form, then $\mu = 2 - 2n$.

From the above proposition we can state the following:

Theorem 4.1. A three dimensional ϕ-projectively semisymmetric (k, μ)-contact space form reduces to an $N(k)$-contact space form.

5. (k, μ)-Contact Space Forms Satisfying $Q \cdot P = 0$

In this section we characterize a (k, μ)-contact space forms satisfying $Q \cdot P = 0$, where Q is the Ricci operator defined by $S(X, Y) = g(QX, Y)$. Suppose M(c) be a (k, μ)-contact space form satisfying $Q \cdot P = 0$. Then

$$Q(P(X,Y)Z) - P(QX,Y)Z - P(X,QY)Z - P(X,Y)QZ = 0 \tag{5.1}$$

Putting $Z = \xi$ in (5.1) and using (2.23), we have

$$\mu\eta(Y)Q(hX) - \mu\eta(X)Q(hY) - \mu\eta(Y)hQX + \mu\eta(QX)hY - \mu\eta(QY)hX + \mu\eta(X)hQY - 2nk\mu\eta(Y)hX + 2nk\mu\eta(Y)hX = 0. \tag{5.2}$$

Using (2.21), we obtain

$$\begin{aligned} Q(hY) - hQY &= \frac{1}{2}[c(n+1) + 3(n-1) + 2k]hY \\ &+ \frac{1}{2}[-c(n+1) - 3(n-1) + 2k(2n-1)]\eta(hY)\xi \\ &+ [2n - 2 + \mu]h^2 Y - \frac{1}{2}[c(n+1) + 3(n-1) + 2k]hY \end{aligned} \tag{5.3}$$

$$-\frac{1}{2}[-c(n+1)-3(n-1)+2k(2n-1)]$$
$$\eta(Y)h\xi-[2n-2+\mu]h^2Y$$
$$=0.$$

Using (5.3) in (5.2), we have

$$2nk\mu[\eta(X)hY-\eta(Y)hX]=0. \tag{5.4}$$

From (5.4), we may conclude that either $k=0$ or $\mu=0$ or,

$$[\eta(X)hY-\eta(Y)hX]=0. \tag{5.5}$$

Putting $Y=\xi$ in the above equation yields

$$hX=0$$

If $k=0$, then from (2.11) we have $\mu=1$ and constant ϕ-sectional curvature $c=-1$.

If $\mu=0$, then M(c) is an $N(k)$-contact space form.

If $h=0$, then M(c) is a Sasakian space form.

Thus we can state the following:

Theorem 5.1. A $(k,\ \mu)$-contact space form satisfying $Q\cdot P=0$ is either (0,1)-contact space form of constant ϕ-sectional curvature -1 or an $N(k)$-contact space form or, a Sasakian space form.

6. Example

Let us consider the 3-dimensional manifold $M=\{(x,y,z)\in\mathbb{R}^3:(x,y,z)\neq(0,0,0)\}$, where (x,y,z) are the standard coordinates in $\mathbb{R}^3$. The vector fields $e_1=e^{z-x}\frac{\partial}{\partial x}$, $e_2=e^{z-y}\frac{\partial}{\partial y}$, $e_3=\frac{\partial}{\partial z}$,
are linear independent at each point of M. Let g be the metric defined by

$$\begin{aligned} g(e_i,e_j) &= 1, \qquad for\ i=j,\\ &= 0, \qquad for\ i\neq j. \end{aligned}$$

Here i and j runs from 1 to 3.
Let η be the 1-form defined by $\eta(Z)=g(Z,e_1)$, for any vector field Z tangent

to M. Let ϕ be the $(1,1)$ tensor field defined by
$\phi e_2 = -e_3$, $\phi e_3 = e_2$, $\phi e_1 = 0$. From the properties of ϕ and η we obtain $g(e_i, \phi e_j) = d\eta(e_i, e_j)$, i and j runs from 1 to 3.
Using the linearity property of ϕ and g, we have

$$\eta(e_1) = 1,$$

$$\phi^2 Z = -Z + \eta(Z)e_1,$$
$$g(\phi Z, \phi W) = g(Z,W) - \eta(Z)\eta(W),$$

for any vector field Z, W.

Then for $e_1 = \xi$, the structure (ϕ, ξ, η, g) defines a contact metric structure on M.

Let ∇ be the Levi-Civita connection on M with respect to the metric g. Then

$$\begin{aligned}[e_1, e_2] &= e^{z-x}\frac{\partial}{\partial x}(e^{z-y}\frac{\partial}{\partial y} - e^{z-y}\frac{\partial}{\partial y}(e^{z-x}\frac{\partial}{\partial x}) \\ &= e^{z-x}e^{z-y}\frac{\partial^2}{\partial x \partial y} - e^{z-x}e^{z-y}\frac{\partial^2}{\partial x \partial y} \\ &= 0.\end{aligned}$$

Similarly,

$$[e_1, e_3] = -e_1, \quad [e_2, e_3] = -e_2, \quad [e_2, e_1] = 0,$$
$$[e_3, e_1] = e_1, \quad [e_3, e_2] = e_2,$$

From Koszul's formula, the Riemannian connection ∇ of the metric g is given by

$$\begin{aligned}2g(\nabla_X Y, Z) &= Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) \\ &\quad -g(X,[Y,Z]) - g(Y,[X,Z]) + g(Z,[X,Y]).\end{aligned} \tag{6.1}$$

Using (6.1) we deduce the following:

$$\nabla_{e_1} e_1 = e_3, \quad \nabla_{e_1} e_2 = 0, \quad \nabla_{e_1} e_3 = -e_1,$$

$$\nabla_{e_2} e_1 = 0, \quad \nabla_{e_2} e_2 = e_3, \quad \nabla_{e_2} e_3 = -e_2,$$
$$\nabla_{e_3} e_1 = 0, \quad \nabla_{e_3} e_2 = 0, \quad \nabla_{e_3} e_3 = 0.$$

We also know that

$$\nabla_{e_2} e_1 = -\phi e_2 - \phi h e_2.$$

Comparing two relations for $\nabla_{e_2}e_1$ and using $\phi e_1 = 0,\ \phi e_3 = e_2$ and $\phi e_2 = -e_3$ we have

$$he_2 = -e_2.$$

Similarly, we obtain

$$he_3 = -e_3 \ and \ he_1 = 0.$$

It is known that Riemannian curvature tensor

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z. \tag{6.2}$$

Using (6.2) we obtain

$$R(e_2, e_1)e_1 = -e_2,$$

$$R(e_3, e_1)e_1 = -e_3,$$

$$R(e_2, e_3)e_1 = 0.$$

Putting $k = -1$ and $\mu = 0$, we conclude that e_1 belong to the $(k,\ \mu)$-nullity distribution.

The nonzero components of the curvature tensor can be obtained as follows:

$$R(e_1, e_2)e_2 = -e_1, \quad R(e_1, e_3)e_3 = -e_1,$$

$$R(e_2, e_3)e_3 = -e_2, \quad R(e_2, e_3)e_2 = e_3,$$

$$R(e_1, e_3)e_3 = -e_1, \quad R(e_3, e_1)e_1 = -e_3, \quad R(e_2, e_1)e_1 = -e_2,$$

From the above expressions of the curvature tensor we get

$$\begin{aligned} S(e_1, e_1) \ &= \ g(R(e_2, e_1)e_1, e_2) + g(R(e_3, e_1)e_1, e_3) \\ &= -1 - 1 \\ &= -2. \end{aligned}$$

Similarly, we obtain $S(e_2, e_2) = -2,\ \ S(e_3, e_3) = -2$ and scalar curvature $r = S(e_1, e_1) + S(e_2, e_2) + S(e_3, e_3 = -6$.

From the above calculation we conclude that $S(X,Y) = -2g(X,Y)$.

For 3-dimensional $(k,\ \mu)$-contact metric manifolds, Riemannian curvature tensor can be written as follows:

$$\begin{aligned} R(X,Y)Z \ &= \ [S(Y,Z)X - S(X,Z)Y + \\ & g(X,Z)QX - g(X,Z)QY] - \frac{r}{2} \\ & [g(Y,Z)X - g(X,Z)Y]. \end{aligned} \tag{6.3}$$

Taking inner product with W of (6.3) we have

$$g(R(X,Y)Z,W) = [S(Y,Z)g(X,W) - S(X,Z)g(Y,W) + g(X,Z)S(X,W) - g(X,Z)S(Y,W)] - \frac{r}{2} [g(Y,Z)g(X,W) - g(X,Z)g(Y,W)]. \quad (6.4)$$

Using the values of the Ricci tensors and the scalar curvature we obtain

$$g(R(X,Y)Z,W) = -[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)]. \quad (6.5)$$

Putting $Y = Z = \phi X$ and $W = X$, we get

$$g(R(X,\phi X)\phi X, X) = -1. \quad (6.6)$$

From (6.6) we can conclude that $(k\ \mu)$-contact metric manifold M has constant $\phi - sectional$ curvature -1.

Thus M is a $(k,\ \mu)$-contact space form.

Now we like to verify Theorem 4.1, that is, a three dimensional ϕ-projectively semisymmetric $(k,\ \mu)$-contact space form reduces to an $N(k)$-contact space form. From the definition of ϕ-projectively semisymmetric manifold we obtain

$$(P(X,Y)\cdot\phi)Z = P(X,Y)\phi Z - \phi P(X,Y)Z = R(X,Y)\phi Z - \phi R(X,Y)Z - \frac{1}{2}[S(Y,\phi Z)X - S(X,\phi Z)Y - S(Y,Z)\phi X + S(X,Z)\phi Y]. \quad (6.7)$$

Using (6.5) and the value of Ricci tensor, (6.7) yields

$$(P(X,Y)\cdot\phi)Z = -[g(Y,\phi Z)X - g(X,\phi Z)Y] + [g(Y,Z)\phi X - g(X,Z)\phi Y] - \frac{1}{2}[-2g(Y,\phi Z)X + 2S(X,\phi Z)Y + 2g(Y,Z)\phi X - 2g(X,Z)\phi Y] = 0 \quad (6.8)$$

Thus the Theorem 4.1 is verified.

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