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MATHEMATICAL MODELLING OF THE GLACIATION PROCESS

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Abstract: The problem of mathematical modelling of gas-pipeline glaciation process is considered. The one-dimensional case is presented. Two methods of numerical solution are realized. The first is a front-tracking method with variable time step. The method is realized for initial-boundary value problem with Stefan condition for linear heat equation with unknown moving boundary. Another method is a continuous method based on the solution of the problem for nonlinear heat equation in domain with fixed boundaries and the Dirichlet condition. By the comparison of the results obtained by two methods the optimal values of continuous method parameters are defined. These values may be realized in application of continuous method in multidimensional cases.

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1. Introduction

In northern seas gas-pipelines are operated at hyper-pressure conditions and high gas rate in pipes. In these conditions gas temperature in the pipeline may be lower than temperature of phase transition, so the glaciation of the pipeline become possible. This process can be modelled by linear heat equation in the domain with unknown moving boundary of phase change and the Stefan condition on this boundary [1].

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There are two main approaches to the numerical solution of problems with Stefan condition. The first approach is based on the *tracking* of phase change boundary. There are many realizations of this approach. In [2], [3] a variable time-stepping method, when phase change boundary moves from one space node to another is proposed. In [4] authors proposed an approach based on space grid with variable step, but with fixed number of space intervals between fixed and moving boundaries. In [5], [6] a front-tracking method is considered. As it is discussed in [7], these methods may be considered as effective only for one-dimensional problems, their extensions to two- and three-dimensional problems lead to serious difficulties.

Another approach is known as a *continuous* method. This method is based on the transformation of the Stefan problem for linear equation with unknown moving boundary to the problem for the nonlinear equation in domain with fixed boundaries with Dirichlet or Neumann boundary conditions. The coefficients of this equation are considered as discontinuous. The discontinuities are smoothed in some small interval in the space of temperature values. The approach is proposed in [8], [9], its enthalpy formulation is described and discussed in [1], [10]. In applications of continuous method the Stefan condition on the boundary of phase transition is automatically satisfied and the coordinates of this boundary are defined by the values of temperature equal to the temperature of phase transition.

Presented paper is dedicated to the comparison of two computational procedures for modelling of the glaciation process in one-dimensional case. In Section 2 the model of glaciation process is presented. In Section 3 numerical methods for the Stefan problem solution are described. In Section 4 the numerical results are analyzed. Some concluding remarks are made in Section 5.

2. Model of Glaciation Process

The model is constructed with the following main assumptions: the process only in cross-section of the pipeline is considered, extended hydrodynamical characteristics are considered as constants and temperature in the cross-section of gas flow is dependent only on time. According to these assumptions, mathematical model of glaciation process is presented as:

$$\frac{\partial T}{\partial t} = \frac{a}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right), \quad r \in (R, R+y), \quad t > 0, \tag{1}$$

$$T(R,t) = T_0(t), \quad T(R+y,t) = T_*, \quad t > 0, \tag{2}$$

$$\lambda \frac{\partial T}{\partial r}(R+y,t) - q = Q\rho \frac{dy}{dt}, \quad t > 0, \tag{3}$$

$$y(0) = 0, \tag{4}$$

where $T = T(r,t)$ is a temperature in the ice layer, t is a time, r is a radial coordinate, R is a outer radius of a pipeline, $a = \lambda/(c\rho)$, where ρ is the ice density, λ is a coefficient of heat conductivity, c is a specific heat, $y = y(t)$ is the thickness of ice layer, T_* is a phase transition temperature. Eq. (3) represents the Stefan condition, where Q is the latent heat of fusion for ice, q is a heat flux from seawater to the ice layer. The coordinate of the phase change boundary is obtained as: $r = R + y(t)$.

3. Numerical Methods for the Solution of Stefan Problem

3.1. Front-Tracking Method

For the numerical solution eq. (1) is approximated on the constructed grid by the following implicit finite-difference scheme:

$$\frac{T_i^{n+1} - T_i^n}{\tau^{n+1}} = \frac{a}{hr_i}\left(\frac{T_{i+1}^{n+1} - T_i^{n+1}}{h} - \frac{T_i^{n+1} - T_{i-1}^{n+1}}{h}\right), \tag{5}$$

where $T_i^n \approx T(r_i, t_n)$ are the approximated values of temperature at grid nodes, h is a step of grid on spatial variable r, r_i is a node of this grid.

Scheme (5) is realized with variable time step τ_{n+1}, which is defined by the following rule: ice layer on time interval $[t_n, t_n + \tau_{n+1}]$ should be increased on the length h. This grid is constructed on the space interval $[R, R + y_*]$, where the value of y_* is set in advance. Let this grid consists of $(N+1)$ nodes r_i, $i = \overline{0, N}$.

The procedure for computation of τ_{n+1} is realized as a special iterative process. The initial value of time step τ_{n+1}^0 is set in advance. After that iterative cycle on $s = 0, 1, 2, \ldots$ is realized by following way. Problem for the heat equation (1) with boundary conditions (2) is solved on the interval $(R, R + (n+1)h)$ by scheme (5). Values in the boundary nodes r_0 and r_N are obtained from boundary conditions: $T_0^{n+1,s} = T_0(t_n + \tau_s)$, $T_N^{n+1,s} = T_*$. The linear system on values of T_i^{n+1}, $i = \overline{1, N-1}$ is solved by tridiagonal matrix algorithm. After the solution of this system the values of temperature $T_i^{n+1,s}$ on the s-th stage are obtained. After that the values of heat fluxes at boundary points are computed:

$$q_1 = \frac{\lambda\left(T_1^{n+1,s} - T_0^{n+1,s}\right)}{h}, \quad q_2 = \frac{\lambda\left(T_N^{n+1,s} - T_{N-1}^{n+1,s}\right)}{h}.$$

The thickness of the ice layer at s-th step is obtained from eq. (3) as:

$$h^s = \frac{\tau_{n+1}^s(q_2 - q)}{\rho Q}.$$

The value of stepsize on next iteration is defined as [3]:

$$\tau_{n+1}^{s+1} = \tau_{n+1}^s + \frac{\rho Q(h - h^s)}{q_1 - q}. \tag{6}$$

Iterative process (6) at every n is stopped according to the following condition: $|\tau_{n+1}^{s+1} - \tau_{n+1}^s| \leq \varepsilon$, where the value of ε is set in advance.

The first step τ_1 is computed by following formulas:

$$q_1 = \frac{\lambda(T_* - T_0^0)}{h}, \quad \tau_1^0 = \frac{\rho Q}{\lambda(T_* - T_0^0) - qh},$$

$$h^0 = \frac{\tau_1^0(q_1 - q)}{\rho Q}, \quad \tau_1 = \tau_1^0 + \frac{\rho Q(h - h^0)}{q_1 - q}.$$

For other time steps τ_n, $n \geq 2$ the starting value for iterative process τ_n^0 is computed as: $\tau_n^0 = \tau_{n-1}$. The length of the ice layer at moment $t = t_n$ is equal to nh.

3.2. Continuous Method

According to the approach, presented in [1], [8], in radial layer $[R, R_1]$, where $R_1 = R + \Delta R$, two phases — ice and seawater are defined by constant densities ρ_1 and ρ_2, specific heat coefficients c_1 and c_2, coefficients of heat conductivity λ_1 and λ_2 are considered.

The process of heat conduction in this layer is described by the following nonlinear equation [8]:

$$S\frac{\partial T}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\lambda\frac{\partial T}{\partial r}\right), \quad r \in (R, R_1), \quad t > 0, \tag{7}$$

where the expression for S is presented as: $S = c\rho + \rho Q\delta(T - T_*)$, where $\delta(T - T_*)$ is a Dirac delta function, c and ρ are considered as discontinuous functions in the layer. Delta function is approximated by function $\delta(T - T_*, \Delta) \geq 0$, where $\Delta > 0$ is a half of the length of the following interval on temperature values: $(T_* - \Delta, T_* + \Delta)$, where $T < T_* - \Delta$ corresponds to ice and $T > T_* + \Delta$ corresponds to the seawater. In other values of T $\delta(T - T_*, \Delta) \equiv 0$.

According to the described above representation S is written as:

$$S(T) = \begin{cases} c_1\rho_1, \quad T < T_* - \Delta, \\ \frac{c_1\rho_1 + c_2\rho_2}{2} + \rho_1 Q \delta(T - T_*, \Delta), \quad |T - T_*| < \Delta, \\ c_2\rho_2, \quad T > T_* + \Delta, \end{cases}$$

and the coefficient λ in eq. (7) is expressed by following formulae:

$$\lambda(T) = \begin{cases} \lambda_1, \quad T < T_* - \Delta, \\ \frac{\lambda_1 + \lambda_2}{2}, \quad |T - T_*| < \Delta, \\ \lambda_2, \quad T > T_* + \Delta. \end{cases}$$

Boundary conditions for nonlinear eq. (7) are written as:

$$T(R, t) = T_0(t), \quad T(R_1, t) = T_w, \tag{8}$$

where T_w is a temperature of seawater.

The initial condition is written as:

$$T(0, r) = f(r), \quad r \in [R, R_1], \tag{9}$$

where $f(r)$ is a solution of stationary problem for temperature distribution in radial layer $[R, R_1]$:

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right) = 0, \quad T(R) = T_0(0), \quad T(R_1) = T_w. \tag{10}$$

Solution of (10) is written as: $T(r) = A\ln(r) + B$, where A and B are defined from boundary conditions. The value of ΔR is defined as $\Delta R = l + R_2$, where the thickness of ice layer (e.g. 2 cm) is defined by interval $[R, R + l]$ and the value of R_2 is varied.

There are two types of the function $\delta(T - T_*, \Delta)$ for approximation of the delta function are considered:

$$\delta(T - T_*, \Delta) = \begin{cases} \frac{1}{2\Delta}, \quad |T - T_*| < \Delta, \\ 0, \quad |T - T_*| > \Delta, \end{cases} \tag{11.1}$$

$$\delta(T-T_*,\Delta)=\begin{cases}\frac{3}{4\Delta^3}(\Delta^2-(T-T_*)^2), & |T-T_*|<\Delta,\\ 0, \quad |T-T_*|>\Delta.\end{cases} \tag{11.2}$$

The nonlinear eq. (7) is approximated by implicit scheme:

$$S(T_i^n)\frac{T_i^{n+1}-T_i^n}{\tau}=\frac{1}{hr_i}\left(a_{i+1}\frac{T_{i+1}^{n+1}-T_i^{n+1}}{h}-a_i\frac{T_i^{n+1}-T_{i-1}^{n+1}}{h}\right), \tag{12}$$

where $a_i = 0{,}5(r_i\lambda(T_i^n)+r_{i-1}\lambda(T_{i-1}^n))$. Linear system (12) is solved by tridiagonal matrix algorithm.

The time evolution of the gas temperature in the pipeline may be approximated by following function [11]:

$$T_0(t)=\frac{m_1}{m_2+t}+m_3,$$

where $m_1 = 15120$, $m_2 = 5040$, $m_3 = 268$.

The values of the parameters of ice-seawater system are presented in SI system: $R = 0.67$, $c_1 = 2100$, $\rho_1 = 928$, $Q = 335000$, $k_1 = 2.3$, $c_2 = 4200$, $\rho_2 = 1025$, $k_2 = 0.56$, $T_* = 271$.

The results of the numerical solution of problem (1)–(4) by the front-tracking method for $h = 0.001$ m and $\varepsilon = 10^{-5}$ as the values of time of glaciation of ice layers are presented at table 1.

Table 1. The time of glaciation of spatial layers $[R, R+l]$ obtained by the front-tracking method (min.)

Layer (l)	1 mm	1 cm	2 cm	3 cm	4 cm	5 cm	6 cm
Time	8.6	113.5	298.9	571.0	944.6	1435.9	2063.6

Table 2. The time of glaciation of spatial layers of length $R + l$ obtained by the continuous method at various R_2 and delta-functions approximations.

R_2	1 cm		1.25 cm		1.5 cm	
Layer (l)	(11.1)	(11.2)	(11.1)	(11.2)	(11.1)	(11.2)
1 mm	8.7450	8.8110	8.6460	8.7120	8.7780	8.8330
1 cm	114.96	115.61	113.55	114.09	112.82	113.35
2 cm	302.54	304.16	299.03	300.19	295.92	296.88
3 cm	578.25	582.66	567.38	569.92	558.61	560.43
4 cm	953.77	962.15	931.09	936.60	912.25	915.90
5 cm	1438.3	1454.4	1400.4	1411.6	1367.4	1375.1
6 cm	2061.2	2088.2	1978.0	1998.4	1939.5	1954.5

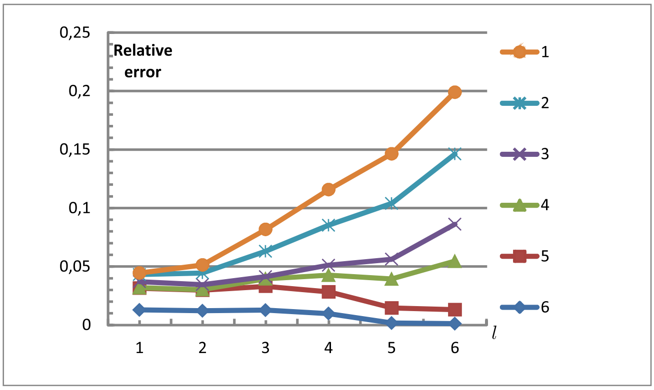

Figure 1: The relative errors of the time of glaciation of layers $[R, R+l]$. 1 and 2 — case of $R_2 = 1.5$ cm, 3, 4 — $R_2 = 1.25$ cm, 5, 6 — $R_2 = 1$ cm. 1, 3, 5 corresponds to quadratic approximation of Dirac delta function, 2, 4, 6 — to constant approximation.

Calculations by the front-tracking method are realized at different grids on time and space variables. To compare with the results presented at table 1, the interval of 2200 min. is considered. At table 2 the results of computations on spatial grid with 500 nodes and time grid with $2 \cdot 10^5$ nodes are presented. Various values of R_2 and two types of approximation for delta functions are considered — the case of constant function (11.1) and the case of quadratic function (11.2).

As it can be seen, in most cases the results obtained by continuous method at some values of R_2 and approximation (11.1) are closer to the results presented at table 1. On fig. 1 the plots of relative errors are presented. As it can be seen, the results obtained for the case of approximation (11.2) are closer only for $l = 3$ cm and $R_2 = 1.25$ cm.

The main advantage of the continuous method is its simple spreading to multidimensional case. For 2D and 3D cases presented values of Δ, l and R_2 with approximation (11.1) may be applied.

4. Conclusion

The problem of the modelling of gas-pipeline glaciation process is considered. The one-dimensional case is presented. Two methods of numerical solution are realized. The first is a front-tracking method with variable time step. The method is realized for initial-boundary value problem with Stefan condition for linear heat equation with unknown moving boundary. Another method is a continuous method based on the solution of the problem for nonlinear heat equation in domain with fixed boundaries and the Dirichlet condition. By the comparison of the results obtained by two methods the optimal values of continuous method parameters are defined. These values may be realized in application of continuous method in multidimensional cases.

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