SOLVABILITY CONDITIONS OF THE SECOND ORDER DIFFERENTIAL EQUATION WITH DRIFT

Kordan N. Ospanov¹, Danagul R. Beisenova²

¹Institute of Mathematics and Mathematical Modeling
Almaty, KAZAKHSTAN

²L.N. Gumilyov Eurasian National University
Astana, KAZAKHSTAN

Abstract: We proved the unique solvability in Hilbert space of the second order differential equation with complex and unbounded intermediate coefficients.

AMS Subject Classification: 34A34

Key Words: differential equation, complex coefficient, solvability, unbounded drift

1. Introduction and Main Result

We consider the following equation

\begin{equation}
ly = -y'' + r(x)y' + q(x)y' + s(x)y + p(x)y = f,
\end{equation}

where \( f \in L_2 := L_2(\mathbb{R}) \), and \( r \) and \( q \) are continuously differentiable, and \( s \) and \( p \) are continuous functions, \( y = y_1 + iy_2 \) and \( \bar{y} = y_1 - iy_2 \).

When \( r \) and \( q \) are unbounded functions, properties of (1.1) are very different from the properties of the Sturm-Liouville equation. So, the study of (1.1) has a theoretical value. On the other hand, a number of problems of stochastic
processes related to the description of particle motion, and wave propagation in a compressible medium and in the medium with resistance lead to the equation (1.1) (see [1-5]). In the case \( s = p = 0 \) (1.1) was studied in [6]. When \( q = p = 0 \), \( r \) and \( s \) are real-valued, and \( f \in L_1(\mathbb{R}) \) (this is a more simple case) it was investigated in [7]. For more details on the singular differential equations, see [8-13].

Let \( g \) and \( h \) be given functions. We put
\[
\alpha_{g,h}(t) = \|g\|_{L_2(0, t)} \|h^{-1}\|_{L_2(t, +\infty)} \quad (t > 0),
\]
\[
\beta_{g,h}(\tau) = \|g\|_{L_2(\tau, 0)} \|h^{-1}\|_{L_2(-\infty, \tau)} \quad (\tau < 0),
\]
\[
\gamma_{g,h} = \max \left( \sup_{t>0} \alpha_{g,h}(t), \sup_{\tau<0} \beta_{g,h}(\tau) \right).
\]
We consider the operator
\[
l y = -y'' + r(x)y' + q(x)y + s(x)y + p(x)y,
\]
which is defined on the set \( C^{(2)}_0(\mathbb{R}) \) of twice continuously differentiable functions with compact support.

**Definition 1.** We say that \( y \in L_2 \) is a solution of (1.1), if there exists a sequence \( \{y_n\}_{n=1}^{+\infty} \subset C^{(2)}_0(\mathbb{R}) \) such that \( \|y_n - y\|_2 \to 0 \) and \( \|ly_n - f\|_2 \to 0 \) as \( n \to +\infty \).

**Theorem 2.** Let \( r \) and \( q \) be continuously differentiable, and let \( s, p \) be continuous functions with the following properties:
\[
(1.2) \quad \sqrt{|\text{Re} r|} - \omega(|\text{Im} r| + |q|) \geq 1 \quad (1 < \omega < 2)
\]
and
\[
(1.3) \quad \gamma_{1+|s|+|p|,\sqrt{\text{Re} r}} < \infty.
\]
Then for any \( f \in L_2 \) there exists a unique solution \( y \) of the equation (1.1).

2. **Auxiliary Statements**

**Lemma 1.** [6] Let functions \( g \) and \( h \) satisfy \( \gamma_{g,h} < \infty \). Then for \( y \in C^{(1)}_0(\mathbb{R}) \) the following inequality holds:
\[
(2.1) \quad \int_{\mathbb{R}} |g(x)y(x)|^2 \, dx \leq c_1 \int_{\mathbb{R}} |h(x)y'(x)|^2 \, dx.
\]
Moreover, if \( c_1 \) is the smallest constant satisfying (2.1), then
\[
\gamma_{g,h} \leq c_1 \leq 2\gamma_{g,h}.
\]

Let \( r_0 = Re r \) and \( l_0 y = -y'' + r_0(x)y' \) be defined on \( C^2_0(\mathbb{R}) \).
Using Lemma 1, we prove the following result.

**Lemma 2.** Let \( r \) be continuously differentiable function and
\[
(2.2) \quad \gamma_{1,\sqrt{r_0}} < \infty.
\]
Then for any \( y \in C^2_0(\mathbb{R}) \) the following estimate holds:
\[
(2.3) \quad \|\sqrt{|r_0|}y'\|_2 + \|y\|_2 \leq c_2\|l_0 y\|_2.
\]

*Proof.* Let \( y \in C^2_0(\mathbb{R}) \). Using integration by parts, we have
\[
(l_0 y, y') = \int_\mathbb{R} r_0(x)|y'|^2 dx.
\]
By Holder inequality,
\[
\|\sqrt{|r_0|}y'\|_2 \leq \left\| \frac{1}{\sqrt{|r_0|}} l_0 y \right\|_2.
\]
Then using (2.2) and Lemma 2.1, we find that
\[
\|\sqrt{|r_0|}y'\|_2 + \|y\|_2 \leq \left( 1 + 2\gamma_{1,\sqrt{|r_0|}} \right) \|l_0 y\|_2.
\]
From this follows (2.3), where \( c_2 = 1 + 2\gamma_{1,\sqrt{|r_0|}} \).

If (2.2) holds, then using Lemma 2, we can prove that the operator \( l_0 \) is closable in the space \( L_2 \). We denote its closure by \( L_0 \). It is easy to see that for any \( y \in D(L_0) \) the following inequality holds:
\[
(2.4) \quad \|\sqrt{|r_0|}y'\|_2 + \|y\|_2 \leq c_2\|L_0 y\|_2.
\]

**Lemma 3.** Assume that the conditions of Lemma 2.2 are fulfilled. Then \( R(L_0) = L_2 \).
Proof. From (2.4) follows that \( L_0 \) has the inverse \( L_0^{-1} \). Let \( R(L_0) \neq L_2 \). Then there exists nonzero \( z_0 \in L_2 \) which is orthogonal to \( R(L_0) \). Hence

\[
(z_0' + r_0 z_0)' = 0,
\]
or

\[
z_0' + r_0 z_0 = c_3,
\]
where \( c_3 \) is const. It is easy to prove that \( z_0 \) is the continuously differentiable function and the following equality holds:

\[
(z_0' + r_0 z_0)' = c_3 \exp \int_a^x r_0(t)dt.
\]

(2.5)

1. If \( c_3 \neq 0 \), then without loss of generality, we can put \( c_3 = -1 \). Hence

\[
(z_0' + r_0 z_0)' < 0.
\]

Then for \( x_1, x_2 \) with \( x_1 < x_2 \),

\[
z_0(x_1) > z_0(x_2) \exp \int_{x_1}^{x_2} r_0(t)dt.
\]

So

\[
z_0(x_1) > z_0(x_2)
\]
for all \( x_1, x_2 \in \mathbb{R} \) satisfying \( x_1 < x_2 \). Therefore \( z_0(x) \notin L_2(R) \).

2. Let \( c_3 = 0 \). Then from (2.5) we have

\[
z_0(x) = c_4 \exp \left[ - \int_a^x r_0(t)dt \right],
\]
hence if \( c_4 \neq 0 \), then \( |z_0(x_0)| \geq |c_4| \) for all \( x_0 < a \). Therefore, \( z_0(x) \notin L_2(R) \).

This is a contradiction. \( \square \)
3. Proof of Theorem 1.1

Let \( x = at \) (\( a > 1 \)), then (1.1) becomes that

\[
L_\alpha z = -z'' + a^{-1}r_1(t)z' + a^{-1}q_1(t)\bar{z}' + a^{-2}s_1(t)z + a^{-2}p_1(t)\bar{z} = f(t),
\]

where \( z = z(t) = y(at), \) \( r_1(t) = r(at), \) \( q_1(t) = q(at), \) \( s_1(t) = s(at), \) \( p_1(t) = p(at) \)
and \( f(t) = a^{-2}f(at). \)

Let \( l_{0a}z = -z'' + a^{-1}Re r_1 z', \) \( z \in C^{(2)}_0(\mathbb{R}). \) By Lemma 2, the operator \( l_{0a} \)
is closable in \( L_2. \) We denote by \( l_\alpha \) the closure of \( l_{0a}. \) Since \( a^{-1}Re r_1(t) \) satisfies
the conditions of Lemma 3, it follows that \( l_\alpha \) is continuously invertible and for
any \( z \in D(l_\alpha), \) the following inequality holds:

\[
\|a^{-1}Re r_1 z'\|_2 \leq \|l_\alpha z\|_2.
\]

This estimate by (1.2) implies

\[
\|a^{-1}|i Im r_1 z' + |q_1 z'||_2 \leq \frac{1}{\omega l_\alpha z_2}, \forall z \in D(l_\alpha).
\]

Hence by Theorem 1.16 in Chapter 4 of [14], we have that the operator \( \tilde{l}_\alpha z = -y'' + a^{-1}r_1(t)z' + a^{-1}q_1(t)\bar{z}' \) is invertible and \( R(\tilde{l}_\alpha) = L_2. \) Moreover, it is
easy to calculate that

\[
\|l_\alpha z\|_2 \leq \frac{\omega}{\omega - 1} \|\tilde{l}_\alpha z\|_2, \forall z \in D(\tilde{l}_\alpha).
\]

Further by condition (1.3) and Lemma 1, for any \( z \in D(l_\alpha) \) the following inequalines hold:

\[
\|a^{-2}s_1 z\|_2 \leq 2a^{-3/2}\gamma_{s_1,\sqrt{Re r_1}} \|a^{-1}Re r_1 z'\|_2,
\]
\[
\|a^{-2}p_1 z\|_2 \leq 2a^{-3/2}\gamma_{p_1,\sqrt{Re r_1}} \|a^{-1}Re r_1 z'\|_2.
\]

We choose

\[
a = \omega \left[ \frac{2\omega}{\omega - 1} \left( \gamma_{s_1,\sqrt{Re r_1}} + \gamma_{p_1,\sqrt{Re r_1}} \right) \right]^{2/3}.
\]

Then by (3.2), (3.3) and (3.4), we have

\[
\|a^{-2}s_1 z + a^{-2}p_1 \bar{z}\|_2 \leq \frac{1}{\omega} \|\tilde{l}_\alpha z\|_2, \forall z \in D(\tilde{l}_\alpha).
\]
The estimate (3.5), by Theorem 1.16 in Chapter 4 of [14], implies that the operator $L_a = \tilde{l}_a + a^{-2}s_1(t)E + a^{-2}p_1(t)\bar{E}$ corresponding to (3.1) is invertible and $R(L_a) = L_2$. Here $\bar{E}z = \bar{z}$. Taking $t = x/a$, we get that $R(L) = L_2$. Then by Definition 1.1, for any $f \in L_2$, (1.1) has a unique solution $y$. □

Brief abstract of this work has been published in the Materials of the workshop ”Differential operators and modeling of complex systems” (April 7-8, 2017, Almaty, Kazakhstan) [15].

Acknowledgements

This research was supported by the grant 5132/GF4 and the target program 0085/PTSF-14 of the Ministry of Science and Education of the Republic of Kazakhstan.

References


