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SYMMETRIC IDENTITIES FOR THE HIGHER-ORDER TWISTED (h, q)-EULER NUMBERS AND POLYNOMIALS

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Abstract: In this paper we investigate some interesting symmetric identities for twisted (h, q)-Euler polynomials of higher order in complex field.

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1. Introduction

Euler numbers and polynomials possess many interesting properties and arising in many areas of mathematics, mathematical physics and statistical physics. Many mathematicians have studied in the area of the q- extension of Euler numbers and polynomials(see [1, 2, 3, 5, 6, 7, 8, 9, 11, 13]). Recently, Y. He studied several identities of symmetry for Carlitz's q-Bernoulli numbers and polynomials in complex field(see [3]). D. Kim *et al.*[4] derived some identities of symmetry for (h, q)-extension of higher-order Euler numbers and polynomials. D. V. Dolgy *et al.*[2] derived some identities of symmetry for higher-order generalized q-Euler polynomials. In this paper, we establish some interesting symmetric identities for twisted (h, q)-Euler polynomials of higher order in complex field.

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The purpose of this paper is to present a systemic study of the twisted (h,q)-Euler numbers and polynomials of higher-order by using the multiple (h,q)-Euler zeta function. Throughout this paper, the notations $\mathbb{N},\mathbb{Z},\mathbb{R}$, and $\mathbb{C}$ denote the sets of positive integers, integers, real numbers, and complex numbers, respectively, and $\mathbb{Z}_+ := \mathbb{N}\cup\{0\}$. We assume that $q\in\mathbb{C}$ with $|q|<1$. Throughout this paper we use the notation:

$$[x]_q = \frac{1-q^x}{1-q} \text{ (cf. [1, 2, 3, 5])} .$$

Note that $\lim_{q\to 1}[x] = x$. Let ε be the p^N-th root of unity(see [10, 12, 13]).

In [5], T. Kim introduced the multiple q-Euler zeta function which interpolates higher-order q-Euler polynomials at negative integers as follows:

$$\zeta_{q,r}(s,x) = [2]_q^r \sum_{m_1,\cdots,m_r=0}^{\infty} \frac{(-1)^{\sum_{j=1}^r m_j} q^{\sum_{j=1}^r m_j}}{[m_1+\cdots+m_r+x]_q^s}, \tag{1}$$

where $s\in\mathbb{C}$ and $x\in\mathbb{R}$, with $x\neq 0,-1,-2,\ldots$.

Recently, D. V. Dolgy *et al.*[2] considered some symmetric identities for higher-order generalized q-Euler polynomials. The Euler polynomials of order $r\in\mathbb{N}$ attached to χ are also defined by the generating function:

$$\left(2\sum_{l=0}^{d-1}\frac{\chi(l)(-1)^l e^{(x+l)t}}{e^{dt}+1}\right)^r = \sum_{m=0}^{\infty} E_{m,\chi}^{(r)}(x)\frac{t^m}{m!}. \tag{2}$$

When $x=0, E_{n,\chi}^{(r)} = E_{n,\chi}^{(r)}(0)$ are called the Euler numbers $E_{n,\chi}^{(r)}$ attached to χ(see [2, 4]).

For $h\in\mathbb{Z},\alpha,k\in\mathbb{N}$, and $n\in\mathbb{Z}_+$, we introduced the higher order twisted q-Euler polynomials with weight α as follows(see [7]):

$$\widetilde{E}_{n,q,\varepsilon}^{(\alpha)}(h,k|x) = \frac{[2]_q^k}{(1-q^\alpha)^n}\sum_{l=0}^{n}\binom{n}{l}(-1)^l\frac{q^{\alpha l x}}{(1+\varepsilon q^{\alpha l+h})\cdots(1+\varepsilon q^{\alpha l+h-k+1})}.$$

In the special case, $x=0$, $\widetilde{E}_{n,q,w}^{(\alpha)}(h,k|0) = \widetilde{E}_{n,q,w}^{(\alpha)}(h,k)$ are called the higher-order twisted q-Euler numbers with weight α.

We consider the higher order q-Euler polynomials of order r attached to χ twisted by ramified roots of unity as follows(see [10]):

$$\sum_{n=0}^{\infty} E_{n,\chi,\zeta,q}^{(r)}(x)\frac{t^n}{n!} = 2^r \sum_{m_1,\ldots,m_r=0}^{\infty} (-\zeta)^{\sum_{j=0}^r m_j}\left(\prod_{i=1}^r \chi(m_i)\right) e^{[x+\sum_{j=1}^r m_j]_q t}.$$

In the special case $x=0$, the sequence $E^{(r)}_{n,\chi,\zeta,q}(0)=E^{(r)}_{n,\chi,\zeta,q}$ are called the n-th q-Euler numbers of order r attached to χ twisted by ramified roots of unity.

As is well known, the higher-order twisted (h,q)-Euler polynomials $E^{(h,k)}_{n,q,\varepsilon}(x)$ are defined by the following generating function to be

$$\begin{aligned}
&\widetilde{F}^{(h,k)}_{q,\varepsilon}(t,x)\\
&=[2]_q^k\sum_{m_1,\cdots,m_k=0}^{\infty}(-1)^{m_1+\cdots+m_k}q^{\sum_{j=1}^{k}(h-j+1)m_j}\varepsilon^{m_1+\cdots+m_k}\\
&\qquad\times e^{[m_1+\cdots+m_k+x]_qt}\\
&=\sum_{n=0}^{\infty}E^{(h,k)}_{n,q,\varepsilon}(x)\frac{t^n}{n!},
\end{aligned}\tag{3}$$

where $h\in\mathbb{Z}$ and $k\in\mathbb{N}$. When $x=0, E^{(h,k)}_{n,q,\varepsilon}=E^{(h,k)}_{n,q,\varepsilon}(0)$ are called the higher-order twisted (h,q)-Euler numbers $E^{(h,k)}_{n,q,\varepsilon}$ attached to χ. Observe that if $q\to1,\varepsilon\to1$, then $E^{(h,k)}_{n,q,\varepsilon}\to E^{(k)}_{n,\chi}$ and $E^{(h,k)}_{n,q,\varepsilon}(x)\to E^{(k)}_{n,\chi}(x)$.

By using (3) and Cauchy product, we have

$$\begin{aligned}
E^{(h,k)}_{n,q,\varepsilon}(x)&=\sum_{l=0}^{n}\binom{n}{l}q^{lx}E^{(h,k)}_{l,q,\varepsilon}[x]_q^{n-l}\\
&=(q^xE^{(h,k)}_{q,\varepsilon}+[x]_q)^n,
\end{aligned}\tag{4}$$

with the usual convention about replacing $(E^{(h,k)}_{q,\varepsilon})^n$ by $E^{(h,k)}_{n,q,\varepsilon}$.

By using complex integral and (3), we can also obtain the Dirichlet-type multiple twisted (h,q)-l-function as follows:

$$\begin{aligned}
l^{(h,k)}_{q,\varepsilon}(s,x)&=\frac{1}{\Gamma(s)}\int_0^{\infty}\widetilde{F}^{(h,k)}_{q,\varepsilon}(-t,x)t^{s-1}dt\\
&=[2]_q^k\sum_{m_1,\cdots,m_k=0}^{\infty}\frac{(-1)^{\sum_{j=1}^{k}m_j}q^{\sum_{j=1}^{k}(h-j+1)m_j}\varepsilon^{\sum_{j=1}^{k}m_j}}{[m_1+\cdots+m_k+x]_q^s},
\end{aligned}\tag{5}$$

where $s\in\mathbb{C}$ and $x\in\mathbb{R}$, with $x\neq0,-1,-2,\ldots$.

By using Cauchy residue theorem, the value of Dirichlet-type multiple twisted (h,q)-l-function at negative integers is given explicitly by the following theorem:

Theorem 1. Let $k\in\mathbb{N}$ and $n\in\mathbb{Z}_+$. We obtain

$$l^{(h,k)}_{q,\varepsilon}(-n,x)=E^{(h,k)}_{n,q,\varepsilon}(x).$$

The purpose of this paper is to obtain some interesting identities of the power sums and the higher-order twisted (h, q)-Euler polynomials $E^{(h,k)}_{n,q,\varepsilon}(x)$ attached to χ using the symmetric properties for Dirichlet-type multiple twisted (h, q)-l-function. In this paper, if we take $\chi^0 = 1, \varepsilon = 1$, then [4] is the special case of this paper. If we take $\varepsilon = 1$ in all equations of this article, then [2] are the special case of our results.

2. Symmetry Identities for Dirichlet-Type Multiple Twisted (h, q)-l-Function

In this section, by using the similar method of [2, 3, 4], expect for obvious modifications, we investigate some symmetric identities for higher-order twisted (h, q)-Euler polynomials $E^{(h,k)}_{n,q,\varepsilon}(x)$ attached to χ using the symmetric properties for Dirichlet-type multiple twisted (h, q)-l-function. We assume that χ is a Dirichlet character with conductor $d \in \mathbb{N}$ with $d \equiv 1 \pmod 2$ and ε be the p^N-th root of unity. Let $w_1, w_2 \in \mathbb{N}$ with $w_1 \equiv 1 \pmod 2$, $w_2 \equiv 1 \pmod 2$. For $h \in \mathbb{Z}, k \in \mathbb{N}$ and $n \in \mathbb{Z}_+$, we obtain certain symmetry identities for Dirichlet-type multiple twisted (h, q)-l-function. Observe that $[xy]_q = [x]_{q^y}[y]_q$ for any $x, y \in \mathbb{C}$. In (5), we derive next result by substitute $w_2x + \dfrac{w_2}{w_1}(j_1 + \cdots + j_k)$ for x in and replace q and ε by q^{w_1} and ε^{w_1}, respectively,

$$
\begin{aligned}
&\frac{1}{[2]^k_{q^{w_1}}} l^{(h,k)}_{q^{w_1},\varepsilon^{w_1}}\left(s, w_2x + \frac{w_2}{w_1}(j_1 + \cdots + j_k)\right) \\
&= \sum_{m_1,\cdots,m_k=0}^{\infty} \frac{(-1)^{\sum_{j=1}^k m_j} q^{w_1\sum_{j=1}^k (h-j+1)m_j} \varepsilon^{w_1\sum_{j=1}^k m_j}}{[m_1 + \cdots + m_k + w_2x + \frac{w_2}{w_1}(j_1 + \cdots + j_k)]^s_{q^{w_1}}} \\
&= \sum_{m_1,\cdots,m_k=0}^{\infty} \frac{(-1)^{\sum_{j=1}^k m_j} q^{w_1\sum_{j=1}^k (h-j+1)m_j} \varepsilon^{w_1\sum_{j=1}^k m_j}}{\left[\dfrac{w_1(m_1 + \cdots + m_k) + w_1w_2x + w_2(j_1 + \cdots + j_k)}{w_1}\right]^s_{q^{w_1}}} \\
&= \sum_{m_1,\cdots,m_k=0}^{\infty} \frac{(-1)^{\sum_{j=1}^k m_j} q^{w_1\sum_{j=1}^k (h-j+1)m_j} \varepsilon^{w_1\sum_{j=1}^k m_j}}{\dfrac{[w_1(m_1 + \cdots + m_k) + w_1w_2x + w_2(j_1 + \cdots + j_k)]^s_q}{[w_1]^s_q}} \\
&= [w_1]^s_q \sum_{m_1,\cdots,m_k=0}^{\infty} \frac{(-1)^{\sum_{j=1}^k m_j} q^{w_1\sum_{j=1}^k (h-j+1)m_j} \varepsilon^{w_1\sum_{j=1}^k m_j}}{[w_1(m_1 + \cdots + m_k) + w_1w_2x + w_2(j_1 + \cdots + j_k)]^s_q}
\end{aligned}
$$

$$= [w_1]_q^s \sum_{m_1,\cdots,m_k=0}^{\infty} \sum_{i_1,\cdots,i_k=0}^{w_2-1} \frac{(-1)^{\sum_{j=1}^k m_j} q^{w_1\sum_{j=1}^k (h-j+1)m_j} \varepsilon^{w_1\sum_{j=1}^k m_j}}{[w_1(m_1+\cdots+m_k)+w_1w_2x+w_2(j_1+\cdots+j_k)]_q^s}$$

$$= [w_1]_q^s \sum_{m_1,\cdots,m_k=0}^{\infty} \sum_{i_1,\cdots,i_k=0}^{w_2-1} (-1)^{\sum_{j=1}^k (dw_2m_j+i_j)}$$

$$\times q^{w_1\sum_{j=1}^k (h-j+1)(dw_2m_j+i_j)} \varepsilon^{w_1\sum_{j=1}^k (dw_2m_j+i_j)}$$

$$\times \left([w_1(dw_2m_1+i_1)+\cdots+w_1(dw_2m_k+i_k)+w_1w_2x+w_2(j_1+\cdots+j_k)]_q^s\right)^{-1}$$

$$= [w_1]_q^s \sum_{m_1,\cdots,m_k=0}^{\infty} \sum_{i_1,\cdots,i_k=0}^{w_2-1} (-1)^{\sum_{j=1}^k m_j}(-1)^{\sum_{j=1}^k i_j}$$

$$\times q^{dw_1w_2\sum_{j=1}^k (h-j+1)m_j} q^{w_1\sum_{j=1}^k (h-j+1)i_j} \varepsilon^{dw_1w_2\sum_{j=1}^k m_j} \varepsilon^{w_1\sum_{j=1}^k i_j}$$

$$\times \left([w_1w_2(x+dm_1+\cdots+dm_k)+w_1(i_1+\cdots+i_k)+w_2(j_1+\cdots+j_k)]_q^s\right)^{-1}. \tag{6}$$

Thus, from (6), we can derive the following equation.

$$\frac{[w_2]_q^s}{[2]_{q^{w_1}}^k} \sum_{j_1,\cdots,j_k=0}^{w_1-1} (-1)^{\sum_{l=1}^k j_l} q^{w_2\sum_{l=1}^k (h-l+1)j_l} \varepsilon^{w_2\sum_{l=1}^k j_l}$$

$$\times\, l_{q^{w_1},\varepsilon^{w_1}}^{(h,k)}(s, w_2x+\frac{w_2}{w_1}(j_1+\cdots+j_k))$$

$$= [w_1]_q^s[w_2]_q^s \sum_{m_1,\cdots,m_k=0}^{\infty} \sum_{i_1,\cdots,i_k=0}^{w_2-1} \sum_{j_1,\cdots,j_k=0}^{w_1-1} (-1)^{\sum_{l=1}^k (j_l+i_l+m_l)}$$

$$\times q^{dw_1w_2\sum_{l=1}^k (h-l+1)m_l} q^{w_1\sum_{l=1}^k (h-l+1)i_l} q^{w_2\sum_{l=1}^k (h-l+1)j_l}$$

$$\times \varepsilon^{dw_1w_2\sum_{l=1}^k m_l} \varepsilon^{w_1\sum_{l=1}^k i_l} \varepsilon^{w_2\sum_{l=1}^k j_l}$$

$$\times \left([w_1w_2(x+dm_1+\cdots+dm_k)+w_1(i_1+\cdots+i_k)+w_2(j_1+\cdots+j_k)]_q^s\right)^{-1} \tag{7}$$

By using the same method as (7), we have

$$\frac{[w_1]_q^s}{[2]_{q^{w_2}}^k} \sum_{j_1,\cdots,j_k=0}^{w_2-1} (-1)^{\sum_{l=1}^k j_l} q^{w_1\sum_{l=1}^k (h-l+1)j_l} \varepsilon^{w_1\sum_{l=1}^k j_l}$$

$$\times\, l_{q^{w_2},\varepsilon^{w_2}}^{(h,k)}(s, w_1x+\frac{w_1}{w_2}(j_1+\cdots+j_k))$$

$$= [w_1]_q^s[w_2]_q^s \sum_{m_1,\cdots,m_k=0}^{\infty} \sum_{j_1,\cdots,j_k=0}^{w_2-1} \sum_{i_1,\cdots,i_k=0}^{w_1-1} (-1)^{\sum_{l=1}^k (j_l+i_l+m_l)}$$

$$\times q^{dw_1w_2\sum_{l=1}^{k}(h-l+1)m_l}q^{w_2\sum_{l=1}^{k}(h-l+1)i_l}q^{w_1\sum_{l=1}^{k}(h-l+1)j_l}$$
$$\times \varepsilon^{dw_1w_2\sum_{l=1}^{k}m_l}\varepsilon^{w_2\sum_{l=1}^{k}i_l}\varepsilon^{w_1\sum_{l=1}^{k}j_l}$$
$$\times \left([w_1w_2(x+dm_1+\cdots+dm_k)+w_1(j_1+\cdots+j_k)+w_2(i_1+\cdots+i_k)]_q^s\right)^{-1}. \tag{8}$$

Therefore, by (7) and (8), we have the following theorem.

Theorem 2. Let $w_1, w_2 \in \mathbb{N}$ with $w_1 \equiv 1 \pmod 2$, $w_2 \equiv 1 \pmod 2$. For $h \in \mathbb{Z}$, we obtain

$$\begin{aligned}
&[w_2]_q^s[2]_{q^{w_2}}^k \sum_{j_1,\cdots,j_k=0}^{w_1-1}(-1)^{\sum_{l=1}^{k}j_l}q^{w_2\sum_{l=1}^{k}(h-l+1)j_l}\varepsilon^{w_2\sum_{l=1}^{k}j_l}\\
&\qquad\times l_{q^{w_1},\varepsilon^{w_1}}^{(h,k)}\left(s, w_2x+\frac{w_2}{w_1}(j_1+\cdots+j_k)\right)\\
&=[w_1]_q^s[2]_{q^{w_1}}^k \sum_{j_1,\cdots,j_k=0}^{w_2-1}(-1)^{\sum_{l=1}^{k}j_l}q^{w_1\sum_{l=1}^{k}(h-l+1)j_l}\varepsilon^{w_1\sum_{l=1}^{k}j_l}\\
&\qquad\times l_{q^{w_2},\varepsilon^{w_2}}^{(h,k)}\left(s, w_1x+\frac{w_1}{w_2}(j_1+\cdots+j_k)\right)
\end{aligned} \tag{9}$$

By (9) and Theorem 1, we obtain the following theorem.

Theorem 3. Let $w_1, w_2 \in \mathbb{N}$ with $w_1 \equiv 1 \pmod 2$, $w_2 \equiv 1 \pmod 2$. For $h \in \mathbb{Z}, k \in \mathbb{N}$ and $n \in \mathbb{Z}_+$, we obtain

$$\begin{aligned}
&[w_2]_q^s[2]_{q^{w_2}}^k \sum_{j_1,\cdots,j_k=0}^{w_1-1}(-1)^{\sum_{l=1}^{k}j_l}q^{w_2\sum_{l=1}^{k}(h-l+1)j_l}\varepsilon^{w_2\sum_{l=1}^{k}j_l}\\
&\qquad\times E_{n,q^{w_1},\varepsilon^{w_1}}^{(h,k)}\left(w_2x+\frac{w_2}{w_1}(j_1+\cdots+j_k)\right)\\
&=[w_1]_q^s[2]_{q^{w_1}}^k \sum_{j_1,\cdots,j_k=0}^{w_2-1}(-1)^{\sum_{l=1}^{k}j_l}q^{w_1\sum_{l=1}^{k}(h-l+1)j_l}\varepsilon^{w_1\sum_{l=1}^{k}j_l}\\
&\qquad\times E_{n,q^{w_2},\varepsilon^{w_2}}^{(h,k)}\left(w_1x+\frac{w_1}{w_2}(j_1+\cdots+j_k)\right).
\end{aligned} \tag{10}$$

From (4), we note that

$$E^{(h,k)}_{n,q,\varepsilon}(x+y) = (q^{x+y}E^{(h,k)}_{n,q,\varepsilon} + [x+y]_q)^n = \sum_{i=0}^{n}\binom{n}{i} q^{xi} E^{(h,k)}_{i,q,\varepsilon}(y)[x]_q^{n-i}. \tag{11}$$

with the usual convention about replacing $(E^{(h,k)}_{q,\varepsilon})^n$ by $E^{(h,k)}_{n,q,\varepsilon}$.

By (11), we have

$$\begin{aligned}
&\sum_{j_1,\cdots,j_k=0}^{w_1-1} (-1)^{\sum_{l=1}^{k} j_l} q^{w_2\sum_{l=1}^{k}(h-l+1)j_l} \varepsilon^{w_2\sum_{l=1}^{k} j_l} \\
&\qquad \times E^{(h,k)}_{n,q^{w_1},\varepsilon^{w_1}}\left(w_2 x + \frac{w_2}{w_1}(j_1+\cdots+j_k)\right) \\
&= \sum_{j_1,\cdots,j_k=0}^{w_1-1} (-1)^{\sum_{l=1}^{k} j_l} q^{w_2\sum_{l=1}^{k}(h-l+1)j_l} \varepsilon^{w_2\sum_{l=1}^{k} j_l} \\
&\times \sum_{i=0}^{n}\binom{n}{i} q^{w_2 i(j_1+\cdots+j_k)} E^{(h,k)}_{i,q^{w_1},\varepsilon^{w_1}}(w_2 x)\left[\frac{w_2}{w_1}(j_1+\cdots+j_k)\right]_{q^{w_1}}^{n-i} \\
&= \sum_{j_1,\cdots,j_k=0}^{w_1-1} (-1)^{\sum_{l=1}^{k} j_l} q^{w_2\sum_{l=1}^{k}(h-l+1)j_l} \varepsilon^{w_2\sum_{l=1}^{k} j_l} \\
&\times \sum_{i=0}^{n}\binom{n}{i} q^{w_2(n-i)\sum_{l=1}^{k} j_l} E^{(h,k)}_{n-i,q^{w_1},\varepsilon^{w_1}}(w_2 x)\left[\frac{w_2}{w_1}(j_1+\cdots+j_k)\right]_{q^{w_1}}^{i}
\end{aligned} \tag{12}$$

Hence we have the following theorem.

Theorem 4. Let $w_1, w_2 \in \mathbb{N}$ with $w_1 \equiv 1 \pmod 2$, $w_2 \equiv 1 \pmod 2$. For $h \in \mathbb{Z}, k \in \mathbb{N}$ and $n \in \mathbb{Z}_+$, we obtain

$$\begin{aligned}
&\sum_{j_1,\cdots,j_k=0}^{w_1-1} (-1)^{\sum_{l=1}^{k} j_l} q^{w_2\sum_{l=1}^{k}(h-l+1)j_l} \varepsilon^{w_2\sum_{l=1}^{k} j_l} \\
&\qquad \times E^{(h,k)}_{n,q^{w_1},\varepsilon^{w_1}}\left(w_2 x + \frac{w_2}{w_1}(j_1+\cdots+j_k)\right) \\
&= \sum_{i=0}^{n}\binom{n}{i} [w_2]_q^i [w_1]_q^{-i} E^{(h,k)}_{n-i,q^{w_1},\varepsilon^{w_1}}(w_2 x) \\
&\times \sum_{j_1,\cdots,j_k=0}^{w_1-1} (-1)^{\sum_{l=1}^{k} j_l} q^{w_2\sum_{l=1}^{k}(n+h-l-i+1)j_l} \varepsilon^{w_2\sum_{l=1}^{k} j_l} [j_1\cdots+j_k]^i_{q^{w_2}}.
\end{aligned}$$

For each integer $n \geq 0$, let

$$\mathcal{S}_{n,i,q,\varepsilon}^{(h,k)}(w) = \sum_{j_1,\cdots,j_k=0}^{w-1} (-1)^{\sum_{l=1}^k j_l} q^{\sum_{l=1}^k (n+h-l-i+1)j_l} \varepsilon^{\sum_{l=1}^k j_l} [j_1 \cdots + j_k]_q^i.$$

The above sum $\mathcal{S}_{n,i,q,\varepsilon}^{(h,k)}(w)$ is called the alternating (h,q)-power sums.

By Theorem 4, we have

$$\begin{aligned}
&[2]_{q^{w_2}}^k [w_1]_q^n \sum_{j_1,\cdots,j_k=0}^{w_1-1} (-1)^{\sum_{l=1}^k j_l} q^{w_2 \sum_{l=1}^k (h-l+1)j_l} \varepsilon^{w_2 \sum_{l=1}^k j_l} \\
&\qquad \times E_{n,q^{w_1},\varepsilon^{w_1}}^{(h,k)} \left(w_2 x + \frac{w_2}{w_1}(j_1 + \cdots + j_k) \right) \qquad (13) \\
&= [2]_{q^{w_2}}^k \sum_{i=0}^n \binom{n}{i} [w_2]_q^i [w_1]_q^{n-i} E_{n-i,q^{w_1},\varepsilon^{w_1}}^{(h,k)}(w_2 x) \mathcal{S}_{n,i,q^{w_2},\varepsilon^{w_2}}^{(h,k)}(w_1)
\end{aligned}$$

By using the same method as in (13), we have

$$\begin{aligned}
&[2]_{q^{w_1}}^k [w_2]_q^n \sum_{j_1,\cdots,j_k=0}^{w_2-1} (-1)^{\sum_{l=1}^k j_l} q^{w_1 \sum_{l=1}^k (h-l+1)j_l} \varepsilon^{w_1 \sum_{l=1}^k j_l} \\
&\qquad \times E_{n,q^{w_2},\varepsilon^{w_2}}^{(h,k)} \left(w_1 x + \frac{w_1}{w_2}(j_1 + \cdots + j_k) \right) \qquad (14) \\
&= [2]_{q^{w_1}}^k \sum_{i=0}^n \binom{n}{i} [w_1]_q^i [w_2]_q^{n-i} E_{n-i,q^{w_2},\varepsilon^{w_2}}^{(h,k)}(w_1 x) \mathcal{S}_{n,i,q^{w_1},\varepsilon^{w_1}}^{(h,k)}(w_2)
\end{aligned}$$

Therefore, by (13) and (14) and Theorem 3, we have the following theorem.

Theorem 5. Let $w_1, w_2 \in \mathbb{N}$ with $w_1 \equiv 1 \pmod 2$, $w_2 \equiv 1 \pmod 2$. For $h \in \mathbb{Z}, k \in \mathbb{N}$ and $n \in \mathbb{Z}_+$, we obtain

$$\begin{aligned}
&[2]_{q^{w_2}}^k \sum_{i=0}^n \binom{n}{i} [w_2]_q^i [w_1]_q^{n-i} E_{n-i,q^{w_1},\varepsilon^{w_1}}^{(h,k)}(w_2 x) \mathcal{S}_{n,i,q^{w_2},\varepsilon^{w_2}}^{(h,k)}(w_1) \\
&= [2]_{q^{w_1}}^k \sum_{i=0}^n \binom{n}{i} [w_1]_q^i [w_2]_q^{n-i} E_{n-i,q^{w_2},\varepsilon^{w_2}}^{(h,k)}(w_1 x) \mathcal{S}_{n,i,q^{w_1},\varepsilon^{w_1}}^{(h,k)}(w_2).
\end{aligned}$$

By Theorem 5, we obtain the interesting symmetric identity for the higher-order twisted (h,q)-Euler numbers $E_{n,q,\varepsilon}^{(h,k)}$ in complex field.

Corollary 6. Let $w_1, w_2 \in \mathbb{N}$ with $w_1 \equiv 1 \pmod 2$, $w_2 \equiv 1 \pmod 2$. For $h \in \mathbb{Z}, k \in \mathbb{N}$ and $n \in \mathbb{Z}_+$, we obtain

$$[2]^k_{q^{w_2}} \sum_{i=0}^{n} \binom{n}{i} [w_2]^i_q [w_1]^{n-i}_q \mathcal{S}^{(h,k)}_{n,i,q^{w_2},\varepsilon^{w_2}}(w_1) E^{(h,k)}_{n-i,q^{w_1},\varepsilon^{w_1}}$$
$$= [2]^k_{q^{w_1}} \sum_{i=0}^{n} \binom{n}{i} [w_1]^i_q [w_2]^{n-i}_q \mathcal{S}^{(h,k)}_{n,i,q^{w_1},\varepsilon^{w_1}}(w_2) E^{(h,k)}_{n-i,q^{w_2},\varepsilon^{w_2}}.$$

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