International Journal of Pure and Applied Mathematics
Volume 113 No. 5 2017, 575-581
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: 10.12732/ijpam.v113i5.5

ON THE STABILITY OF ALMOST BI-MULTIPLICATIVE FUNCTIONALS

A. Zivari-Kazempour[1 §], Z. Fallah[2]
[1,2]Department of Mathematics
Ayatollah Borujerdi University
Borujerd, IRAN

Abstract: The goal of this paper is to investigate the stability of bi-multiplicative functionals $\phi : \mathcal{A} \times \mathcal{B} \longrightarrow \mathbb{C}$, where $\mathcal{A}$ and $\mathcal{B}$ are two normed algebra over a complex filed $\mathbb{C}$.

AMS Subject Classification: 46L05, 46H25.
Key Words: Bi-multiplicative, Bi-additive, Bi-ring homomorphism.

1. Introduction

Let X be real normed space and Y be real Banach spaces. S. M. Ulam [10] posed the problem: When does a linear mapping near an approximately additive mapping $f : X \longrightarrow Y$ exist?

In 1941, Hyers [5] gave an affirmative answer to the question of Ulam for additive Cauchy equation in Banach spaces.

Let X and Y be two Banach spaces and let $f : X \longrightarrow Y$ be a mapping satisfying:

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon,$$

for all $x, y \in X$ and $\epsilon > 0$. Then there is a unique additive mapping $F : X \longrightarrow Y$ which satisfies

Received: November 22, 2016
Revised: March 7, 2017
Published: March 31, 2017

§Correspondence author

$$\|F(x) - f(x)\| \le \epsilon,$$

for all $x \in X$.

Hyers in [5] called any solution of this equation a *linear* function, and he considered only the bounded Cauchy difference $f(x+y) - f(x) - f(y)$.

Th. M. Rassias [7] considered a generalized version of the Hyers's result which permitted the Cauchy difference to become unbounded. That is, he proved:

Theorem 1. *Let X and Y be two real Banach spaces, $\epsilon \ge 0$ and $0 \le p < 1$. If a mapping $f : X \longrightarrow Y$ satisfies*

$$\|f(x+y) - f(x) - f(y)\| \le \epsilon(\|x\|^p + \|y\|^p),$$

for all $x, y \in X$, then there is a unique additive mapping $F : X \longrightarrow Y$ such that

$$\|F(x) - f(x)\| \le \frac{2\epsilon}{|2 - 2^p|}\|x\|^p,$$

for all $x \in X$. If, in addition, for each fixed $x \in X$ the function $t \longmapsto f(tx)$ is continuous in $t \in \mathbb{R}$, then F is linear.

In [3], Gajda proved that Theorem 1 is valid for $p > 1$. He also gave an example showing that a similar result to the above does not hold for $p = 1$.

A generalization of Rassias's Theorem is also obtained by Gavruta [4], who replaced $\epsilon(\|x\|^p + \|y\|^p)$ by a general control function $\varphi(x, y)$. For more details about the stability problems the reader is referred to [1], [6], [8] and [9].

Baker in [2] considered the stability of the multiplicative Cauchy equation. He demonstrated the following:

Theorem 2. *Let $\delta > 0$ and let f be a complex-valued function on a semigroup S such that $|f(xy) - f(x)f(y)| \le \delta$, for all $x, y \in S$, then either f is multiplicative, or*

$$|f(x)| \le \frac{1}{2}(1 + \sqrt{1 + 4\delta}),$$

for all $x \in S$.

Let $\mathcal{A}$ and $\mathcal{B}$ be a two normed algebra over a complex filed $\mathbb{C}$ and set $\mathcal{U} = \mathcal{A} \times \mathcal{B}$. Then $\mathcal{U}$ is a normed algebra for the multiplication

$$(a, b)(x, y) = (ax, by), \qquad (a, b), (x, y) \in \mathcal{U},$$

and with norm

$$\|(a, b)\| = \|a\| + \|b\|.$$

Let $\mathcal{D}$ be a complex normed algebra and let $\varphi : \mathcal{U} \longrightarrow \mathcal{D}$ be a map. Then φ is called bi-multiplicative, if

$$\varphi(ax, by) = \varphi(a, b)\varphi(x, y), \qquad (a, b), (x, y) \in \mathcal{U},$$

and it is called almost bi-multiplicative, if

$$\|\varphi(ax, by) - \varphi(a, b)\varphi(x, y)\| \leq \delta\|(a, b)\|^p\|(x, y)\|^p, \tag{1}$$

for some $\delta \geq 0$ and $p \geq 0$. The map φ is said to be bi-additive, if

$$\varphi(a + x, b + y) = \varphi(a, b) + \varphi(x, y),$$

for all $(a, b), (x, y) \in \mathcal{U}$. If φ is bi-additive and bi-multiplicative, then it is called bi-ring homomorphism.

In this paper, we shall prove that a Baker type stability result holds for the functionals $\varphi : \mathcal{U} \longrightarrow \mathbb{C}$ satisfying (1) for some $\delta \geq 0$ and $p \geq 0$. Moreover, if φ is almost bi-additive in the sense

$$|\varphi(a + x, b + y) - \varphi(a, b) - \varphi(x, y)| \leq \delta\big(\|(a, b)\|^p + \|(x, y)\|^p\big),$$

for all $(a, b), (x, y) \in \mathcal{U}$, then we show that either φ is a bi-ring homomorphism, or it satisfies

$$|\varphi(x, y)| \leq \frac{1}{2}(1 + \sqrt{1 + 4\delta})\|(x, y)\|^p,$$

for all $(x, y) \in \mathcal{U}$.

2. Stability of Bi-Multiplicative Functionals

We commence with the following Theorem which is the main one in the paper.

Theorem 3. *Let $\delta \geq 0$ and $p \geq 0$. If a functional $\varphi : \mathcal{U} \longrightarrow \mathbb{C}$ satisfies*

$$|\varphi(ax, by) - \varphi(a, b)\varphi(x, y)| \leq \delta\|(a, b)\|^p\|(x, y)\|^p, \tag{2}$$

for all $(a, b), (x, y) \in \mathcal{U}$, then φ is bi-multiplicative or there exists a constant $k > 0$ such that

$$|\varphi(x, y)| \leq k\|(x, y)\|^p,$$

for all $(x, y) \in \mathcal{U}$.

Proof. Suppose that φ is not bi-multiplicative, that is, there are $(a,b),(c,d)\in\mathcal{U}$ such that

$$\varphi(ac,bd)\neq\varphi(a,b)\varphi(c,d).$$

Pick $(x,y)\in\mathcal{U}$ arbitrarily. It follows from (2) that

$$\begin{aligned}|\varphi(x,y)||\varphi(ac,bd)-\varphi(a,b)\varphi(c,d)| &\leq |\varphi(x,y)\varphi(ac,bd)-\varphi(a,b)\varphi(c,d)|\\ &+|\varphi((x,y)(ac,bd))-\varphi((x,y)(a,b))\varphi(c,d)|\\ &+|\varphi((x,y)(a,b))\varphi(c,d)-\varphi(x,y)\varphi(a,b)\varphi(c,d)|\\ &\leq \delta\|(a,b)\|^p\|(x,y)\|^p\big(2\|(c,d)\|^p+|\varphi(c,d)|\big).\end{aligned}$$

Thus, if we take

$$k=\frac{\delta\|(a,b)\|^p\big(2\|(c,d)\|^p+|\varphi(c,d)|\big)}{|\varphi(ac,bd)-\varphi(a,b)\varphi(c,d)|},$$

then we get

$$|\varphi(x,y)|\leq k\|(x,y)\|^p,$$

for all $(x,y)\in\mathcal{U}$. This complete the proof. □

As Baker pointed out in his article, the above proof works for functions $\varphi:\mathcal{U}\longrightarrow\mathcal{D}$, where $\mathcal{D}$ is a normed algebra in which the norm is multiplicative, that is,

$$\|xy\|=\|x\|\|y\|,\quad (x,y\in\mathcal{D}).$$

However, Theorem 3 cannot hold for all normed algebras $\mathcal{D}$, as is shown by the following example. Let $\delta>0$ and choose $\varepsilon>0$ so that $|\varepsilon-\varepsilon^2|=\delta$, and define $\varphi:\mathbb{R}\times\mathbb{R}\longrightarrow M_3(\mathbb{R})$ by

$$\varphi(x,y)=\begin{bmatrix}x&0&0\\0&y&0\\0&0&\varepsilon\end{bmatrix}\qquad(x,y\in\mathbb{R}).$$

Then with the usual matrix norm

$$\|\varphi(ax,by)-\varphi(a,b)\varphi(x,y)\|=\delta,$$

for all $(a,b),(x,y)\in\mathbb{R}\times\mathbb{R}$. So φ satisfy (2) for $p=0$, but neither φ is bounded nor it is bi-multiplicative.

Theorem 4. *Suppose that a functional $\varphi:\mathcal{U}\longrightarrow\mathbb{C}$ satisfies (2) for some $\delta\geq 0$ and $p\geq 0$. Then φ is bi-multiplicative or*

$$|\varphi(x,y)|\leq\frac{1}{2}(1+\sqrt{1+4\delta})\|(x,y)\|^p,\tag{3}$$

for all $(x,y)\in\mathcal{U}$.

Proof. Suppose that φ is not bi-multiplicative. Then by Theorem 3 there exists a constant $k > 0$ such that

$$|\varphi(x,y)| \leq k\|(x,y)\|^p, \tag{4}$$

for all $(x,y) \in \mathcal{U}$. Set

$$m = \begin{cases} sup\ |\varphi(x,y)| & if \quad p = 0 \\ \\ sup\ \frac{|\varphi(x,y)|}{\|(x,y)\|^{p}} & if \quad p \neq 0, \text{ and } (x,y) \neq (0,0). \end{cases}$$

Then $m \leq k < \infty$. Let $p \neq 0$, then by (4) we get $\varphi(0,0) = 0$. Thus,

$$|\varphi(x,y)| \leq m\|(x,y)\|^p, \tag{5}$$

for all $(x,y) \in \mathcal{U}$. It follows from (2) that

$$|\varphi(x^2,y^2) - \varphi(x,y)^2| \leq \delta\|(x,y)\|^{2p}, \tag{6}$$

and hence

$$\begin{aligned} |\varphi(x,y)|^2 &\leq \delta\|(x,y)\|^{2p} + |\varphi(x^2,y^2)| \\ &\leq \delta\|(x,y)\|^{2p} + m\|(x^2,y^2)\|^p \\ &\leq (\delta + m)\|(x,y)\|^{2p}. \end{aligned}$$

Therefore $m^2 - m \leq \delta$, which proves that $m \leq \frac{1}{2}(1 + \sqrt{1 + 4\delta})$. Thus, we get (3) for all $(x,y) \in \mathcal{U}$. □

Theorem 5. *Let a functional $\varphi : \mathcal{U} \longrightarrow \mathbb{C}$ satisfies (2) and let*

$$|\varphi(a+x, b+y) - \varphi(a,b) - \varphi(x,y)| \leq \delta\big(\|(a,b)\|^p + \|(x,y)\|^p\big), \tag{7}$$

for some $\delta \geq 0$ and $p \geq 0$. Then φ is bi-ring homomorphism or it satisfies (3).

Proof. Suppose that φ is not bi-ring homomorphism. If φ is not bi-multiplicative, then by above Theorem we get (3). Assume that φ is not bi-additive, then there exist $(a,b), (c,d) \in \mathcal{U}$ such that

$$\varphi(a+c, b+d) \neq \varphi(a,b) + \varphi(c,d).$$

By using a similar argument as in the proof of Theorem 3 we have

$$|\varphi(x,y)||\varphi(a+c, b+d) - \varphi(a,b) - \varphi(c,d)| \leq \delta M\|(x,y)\|^p,$$

where

$$M = \|(a+c, b+d)\|^p + 2\|(a,b)\|^p + 2\|(c,d)\|^p.$$

Take

$$k = \delta M \big(|\varphi(a+c, b+d) - \varphi(a,b)\varphi(c,d)|\big)^{-1}.$$

Then

$$|\varphi(x,y)| \leq k\|(x,y)\|^p, \quad (x,y) \in \mathcal{U}.$$

Now the rest of the proof is similar to the proof of Theorem 3. □

By a same method of the proof of Rassias's Theorem we can prove the next result.

Theorem 6. *Let $\mathcal{U}$ and E be two Banach spaces, $\epsilon \geq 0$ and $0 \leq p < 1$. If a mapping $\varphi : \mathcal{U} \longrightarrow E$ satisfies*

$$\|\varphi(a+x, b+y) - \varphi(a,b) - \varphi(x,y)\| \leq \delta\big(\|(a,b)\|^p + \|(x,y)\|^p\big),$$

for all $(a,b), (x,y) \in \mathcal{U}$, then there is a unique bi-additive mapping $F : \mathcal{U} \longrightarrow E$ such that

$$\|F(x,y) - \varphi(x,y)\| \leq \frac{2\delta}{|2-2^p|}\|(x,y)\|^p,$$

for all $(x,y) \in \mathcal{U}$.

Corollary 7. *Let a functional $\varphi : \mathcal{U} \longrightarrow \mathbb{C}$ satisfies (2) and (7) for some $\delta \geq 0$ and $p \geq 0$ with $p \neq 1$. Then φ is bi-ring homomorphism or*

$$|\varphi(x,y)| \leq \frac{2\delta}{|2-2^p|}\|(x,y)\|^p,$$

for all $(x,y) \in \mathcal{U}$.

Proof. By Theorem 6 there is a unique bi-additive mapping $F : \mathcal{U} \longrightarrow \mathbb{C}$ such that

$$|F(x,y) - \varphi(x,y)| \leq \frac{2\delta}{|2-2^p|}\|(x,y)\|^p, \tag{8}$$

for all $(x,y) \in \mathcal{U}$. Suppose that φ is not bi-ring homomorphism. Then by Theorem 5 we get (3) for all $(x,y) \in \mathcal{U}$. Hence

$$|F(x,y)| \leq |\varphi(x,y)| + \frac{2\delta}{|2-2^p|}\|(x,y)\|^p \leq k\|(x,y)\|^p, \tag{9}$$

for all $(x, y) \in \mathcal{U}$, where

$$k = \frac{1 + \sqrt{1 + 4\delta}}{2} + \frac{2\delta}{|2 - 2^p|}.$$

We claim that $F(x, y) = 0$ for all $(x, y) \in \mathcal{U}$. Let $(x, y) \in \mathcal{U}$ be arbitrarily and set $t = \frac{|1-p|}{1-p}$. Then $t = \pm 1$. If $t = 1$, then by (9) for each $n \in \mathbb{N}$, we have

$$|F(nx, ny)| \leq k\|(nx, ny)\|^p = kn^p\|(x, y)\|^p.$$

Thus,

$$|F(x, y)| \leq kn^{p-1}\|(x, y)\|^p \longrightarrow 0,$$

as $n \longrightarrow \infty$, because $p - 1 < 0$. Since $(x, y) \in \mathcal{U}$ was arbitrary, we have $F(x, y) = 0$ for all $(x, y) \in \mathcal{U}$. So by (8) we get

$$|\varphi(x, y)| \leq \frac{2\delta}{|2 - 2^p|}\|(x, y)\|^p, \quad (x, y) \in \mathcal{U}.$$

This complete the proof for the case $t = 1$. The proof of the case $t = -1$ is completely similar. □

References

[1] R. Badora, On approximate ring homomorphisms, *J. Math. Anal. Appl.*, **276** (2002), 589-597.

[2] J. A. Baker, The stability of the cosine equation, *Proc. Amer. Math. Soc.*, **80** (1980), 411-416.

[3] Z. Gajda, On stability of additive mappings, *Inter. J. Math. Math. Sci.*, **14** (1991), 431-434.

[4] P. Gavruta, A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings, *J. Math. Anal. Appl.*, **184** (1994), 431-436.

[5] D. H. Hyers, On the stability of the linear functional equation, *Proc. Nat. Acad. Sci. U. S. A.*, **27** (1941), 222-224.

[6] K. Jarosz, Almost multiplicative functionals, *Studia Math.*, **124** (1997), 37-58.

[7] Th. M. Rassias, On the stability of the linear mapping in Banach spaces, *Proc. Amer. Math. Soc.*, **72** (1978), 297-300.

[8] Th. M. Rassias, On the stability of functional equations in Banach spaces, *J. Math. Anal. Appl.*, **251** (2000), 264-284.

[9] Th.M. Rassias, On the stability of functional equations and a problem of Ulam, *Acta Appl. Math.*, **62** (2000), 23-130.

[10] S. M. Ulam, *Problems in Modern Mathematics*, Chap. VI, Wiley, New York, 1960.