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EXACT SOLUTIONS FOR A FORCED GENERALIZATION OF THE (2+1)-GARDNER EQUATIONS

Cesar A. Gómez S.

Departamento de Matemáticas
Universidad Nacional de Colombia
Kra. 30 No. 45-03, Bogotá, COLOMBIA

Abstract: We use an improved tanh-coth method to obtain traveling wave solutions for a new model which can be considered as a generalization of the classical (2+1)-Gardner equations with forcing term. The new model considered here, have relevance in the sense that it include the (2+1)-Gardner equation as well as the modified KdV equation and the KdV equation with forcing term. We show that from the solutions obtained for this new model, we can derived solution for classical and well know models of applied physics.

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1. Introduction

Since the apparition of the KdV equation several decades ago, the study of nonlinear partial differential equations (NLPDE's) is an important task for many scientists and researchers, mainly in physics and applied mathematics. In many cases is very difficult the study of its analytic solutions, in this sense the use of computational methods is very useful to obtain certain types of solutions know as traveling solutions or exact solutions. The improved tanh-coth method is a generalization of some used methods such as the tanh method, the tanh-coth method, the G'/G-method and many others equivalent methods. The main idea of this paper is the use of the improved tanh-coth method [1] to obtain traveling solutions of the following generalized model

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$$\begin{cases} u_t + a(t)u_{xxx} + b(t)u^2u_x + c(t)uu_x + d(t)v_y + p(t)u_xv = F(t) \\ v_x = u_y, \end{cases} \quad (1)$$

where u, v depending of the variables x, y, t, the coefficients a, b, c, d, p depending only of the variable time t, and $F(t)$ is a forcing term. Clearly, Eq.(1) can be written as

$$u_t + a(t)u_{xxx} + b(t)u^2u_x + c(t)uu_x + d(t)\partial_x^{-1}u_{yy} + p(t)u_x\partial_x^{-1}u_y = F(t). \quad (2)$$

In the case $F(t) = 0$, we obtain the following generalization of the standard (2+1)-Gardner equation [2],

$$u_t + a(t)u_{xxx} + b(t)u^2u_x + c(t)uu_x + d(t)\partial_x^{-1}u_{yy} + p(t)u_x\partial_x^{-1}u_y = 0. \quad (3)$$

In the same form, if in Eq.(2) we take $u_y = 0$ ($d(t) = 0$, $p(t) = 0$) we have the following new model called forced (1+1)-dimensional KdV equation

$$u_t + a(t)u_{xxx} + b(t)u^2u_x + c(t)uu_x = F(t) \quad (4)$$

which can be called the forced KdV-mKdV equation. Additionally, if we take $F = 0$ in Eq.(4), the following generalization of the standard (1+1)-dimensional Gardner (KdV-mKdV equation) equation is obtained [2], [3],

$$u_t + a(t)u_{xxx} + b(t)u^2u_x + c(t)uu_x = 0. \quad (5)$$

2. Exact solutions to Eq.(1)

We consider the following transformation

$$\begin{cases} u(x, y, t) = u(\xi) \\ v(x, y, t) = v(\xi) \\ \xi = x + y + \int h(t)dt. \end{cases} \quad (6)$$

Taking into account that:

$$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial \xi}\frac{\partial \xi}{\partial x} = \frac{dv}{d\xi} = v',$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial y} = \frac{du}{d\xi} = u',$$

then, after one integration with respect to ξ the Eq.(1) converts to

$$\begin{cases} (h(t) + d(t)u + a(t)u''+ \\ \frac{b(t)}{3}(u + \int F(t)dt))^3 + \frac{c(t)+p(t)}{2}(u + \int F(t)dt))^2 + G = 0, \end{cases} \tag{7}$$

being $G = G(t)$ the constant of integration. The idea of the improved tanh-coth method [1] consists on the search solutions to Eq.(7) in the form

$$v(\xi) = \sum_{i=0}^{M} a_i(t)\phi(\xi)^i + \sum_{i=M+1}^{2M} a_i(t)\phi(\xi)^{M-i}, \tag{8}$$

where M is a positive integer determined by balancing and $\phi = \phi(\xi)$ is solution of the Riccati equation

$$\phi'(\xi) = \alpha(t) + \beta(t)\phi(\xi) + \gamma(t)\phi(\xi)^2. \tag{9}$$

The $a_i(t)$, $i = 1, 2, \ldots, 2M$, $\alpha(t)$, $\beta(t)$, $\gamma(t)$ are functions depending only of the variable t to be determined later.

Substituting (8) into (7), and balancing $u''(\xi)$ with $u(\xi)^3$ is derived the following relation

$$M + 2 = 3M,$$

therefore

$$M = 1.$$

So that (8) reduces to

$$v(\xi) = a_0(t) + a_1(t)\phi(\xi) + a_2(t)(\phi(\xi))^{-1}. \tag{10}$$

Substituting (10) into (7), using (9) and equating to zero the coefficients of all powers of $\phi(\xi)$, we get a set of algebraic equations for the unknowns $a_0(t)$, $a_1(t)$, $a_2(t)$, $h(t)$, $G(t)$, $\alpha(t)$, $\beta(t)$, $\gamma(t)$. We solve the system with aid the Mathematica. For sake of simplicity, we consider only the following two solutions:

Fisrt Case:

$$\begin{cases} G(t) = -\frac{1}{6b(t)}\Big[b(t)(\int F(t)dt)^2\big(2b(t)(\int F(t)dt) + 3c(t) + 3p(t)\big) \\ +\big(2b(t)(\int F(t)dt) + c(t) + p(t)\big) \\ \Big(-i\sqrt{6}\alpha(a_1(t))\sqrt{a(t)}\sqrt{b(t)} + a_0(t)(2b(t)(\int F(t)dt) + c(t) + p(t)) + (a_0(t))^2 b(t)\Big)\Big], \\ \\ h(t) = \frac{1}{6b(t)}\Big[2i\sqrt{6}\alpha a_1(t)\sqrt{a(t)}b(t)^{3/2} - 2(a_0(t))^2 b(t)^2 \\ -2(a_0(t))b(t)(2b(t)(\int F(t)dt) + c(t) + p(t)) + 2c(t)(p(t) - b(t)(\int F(t)dt)) \\ -2b(t)\left(b(t)(\int F(t)dt)^2 + 3D\right) - 2b(t)(\int F(t)dt)p(t) + c(t)^2 + p(t)^2\Big], \\ \\ \beta(t) = -\frac{i(2(a_0(t))b(t) + 2b(t)(\int F(t)dt) + c(t) + p(t))}{\sqrt{6}\sqrt{a(t)}\sqrt{b(t)}}, \\ \\ \gamma(t) = -\frac{i(a_1(t))\sqrt{b(t)}}{\sqrt{6}\sqrt{a(t)}}, \\ \\ a_2(t) = 0. \end{cases} \tag{11}$$

Now, its well known that the solution of (9) in the case $\beta(t)^2 - 4\alpha(t)\gamma(t) \neq 0$ is given by (see [4]):

$$\phi(\xi) = \frac{-\sqrt{\beta(t)^2 - 4\alpha(t)\gamma(t)}\tanh[\frac{1}{2}\sqrt{\beta(t)^2 - 4\alpha(t)\gamma(t)}\xi] - \beta(t)}{2\gamma(t)}. \tag{12}$$

Therefore, using the values given by (11), we have

$$\begin{cases} \phi(\xi) = -\frac{2a_0(t)b(t) + 2b(t)\int F(t)dt + c(t) + p(t)}{2a_1(t)b(t)} \\ \\ -\Bigg(\frac{i\sqrt{\frac{3}{2}}\sqrt{a(t)}\sqrt{-\frac{(2a_0(t)b(t) + 2b(t)\int F(t)dt + c(t) + p(t))^2}{6a(t)b(t)} + \frac{2i\sqrt{\frac{2}{3}}\alpha a_1\sqrt{b(t)}}{\sqrt{a(t)}}}}{a_1(t)\sqrt{b(t)}}\Bigg) \\ \tanh\left(\frac{1}{2}x\sqrt{-\frac{\left(2a_0(t)b(t) + 2b(t)\int F(t)dt + c(t) + p(t)\right)^2}{6a(t)b(t)} + \frac{2i\sqrt{\frac{2}{3}}\alpha a_1(t)\sqrt{b(t)}}{\sqrt{a(t)}}}\right). \end{cases} \tag{13}$$

Finally, by (12), (10) and (6) we have that the solution of (1) corresponding to this values, is given by

$$u(x,t) = u(\xi) = a_0(t) + a_1(t)\phi(\xi) \tag{14}$$

where $a_0(t)$, $a_1(t)$, are arbitrary functions depending on variable t, $\phi(\xi)$ given by (12). If we take $F = 0$, in (11) and (13), we obtain exact solutions to the generalized (2+1)-Gardner equation (3). In the same form, if we take, $u_y = 0$ ($d(t) = 0$, $p(t) = 0$) we have solution to the forced (1+1)-dimensional KdV equation (4) (forced KdV-mKdV equation), if additionally we take $F = 0$, exact solution to standard (1+1)-dimensional Gardner equation (5) is obtained.

Second Case:

$$\begin{cases} G(t) = -\frac{1}{6b}\Big[\big(a_0 b + 2b(\int F) + c + p\big)\big(a_0(2b(\int F) + c + p) + 4a_1a_2b\big) + \\ b(\int F)^2(2b(\int F) + 3c + 3p)\Big], \\ \\ h(t) = \frac{1}{6b}\Big[-4a_1a_2b^2 - 2a_0b\big(a_0b + 2b(\int F) + c + p\big) + \\ 2c(p - b(\int F)) - 2b\big(b(\int F)^2 + 3d\big) - 2b(\int F)p + c^2 + p^2\Big], \\ \\ \alpha(t) = -\frac{ia_2\sqrt{b}}{\sqrt{6}\sqrt{a}}, \\ \\ \beta(t) = -\frac{i\left(2a_0b + 2b(\int F) + c + p\right)}{\sqrt{6}\sqrt{a}\sqrt{b}}, \\ \\ \gamma(t) = -\frac{ia_1\sqrt{b}}{\sqrt{6}\sqrt{a}}. \end{cases} \tag{15}$$

In all cases, $a_0 = a_0(t)$, $a_1 = a_1(t)$, $a_2 = a_2(t)$, $a = a(t)$, $b = b(t)$, $c = c(t)$, $d = d(t)$, $p = p(t)$, $\int F = \int F(t)dt$.

With respect to this set of solutions and according with (12), then we have

$$\begin{cases} \phi(\xi) = \\ \frac{i\sqrt{\frac{3}{2}}\sqrt{A}}{a_1\sqrt{B}}\left[-\frac{\sqrt{-\frac{(2a_0B+2BF+C+P)^2-4a_1a_2B^2}{AB}}\tanh\left(\frac{x\,\square\overline{-\frac{(2a_0B+2BF+C+P)^2-4a_1a_2B^2}{AB}}}{2\sqrt{6}}\right)}{\sqrt{6}} + \right. \\ \left.\frac{i(2a_0B+2BF+C+P)}{\sqrt{6}\sqrt{A}\sqrt{B}}\right]. \end{cases} \tag{16}$$

Finally, the solution to Eq.(1), is given by

$$u(\xi) = a_0(t) + a_1(t)\phi(\xi) + a_2(t)\phi(\xi)^{-1},$$

where ξ is given by (6), $h(t)$ given by (15).
As in the previous case, we can obtain solutions to (3), (4) and (5).

3. Conclusions

We have studied, from of point of view of its exact traveling solutions a forced generalized model, from which we can derived solutions to generalizations of other important models of the applied physics such that: a generalization of the standard (2+1)-Gardner equation, a forced generalization of the standard (1+1)-Gardner equation and a generalization of the KdV-mKdV equation. The results show us the importance of the improved tanh-coth method in the study of this class of models.

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