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CONTROLLABILITY RESULTS FOR NONLINEAR IMPULSIVE FUZZY NEUTRAL INTEGRODIFFERENTIAL EVOLUTION SYSTEMS

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Abstract: In this paper, author's studied the controllability results for nonlinear fuzzy neutral integrodifferential systems. Moreover we study the fuzzy solution for the normal, convex, upper semicontinuous, and compactly supported interval fuzzy number. The results are obtained by using the Banach fixed point theorem and evolution family of functions.

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1. Introduction

In various fields of science and engineering, many problems that are related to linear viscoelasticity, nonlinear elasticity, heat conduction in materials with memory and Newtonian or non-Newtonian fluid mechanics have mathematical models. Popular models essentially fall into two categories: the differential mod-

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els and the integrodifferential models. Our work centers around the problems described by the integrodifferential models. A large class of scientific and engineering problems modelled by partial differential equations can be expressed in various forms of differential or integrodifferential equations in abstract spaces.

Control theory, on the other hand, is that branch of application-oriented mathematics that deals with the basic principles underlying the analysis and design of control systems. To control an object implies the influence of its behaviour so as to accomplish a desired goal. In order to implement this influence, practitioners build devices and their interaction with the object being controlled is the subject of control theory. The controllability problem may be formulated as follows: Consider an evolution system either described in terms of partial or ordinary differential equations. We are allowed to act on the trajectories of the system by means of a suitable control. Then, given a time $t \in [0,T]$ and initial and final states, we have to find a control such that the solution matches both the initial state at time$t = 0$ and the final one at time $t = T$:

The fuzzy set theory was intended to improve the oversimplified model; thereby, developing a more robust and flexible model in order to solve real-world complex systems involving human aspects. It is much more adaptable to diverse problem structures and better suited to model human evaluation and decision making processes, that classical mathematics. It can also be considered as a modeling language, well suited for situations in which fuzzy relations, criteria, and phenomena exist. From the viewpoint of application in science and engineering, it was undoubtedly the book written by Zimmermann ([22]) which played an outstanding role in the development of the subject which can be called fuzzy sets decision making and expert systems.

The role of the membership function is to represent an individual and subjective human perception as a member of a fuzzy set. A fuzzy set has several membership functions $\mu_{\mathcal{A}}$, defined as a function from a well defined universe, X into a unit interval, 0 through 1. The function $\mu_{\mathcal{A}} : X \to [0,1]$ is defined by

$$\mu_{\mathcal{A}}(x) = \begin{cases} 1 & if \;\; x \in \mathcal{A} \\ 0 & if \;\; x \notin \mathcal{A}. \end{cases}$$

The value zero represents complete non-membership, the value one is used to represents complete membership, and the values between are used to represents intermediate degrees of the membership. The mapping $\mathcal{A}$ is called the membershi function of fuzzzy set $\mathcal{A}$.

Example 1.1. The membership function of the fuzzy set of real numbers "close to one" can be defined as

$$\mathcal{A}(t) = exp(-\beta(t-1)^2), \quad \text{where } \beta \text{is a positive real number.}$$

Example 1.2. Consider the membership function of the fuzzy set of real numbers "close to zero" defined as $B(x) = \frac{1}{1-x^3}$. Using this function, we can determine the membership grade of real number in this fuzzy set, which signifies the degree to which that number is close to zero. For instance, the number 3 is assigned a grade of 0.035, the number 1 a grade of 0.5, and the number 0 a grade of 1.

Fuzzy differential and integrodifferential equations are a field of interest, due to their applicability to the analysis of phenomena with memory where imprecision is inherent. However, the concrete example is the radiocardiogram, where the two compartments correspond to the left and right ventricles of the pulmonary and systematic circulation. Pipes coming out from and returning into the same compartment may represent shunts, and the equation representing this model is a nonlinear neutral Volterra integrodifferential equation. These classes of equations also arise, for example, in the study of problems such as heat conduction in materials with memory or population dynamics for spatially distributed populations; The integral of fuzzy mapping was proposed by Dubois and Prade ([6, 7, 8]). The H-differentiability of fuzzy mapping was introduced by Puri and Ralescu ([17]). Especially, one always describes a model which possesses hereditary properties by integrodifferential equations in practice. Generally, several systems are mostly related to uncertainty and inaccuracy. Kaleva ([13]) studied the existence and uniqueness of solution for the fuzzy differential equations on $\mathbb{E}_N$, where $\mathbb{E}_N$ is normal, convex, upper semicontinuous and compactly supported fuzzy sets in $\mathcal{R}^n$. Seikkala ([21]) proved the existence and uniqueness of fuzzy solution for the following equations:

$$\dot{x} = f(t, x(t)), \quad x(0) = x_0,$$

where f is continuous mapping from $\mathcal{R}^+ \times \mathcal{R}$ into $\mathcal{R}$ and x_0 is a fuzzy number in E_1. Alikharni et al. ([1]) proved the existence of global solutions to nonlinear fuzzy Volterra integrodifferential equations. Balachandran and Duar ([3]) established the existence of perturbed fuzzy integral equations and fuzzy delay differential equations with nonlocal conditions. Diamond and Kloeden ([5]) proved the fuzzy optimal control for the following system

$$\dot{x} = a(t)x(t) + u(t), \quad x(0) = x_0,$$

where $x(\cdot)$, $u(\cdot)$ are nonempty compact interval valued functions on $\mathbb{E}_1$. The study of abstract nonlocal semilinear initial value problems was initiated by Byszewski ([4, **?**]). Because it is demonstrated that the nonlocal problems have better effects in applications than the classical Cauchy problems. Ding and

Kandel ([9]) analyzed a way to combine differential equations with fuzzy sets to form a fuzzy logic systems called a fuzzy dynamical system, which can be regarded to form a fuzzy neutral functional differential equations. Kuwun and Park ([14]) proved the existence of fuzzy optimal control for the nonlinear fuzzy differential system with nonlocal initial condition in $\mathbb{E}_N$ using by Kuhn Tucker theorems.

Consider the first order nonlinear impulsive fuzzy neutral integrodifferential system ($v(t) \equiv 0$) with nonlocal conditions of the form

$$\begin{aligned} \frac{d}{dt}(x(t) - h(t, x(t))) &= A(t)\Big[x(t) + \int_0^t G(t,s)x(s)ds\Big] \\ &\quad + f(t, x(t)) + v(t), \quad t \neq t_k, \quad t \in J = [0, b], \quad (1.1) \\ x(0) + g(x) &= x_0, \quad (1.2) \\ \Delta(x(t_k)) &= I_k(x(t_k^-)), \quad (1.3) \end{aligned}$$

where $A(t) : J \to \mathbb{E}_N$ is fuzzy coefficient, $\mathbb{E}_N$ is the fuzzy set of all upper semicontinuous, convex, normal fuzzy numbers with bounded $\alpha-$ level intervals, $f : J \times \mathbb{E}_N \to \mathbb{E}_N$, $h : J \times \mathbb{E}_N \to \mathbb{E}_N$, $g : \mathbb{E}_N \to \mathbb{E}_N$ are all nonlinear functions, $G(t, s)$ is $n \times n$ continuous matrix such that $\frac{dG(t,s)x}{dt}$ is continuous, for $x \in \mathbb{E}_N$ and $t, s \in J$ with $\|G(t, s)\| \leq k$, $k > 0$, $v : J \to \mathbb{E}_N$ is control function. $I_k : \mathbb{E}_N \to \mathbb{E}_N$, $\Delta x|_{t=t_k} = x(t_k^+) - x(t_k^-)$, for all $k = 1, 2, \ldots, m$; $0 = t_0 < t_1 < t_2 < \ldots < t_m < t_{m+1} = b$; and the nonlocal function $g : [\mathcal{PC}([0, b], \mathbb{E}_N)] \to \mathbb{E}_N$ are given appropriate functions.

Denote $J_0 = [0, t_1]$, $J_k = (t_k, t_{k+1}]$, $k = 1, 2, \ldots, m$ and define the following space: Let $\mathcal{PC}([0, b], \mathbb{E}_N)$=\{ $x : x$ is a function from $[0, b]$ into $\mathbb{E}_N$ such that $x(t)$ is continuous at $t \neq t_k$ and left continuous at $t = t_k$ and the right limit $x(t_k^+)$ exists for $k = 1, 2, \ldots, m$\}. Similarly as in ([10]), we see that $\mathcal{PC}([0, b], \mathbb{E}_N)$ is a Banach space with norm $\|x\|_{\mathcal{PC}} = \sup_{t\in[0,b]} \|x(t)\|$.

For the family $\{A(t) : 0 \leq t \leq b\}$ of linear operators, we assume the following hypotheses:

(A1) $A(t)$ is a closed linear operator and the domain $D(A)$ of $\{A(t) : 0 \leq t \leq b\}$ is dense in $\mathbb{E}_N$ and independent of t.

(A2) For each $t \in [0, b]$, the resolvent $R(\lambda, A(t)) = (\lambda I - A(t))^{-1}$ of $A(t)$ exists for all λ with $Re\ \lambda \leq 0$ and $\|R(\lambda, A(t))\| \leq C(|\lambda| + 1)^{-1}$.

(A3) For any $t, s, \tau \in [0, b]$, there exists a $0 < \delta < 1$ and $L > 0$ so that

$$\|(A(t) - A(\tau))A^{-1}(s)\| \leq L|t - \tau|^{\delta}.$$

Statements $(A1)-(A3)$ implies that there exists a family of evolution operator $U(t,s)$.

Motivated by all the above literature approach, the goal of this paper is to use the fixed point theorem to obtain the controllability results for nonlinear impulsive fuzzy neutral integrodifferential systems with nonlocal condition.

2. Preliminaries

In this section, we give some basic definitions, notations, lemmas and result which are used in the sequel.

2.1. Fuzzy Sets and Numbers

Let $\mathcal{F}_k(\mathcal{R}^n)$ denote the family of all nonempty compact convex subset of $\mathcal{R}^n$ and define the addition and scalar multiplication in $\mathcal{F}_k(\mathcal{R}^n)$ as usual. Let A and B be two nonempty bounded subsets of $\mathbb{R}^n$. The distance between A and B is defined by the Hausdorff metric

$$d(A,B)=\max\Big\{\sup_{a\in A}\inf_{b\in B}\|a-b\|,\ \sup_{b\in B}\inf_{a\in A}\|a-b\|\Big\},$$

where $\|\cdot\|$ denotes the usual Euclidean norm in $\mathcal{R}^n$. Then it is clear that $\Big(\mathcal{F}_k(\mathcal{R}^n),\ d\ \Big)$ becomes a complete and separable metric space.

Let x be a point in $\mathcal{R}^n$ and A be a nonempty subsets of $\mathcal{R}^n$. We define the Hausdroff separation of B from A by be $d(x,A)=\inf\{\|x-a\|:a\in A\}$. Now let A and B be nonempty subsets of $\mathbb{R}^n$. We define the Hausdroff separation of B from A by

$$d^*_H(B,A)=\sup\{d(b,A):b\in B\}.$$

In general, $d^*_H(A,B)\neq d^*_H(B,A)$. We define the Hausdroff distance between nonempty subsets of A and B of $\mathcal{R}^n$ by $d_H(A,B)=\max\{d^*_H(A,B),d^*_H(B,A)\}$.

The supremum metric d_∞ on $\mathbb{E}_N$ is defined by $d_\infty(v,w)=\sup\{d_H([v]^\alpha,[w]^\alpha):\alpha\in(0,1]\}$, for all $v,w\in\mathbb{E}_N$, and is obviously metric on $\mathbb{E}_N$. The supremum metric $\mathcal{H}_1$ on $\mathcal{C}(J,\mathbb{E}_N)$ is defined by

$$\mathcal{H}_1(x,y)=\sup\{d_\infty(x(t),y(t):t\in J),\ \text{for all } x,y\in\mathcal{C}(J:\mathbb{E}_N)\}.$$

Let $I=[0,1]\subseteq\mathcal{R}$ be a compact interval and denote $\mathbb{E}_N=\Big\{v:\mathcal{R}^n\to[0,1]|\ v \text{ satisfies } (i)-(iv)\ \text{ below}\Big\};$ where

(i) v is normal i.e. there exists an $x_0 \in \mathbb{R}^n$ such that $v(x_0) = 1$;

(ii) v is fuzzy convex;

(iii) v is upper semicontinuous;

(iv) $[v]^0 = cl\{x \in \mathcal{R}^n : \ v(x) > 0\}$ is compact.

For $0 < \alpha \leq 1$, denote $[v]^\alpha = \{t \in \mathcal{R}^n | \ v(t) \geq \alpha\}$. Then from $(i) - (iv)$, it follows that the $\alpha-$ level set $[v]^\alpha \in \mathcal{F}_k(\mathcal{R}^n)$, for all $0 \leq \alpha \leq 1$.

According to Zadeh's extension principle, we have addition and scalar multiplication in fuzzy number space $\mathbb{E}_N$ as $[v+w]^\alpha = [v]^\alpha + [w]^\alpha$; $[kv]^\alpha = k[v]^\alpha$, where $v, w \in \mathbb{E}_N, \ k \in \mathcal{R}$ and $0 \leq \alpha \leq 1$. Define a mapping $\mathcal{D} : \mathbb{E}_N \times \mathbb{E}_N \to \mathcal{R}^+$ by $\mathcal{D}(v,w) = \sup_{0 \leq \alpha \leq 1} d\Big([v]^\alpha, [w]^\alpha\Big)$, where d is the Hausdorff metric non empty compact sets in $\mathcal{R}^n$. Then it is easy to see that $\mathcal{D}$ is a metric in $\mathbb{E}_N$. Using the results in [18], we know that

(i) $(\mathbb{E}_N, \mathcal{D})$ is a complete metric space;

(ii) $\mathcal{D}(v+z, z+w) = \mathcal{D}(v,w)$, for all $v, z, w \in \mathbb{E}_N$;

(iii) $\mathcal{D}(kv, kw) = |k| \ \mathcal{D}(v,w)$, for all $v, w \in \mathbb{E}_N, \ k \in \mathcal{R}$.

A fuzzy number a in real line $\mathcal{R}$ is a fuzzy set characterized by a membership function $\chi_a : \mathcal{R} \to [0,1]$. A fuzzy number a is expressed as $a = \int_{x \in \mathbb{R}} \frac{\chi_a}{x}$ with the understanding that $\chi_a(x) \in [0,1]$, represents the grade of membership of x in a and $\int$ denotes the union of $\frac{\chi_a}{x}$.

Result 2.1.1. [16] *Let $\mathbb{E}_N$ be the set of all upper semicontinuous convex normal fuzzy numbers with bounded $\alpha-$ level intervals. This means that if $a \in \mathbb{E}_N$, then $\alpha-$ level set $[a]^\alpha = \{x \in \mathcal{R} : a(x) \geq \alpha, 0 \leq \alpha \leq 1\}$, is a closed bounded interval, which we denote by $[a]^\alpha = [a_q^\alpha, a_r^\alpha]$, and there exists $t_0 \in \mathcal{R}$, such that $a(t_0) = 1$.*

Result 2.1.2. [16] *Two fuzzy numbers a and b are called equal $a = b$, if $\chi_a(x) = \chi_b(x)$, for all $x \in \mathbb{R}$. It follows that $a = b \Leftrightarrow [a]^\alpha = [b]^\alpha$, for all $\alpha \in (0,1]$.*

Lemma 2.1.3. [21] *Let $[a_q^\alpha, a_r^\alpha]$, $0 < \alpha \leq 1$, be a given family of nonempty intervals. If*

$$[a_q^\beta, a_r^\beta] \subset [a_q^\alpha, a_r^\alpha], \ for \ 0 < \alpha \leq \beta; \quad [\lim_{k \to \infty} a_q^{\alpha_k}, \lim_{k \to \infty} a_r^{\alpha_k}] = [a_q^\alpha, a_r^\alpha],$$

whenever (α_k) is non-decreasing sequence converting to $\alpha \in (0,1]$, then the family $[a_q^\alpha, a_r^\alpha]$, $0 < \alpha \leq 1$, are the α -level sets of a fuzzy number $a \in \mathbb{E}_N$.

We consider $\mathcal{C}(J,\ \mathbb{E}_N)$ the space of all continuous fuzzy functions defined on $[0,b] \subset \mathbb{R}$ into $\mathbb{E}_N$, where $b > 0$. For $v, w \in \mathcal{C}(J,\ \mathbb{E}_N)$, we define the metric $\mathcal{H}(v,w) = \sup_{t\in[0,b]} \mathcal{D}\Big(v(t), w(t)\Big)$. Then $(\mathcal{C}(J, \mathbb{E}_N), H)$ is a complete metric space.

We recall some measurability, integrability properties for fuzzy set-valued mappings (see [13]). Let $\mathcal{I} = [0,1] \subset \mathbb{R}$ be a compact interval.

Definition 2.1.4. [2] Let $F(t)$ be a nonempty subset of $\mathcal{R}^n$. Let $\mathcal{F}$ be the set of all point-valued functions $f : \mathcal{I} \to \mathcal{R}^n$ such that f is integrable over $\mathcal{I}$ and $f(t) \in F(t)$, for all $t \in \mathcal{I}$. It is denoted by $\displaystyle\int_{\mathcal{I}} F(t)dt$, is defined by the equation

$$\int_{\mathcal{I}} F(t)dt = \Big\{ \int_{\mathcal{I}} f(t)dt \ :\ f \in \mathcal{F} \Big\}.$$

Definition 2.1.5. [13] A mapping $F : \mathcal{I} \to \mathbb{E}_N$ is strongly measurable if, for all $\alpha \in [0,1]$ the set-valued function $F_\alpha : \mathcal{I} \to \mathcal{F}_k(\mathcal{R}^n)$ defined by $F_\alpha(t) = [F(t)]^\alpha$ is Lebesgue measurable when $\mathcal{F}_k(\mathcal{R}^n)$ has the topology induced by the Hausdorff metric d.

Definition 2.1.6. [13] A mapping $F : \mathcal{I} \to \mathbb{E}_N$ is called integrably bounded if there exists an integrable function k such that $\|x\| \leq k(t)$, for all $x \in F_0(t)$.

Definition 2.1.7. [18] The integral of a fuzzy mapping $F : [0,1] \to \mathbb{E}_n$ is defined level wise by

$$\begin{aligned} \Big[\int_{\mathcal{I}} F(t)dt \Big]^\alpha &= \Big\{ \int_{[0,1]} F_\alpha(t)dt \Big\} \\ &= \Big\{ \int_{[0,1]} f(t)dt : f : [0,1] \to \mathcal{R}^n \text{ is a measurable} \\ &\qquad \text{selection for } F_\alpha \Big\}, \forall\ \alpha \in [0,1]. \end{aligned}$$

It has been proved by Puri and Ralescu [18] that a strongly measurable and integrably bounded mapping $F : \mathcal{I} \to \mathbb{E}_N$ is integrable (i.e.$\int_{\mathcal{I}} F(t)dt \in \mathbb{E}_N$).

We assume the following to prove the existence of solution of the equation (1.1) – (1.3).

(H1) The nonlinear function $g : J \times \mathbb{E}_N \to \mathbb{E}_N$ is a continuous function and satisfies the inequality

$$d_H([g(x(\cdot))]^\alpha, [g(y(\cdot))]^\alpha) \leq \delta_g d_H([x(\cdot)]^\alpha, [y(\cdot)]^\alpha),\ x, y \in \mathbb{E}_N.$$

(H2) The inhomomogeneous term $f : J \times \mathbb{E}_N \to \mathbb{E}_N$ is continuous function and satisfies a global Lipschitz

$$d_H([f(s, x(s))]^\alpha, [f(s, y(s))]^\alpha) \leq \delta_f d_H([x(\cdot)]^\alpha, [y(\cdot)]^\alpha),\ x, y \in \mathbb{E}_N.$$

(H3) The nonlinear function $h : J \times \mathbb{E}_N \to \mathbb{E}_N$ is continuous function and satisfies the global Lipschitz condition

$$\begin{aligned}(i)\ \ d_H([h(s,x(\cdot))]^\alpha,[h(s,y(\cdot))]^\alpha) &\leq \delta_h d_H([x(\cdot)]^\alpha,[y(\cdot)]^\alpha)\\ (ii)\ \ d_H([A(s)h(s,x(\cdot))]^\alpha,[A(s)h(s,y(\cdot))]^\alpha) &\leq M_A\delta_h d_H([x(\cdot)]^\alpha,[y(\cdot)]^\alpha).\end{aligned}$$

(H4) There exists $\delta_i > 0$ and $\delta_I > 0$ such that

$$d_H([I_k(x(\cdot))]^\alpha,[I_k(y(\cdot))]^\alpha) \leq \delta_i d_H([x(\cdot)]^\alpha,[y(\cdot)]^\alpha)$$

$$\sum_i \delta_i = \delta_I, \quad \text{where } x(t),\ y(t) \in \mathcal{PC}(J,\mathbb{E}_N).$$

(H5) $U(t,s)$ is the fuzzy number satisfies for $y \in \mathbb{E}_N$, $\frac{d}{dt}U(t,s)y \in \mathcal{PC}(J,\mathbb{E}_N) \cap \mathcal{PC}(J,\mathbb{E}_N)$ the equation

$$\frac{d}{dt}U(t,s)y = A(t)U(t,s)y + \int_0^t U(t,s)A(s)G(t,s)yds, \quad t \in J,$$

such that $[U(t,s)]^\alpha = [U_l^\alpha, U_r^\alpha]$, and $U_i^\alpha(t,s)$, $i = l,r$ are continuous. That is, there exists a constant δ_s such that $\|U_i^\alpha(t,s)\| < \delta_s$.

Lemma 2.1. *If x is an integral solution of* (1.1) – (1.3) *(v ≡ 0), then x is given by*

$$\begin{aligned}x(t) &= U(t,0)(x_0 - g(x) - h(0,x_0 - g(x))) + h(t,x(t))\\ &\quad + \int_0^t U(t,s)A(s)h(s,x(s))ds\\ &\quad + \int_0^t U(t,s)f(s,x(s))ds + \sum_{0<t_k<t} U(t,t_k)I(x(t_k^-)),\ \textit{for } t \in J. \quad (2.1)\end{aligned}$$

3. Existence and Uniqueness of Fuzzy Solution

In this section, we consider the existence and uniqueness of fuzzy solutions for (1.1) – (1.3) $(v \equiv 0)$. We define that

$$\begin{aligned}\Psi x(t) &= U(t,0)\big[x_0 - g(x) - h(0,x(0))\big] + h(t,x(t))\\ &\quad + \int_0^t U(t,s)A(s)h(s,x(s))ds\end{aligned}$$

$$+\int_0^t U(t,s)f(s,x(s))ds + \sum_{0<t<t_k} U(t,t_k)I_k(x_k(t_k^-)), \quad (3.1)$$

where Ψ is a continuous function from $\mathcal{PC}(J,\mathbb{E}_N)$ to itself.

Theorem 3.1. *Suppose that hypotheses $(H1)-(H5)$ are satisfied. Then the equation $(1.1)-(1.3)$ has unique fixed point in $\mathcal{PC}(J,\mathbb{E}_N)$.*

Proof. For $x, y \in \mathcal{PC}(J,\mathbb{E}_N)$,

$$\begin{aligned}
&d_H([\Psi x(t)]^\alpha, [\Psi y(t)]^\alpha) \\
&\quad\leq d_H\Big(\Big[U(t,0)[x_0 - g(x) - h(0,x(0))]\Big]^\alpha, \Big[U(t,0)[y_0 - g(y) - h(0,y(0))]\Big]^\alpha\Big) \\
&\quad\quad + d_H([h(t,x(t))]^\alpha, [h(t,y(t))]^\alpha) \\
&\quad\quad + d_H\Big(\Big[\int_0^t U(t,s)A(s)h(s,x(s))ds\Big]^\alpha, \Big[\int_0^t U(t,s)A(s)h(s,y(s))ds\Big]^\alpha\Big) \\
&\quad\quad + d_H\Big(\Big[\sum_{0<t_k<t} U(t,t_k)I_k(x_k(t_k^-))\Big]^\alpha, \sum_{0<t_k<t} U(t,t_k)I_k(y_k(t_k^-))\Big]^\alpha\Big) \\
&\quad = \Delta_1 d_H([x(\cdot)]^\alpha, [y(\cdot)]^\alpha) + \Delta_2 \int_0^t d_H([x(\cdot)]^\alpha, [y(\cdot)]^\alpha)ds,
\end{aligned}$$

where $\Delta_1 = \delta_s(\delta_g + \delta_{h_1} + \delta_h\delta_g) + \delta_h$ and $\Delta_2 = \delta_s(\delta_f + M_A\delta_h)$. Therefore

$$\begin{aligned}
d_\infty(\Psi x(\cdot), \Psi y(\cdot)) &= \sup_{\alpha\in[0,1]} d_H([\Psi x(\cdot)]^\alpha, [\Psi y(\cdot)]^\alpha) \\
&\leq \Delta_1 d_\infty(x(\cdot), y(\cdot)) + \Delta_2 \int_0^t d_\infty(x(\cdot), y(\cdot))ds.
\end{aligned}$$

Hence

$$\begin{aligned}
H_1(\Psi x(\cdot), \Psi y(\cdot)) &\leq \Delta_1 H_1(x(\cdot), y(\cdot)) + \Delta_2 b H_1(x(\cdot), y(\cdot)) \\
&= (\Delta_1 + \Delta_2 b) H_1(x(\cdot), y(\cdot)),
\end{aligned}$$

where $(\Delta_1 + \Delta_2 b) < 1$. Then by hypotheses, Ψ is a contraction mapping. By using Banach fixed point theorem, equation $(1.1)-(1.3)$ have a unique fixed point, $x \in \mathcal{PC}(J,\mathbb{E}_N)$.

4. Controllability of Fuzzy System

In this section, we show the controllability results for the nonlinear impulsive fuzzy neutral integrodifferential system $(1.1)-(1.3)$ of the form

$$x(t) = U(t,0)\big[x_0 - g(x) - h(0,x(0))\big] + h(t,x(t)) + \int_0^t U(t,s)A(s)h(s,x(s))ds$$
$$+ \int_0^t U(t,s)f(s,x(s))ds + \int_0^t U(t,s)v(s)ds + \sum_{0<t_k<t} U(t,t_k)I_k(x(t_k^-)). \quad (4.1)$$

Definition 4.1. ([19]) System (1.1) – (1.3) is said to be controllable on the interval J, if for every initial functions $x_0 \in \mathbb{E}_N$ and $x_1 \in \mathbb{E}_N$ there exists a control function $v(t)$ such that the fuzzy solution $x(t)$ to (1.1) – (1.3) satisfies $x(b) = x_1$, that is $[x(b)]^\alpha = [x_1]^\alpha$, where x_1 is a target set.

Defined the fuzzy mapping $\zeta : \widehat{P(R)} \to \mathbb{E}_N$ by

$$\eta^\alpha v = \begin{cases} \int_0^t U(b,s)v(s)ds, & v \subset \overline{\Gamma}_u, \\ 0, & \text{otherwise.} \end{cases} \quad (4.2)$$

Then there exists $\eta_i^\alpha (i = l, r)$ such that

$$\eta_q^\alpha v_l = \int_0^t U_l^\alpha(b,s)v_l(s)ds, \ v_l \in [u_l^\alpha, u^1];$$
$$\eta_r^\alpha v_r = \int_0^t U_r^\alpha(b,s)v_r(s)ds, \ v_r \in [u^1, u_r^\alpha].$$

We assume that η_i's are bijective mappings. Hence the α-set of $v(s)$ are

$$\begin{aligned} [v(s)]^\alpha &= [v_l^\alpha(s), v_r^\alpha(s)] \\ &= \Big[(\widehat{\eta}_l^\alpha)^{-1}\Big((x_1)_l^\alpha - U_l^\alpha(b,0)[(x_0)_l^\alpha - g_l^\alpha - h_l^\alpha(0,x(0))] \\ &\quad -h_l^\alpha(t,x(t)) - \int_0^t U_l^\alpha(t,s)A_l^\alpha(s)h_l^\alpha(t,x(s))ds \\ &\quad - \int_0^t U_l^\alpha(t,s)f_l^\alpha(t,x(s))ds - \sum_{0<t_k<t} U_l^\alpha(t,t_k)(I_k)_l^\alpha(x(t_k^-))\Big), \\ &\quad (\widehat{\eta}_r^\alpha)^{-1}\Big((x_1)_r^\alpha - U_r^\alpha(b,0)[(x_0)_r^\alpha - g_r^\alpha(x) - h_r^\alpha(0,x(0))] \\ &\quad -h_r^\alpha(t,x(t)) - \int_0^t U_r^\alpha(t,s)A_r^\alpha(s)h_r^\alpha(t,x(s))ds \\ &\quad - \int_0^t U_r^\alpha(t,s)f_r^\alpha(t,x(s))ds - \sum_{0<t_k<t} U_r^\alpha(t,t_k)(I_k)_r^\alpha(x(t_k^-))\Big)\Big]. \end{aligned}$$

Then substituting this expression into equation (4.1) yields α- level set of $x(b)$ becomes

$$[x(b)]^\alpha = \Big[U_l^\alpha(b,0)[(x_0)_l^\alpha - g_l^\alpha(x) - h_l^\alpha(0,x(0))] + h_l^\alpha(s,x(s))$$

$$
\begin{aligned}
&+\int_0^b U_l^\alpha(b,s)A_l^\alpha(s)h_l^\alpha(t,x(s))ds+\int_0^b U_l^\alpha(b,s)f_l^\alpha(s,x(s))ds\\
&+\sum_{0<t<t_k} U_l^\alpha(t,t_k)(I_k)_l^\alpha(x(t_k^-))+\int_0^b U_l^\alpha(b,s)(\widehat{\eta}_l^\alpha)^{-1}\Big\{(x_1)_l^\alpha-U_l^\alpha(b,0)\\
&\times[(x_0)_l^\alpha-g_l^\alpha(x)-h_l^\alpha(0,x(0))]-h_l^\alpha(s,x(s))\\
&-\int_0^b U_l^\alpha(b,s)A_l^\alpha(s)h_l^\alpha(s,x(s))ds-\int_0^l U_l^\alpha(b,s)f_l^\alpha(s,x(s))ds\\
&-\sum_{0<t_k<t} U_l^\alpha(t,t_k)(I_k)_l^\alpha(x(t_k^-))\Big\}ds, U_r^\alpha(b,0)[(x_0)_r^\alpha-g_r^\alpha(x)-h_r^\alpha(0,x(0))]\\
&+h_r^\alpha(t,x(t))+\int_0^b U_r^\alpha(b,s)A_r^\alpha(s)h_r^\alpha(s,x(s))ds\\
&+\int_0^b U_r^\alpha(b,s)f_r^\alpha(s,x(s))ds\\
&+\sum_{0<t_k<t} U_r^\alpha(t,t_k)(I_k)_r^\alpha(x(t_k^-))\\
&+\int_0^b U_r^\alpha(b,s)(\widehat{\eta}_r^\alpha)^{-1}\Big\{((x^1)_r^\alpha-U_r^\alpha(b,0))[(x_0)_r^\alpha-g_r^\alpha(x)-h_r^\alpha(0,x(0))]\\
&-h_r^\alpha(s,x(s))-\int_0^b U_r^\alpha(b,s)A_q^\alpha(s)h_r^\alpha(s,x(s)ds\big](s)ds\\
&-\int_0^b U_r^\alpha(b,s)f_r^\alpha(s,x(s))ds-\sum_{0<t<t_k} U_r^\alpha(t,t_k)(I_k)_r^\alpha(x(t_k^-))\Big\}ds\Big]\\
=&[(x_1)_l^\alpha,(x_1)_r^\alpha]=[x_1]^\alpha.
\end{aligned}
$$

We now set

$$
\begin{aligned}
\Omega x(t)=&U(t,0)[x_0-g(x)-h(0,x(0))]+h(t,x(t))\\
&+\int_0^t U(t,s)A(s)h(t,x(s))ds\\
&+\int_0^t U(t,s)\widehat{\eta}^{-1}\Big[x_1-U(b,0)[x_0-g(x)-h(0,x(0))]-h(s,x(s))\\
&-\int_0^b U(b,s)A(s)h(s,x(s))ds-\int_0^b U(b,s)f(s,x(s))ds\\
&-\sum_{0<t_k<t} U(t,t_k)I_k(x(t_k^-))\Big](s)ds+\int_0^t U(t,s)f(s,x(s))ds
\end{aligned}
$$

$$+\sum_{0<t_k<t} U(t,t_k)I_k(x(t_k^-)),$$

where the fuzzy mapping $\widehat{\eta}^{-1}$ satisfied above statement.

Now notice that $\Omega x(T) = x_1$, which means that the control $v(t)$ steers the state $x(t)$ from the initial state to the final state x_1 in J provided we can obtain a fixed point of the nonlinear operator Ω.

Assume that the following additional hypotheses:

(H_6) The system $(1.1) - (1.3)$ is linear $f \equiv 0$ is nonlocal controllable.

(H_7) For convenient take, $(\delta_h + (\delta_s(\delta_f + \delta_k b) + \delta_h) < 1$.

Theorem 4.2. *Suppose that the hypotheses $(H_1) - (H_7)$ are satisfied. Then the system $(1.1) - (1.3)$ is a nonlocal controllable.*

Proof. We can easily check that Ω is continuous from $\mathcal{PC}([0,b]:\mathbb{E}_N)$ to itself. For $x, y \in \mathcal{PC}([0,b]:\mathbb{E}_N)$,

$$\begin{aligned} d_H([\Omega x(t)]^\alpha, [\Omega y(t)]^\alpha) \le & (\delta_s(\delta_g + \delta_{h_1} + \delta_h\delta_g) + \delta_h) d_H([x(s)]^\alpha, [y]^\alpha) \\ & + (\delta_s(M_A\delta_h + \delta_f) \int_0^t d_H([x(s)]^\alpha, [y(s)]^\alpha) \\ & + \delta_h d_H([x(s)]^\alpha, [y]^\alpha) \\ & + (\delta_s(\delta_h + \delta_f) \int_0^b d_H([x(s)]^\alpha, [y(s)]^\alpha). \end{aligned}$$

Let $\kappa_1 = \delta_s(\delta_g + \delta_{h_1} + \delta_h\delta_g) + \delta_h$, and $\kappa_2 = \delta_s(M_A\delta_h + \delta_f)$ then we have

$$\begin{aligned} d_H([\Omega x(t)]^\alpha, [\Omega y(t)]^\alpha) \le & \, 2\kappa_1 d_H([x(s)]^\alpha, [y]^\alpha) \\ & + \kappa_2\Big(\int_0^t d_H([x(s)]^\alpha, [y(s)]^\alpha)ds + \int_0^b d_H([x(s)]^\alpha, [y(s)]^\alpha)ds\Big). \end{aligned}$$

Therefore

$$\begin{aligned} d_\infty(\Omega x(t), \Omega y(t)) \le & \, 2\kappa_1 d_\infty([x(s)]^\alpha, [y]^\alpha) \\ & + \kappa_2\Big(\int_0^t d_H([x(s)]^\alpha, [y(s)]^\alpha)ds + \int_0^b d_\infty([x(s)]^\alpha, [y(s)]^\alpha)ds\Big). \end{aligned}$$

Hence

$$H_1(\Omega x(t), \Omega y(t)) = \sup_{t\in[0,b]} d_H([\Omega x(t)]^\alpha, [\Omega y(t)]^\alpha)$$

$$\leq (2(\kappa_1 + \kappa_2 b))\mathcal{H}_1(x, y).$$

By hypotheses (H_7), *we take sufficiently small* b, Ω *is a contraction mapping. By Banach fixed point theorem the system* $(1.1)-(1.3)$ *has a unique fixed point* $x \in \mathcal{PC}([0, b] : \mathbb{E}_N)$.

5. Example

Consider the fuzzy solution of the nonlinear fuzzy neutral integrodifferential equation of the form:

$$\frac{d}{dt}(x(t) - \mathbf{2}tx(t)^2)) = \mathbf{2}\Big[x(t) - \int_0^t e^{-t}x(s)ds\Big] + \mathbf{3}tx(t)^2 + u(t), \qquad (5.1)$$

$$t \in J,$$

$$x(0) + \sum_{k=1}^{n} c_k x(t_k) = \mathbf{0} \in E_N, \qquad (5.2)$$

$$I_k(x(t_k^-)) = \frac{1}{1 + x(t_k)}, \qquad (5.3)$$

where x_1 is target set, and the α - level set of fuzzy number $\mathbf{0}$, $\mathbf{2}$ and $\mathbf{3}$ are

$[\mathbf{0}]^\alpha = [\alpha - 1, 1 - \alpha]$, $[\mathbf{2}]^\alpha = [\alpha + 1, 3 - \alpha]$, $[\mathbf{3}]^\alpha = [\alpha + 2, 4 - \alpha]$, for $\alpha \in [0, 1]$.

Let $G(t, s) = e^{-t}I, f(t, u(t)) = \mathbf{3}tu(t)^2, h(t, u(t)) = \mathbf{2}tu(t)^2$. Then α - level set of $g(x) = \sum_{k=1}^{n} c_k x(t_k)$ is

$$[g(x)]^\alpha = [\sum_{k=1}^{n} c_k x(t_k)]^\alpha = [\sum_{k=1}^{n} c_k x_q^\alpha(t_k), \sum_{k=1}^{n} c_k x_r^\alpha(t_k)]$$

and

$$d_H([g(x)]^\alpha, [g(y)]^\alpha) = d_H([\sum_{k=1}^{n} c_k x(t_k)]^\alpha, [\sum_{k=1}^{n} c_k y(t_k)]^\alpha)$$
$$\leq \delta_g d_H([x(\cdot)]^\alpha, [y(\cdot)]^\alpha),$$

where $\delta_g = \| \sum_{k=1}^{n} c_k \|$, and

$$[x(t)]^\alpha = [x_l^\alpha(t),\ x_r^\alpha(t)],$$

$$[\mathbf{3}]^\alpha = [\alpha+2, 4-\alpha], \text{ for } \alpha \in [0,1],$$

the α - level set of $f(t, x(t))$ is

$$[f(t,x(t))]^\alpha = [\mathbf{3}tx(t)^2]^\alpha = t[\mathbf{3}]^\alpha [x(t)^2]^\alpha = t[(\alpha+2)(x_l^\alpha(t)^2), (4-\alpha)(x_r^\alpha(t))^2],$$
$$d_H([f(t,x(t))]^\alpha, [f(t,y(t))]^\alpha) \le \delta_f \| d_H([x(t)]^\alpha, [y(t)]^\alpha),$$

where, $\delta_f = 4b\|(x_r^\alpha(t)) + (y_r^\alpha(t))\|$, $[x(t)]^\alpha = [x_l^\alpha(t), x_r^\alpha(t)]$ and $[\mathbf{3}]^\alpha = [\alpha+2, 4-\alpha]$, for $\alpha \in [0,1]$ and the α - level set of $h(t,x(t))$ is

$$[h(t,x(t))]^\alpha = [\mathbf{2}tx(t)^2]^\alpha = t[\mathbf{2}]^\alpha [x(t)^2]^\alpha = t[(\alpha+1)(x_l^\alpha(t))^2, (3-\alpha)(x_r^\alpha(t))^2].$$

We introduce the $\alpha-$ set of equation (5.1) – (5.3). Thus

$$d_H([h(t,x(t))]^\alpha, [h(t,y(t))]^\alpha) \le \delta_g d_H([x(t)]^\alpha, [y(t)]^\alpha),$$

where $\delta_g = 3b\|(x_r^\alpha(t)) + (y_r^\alpha(t))\|$.

$$[I(x(t_k))]^\alpha = \Big[\frac{1}{1+x(t_k)}\Big]^\alpha = \Big[\frac{1}{1+x_l^\alpha(t_k)}, \frac{1}{1+x_r^\alpha(t_k)}\Big].$$

$$\text{Thus } d_H\Big([I(x(t_k))]^\alpha, [I(y(t_k))]^\alpha\Big) \le \delta_I \max_k \{|x_l^\alpha - y_l^\alpha|, |x_r^\alpha - y_r^\alpha|\} = \delta_I d_H([x]^\alpha, [y]^\alpha),$$

where $\delta_I = 1/(1+|x_r^\alpha(t_k)|)(1+|y_r^\alpha(t_k)|)$. The constant term, $\delta_f, \delta_g, \delta_I$ are satisfied in the above hypotheses, and choose b is sufficiently small. Then all conditions satisfied in the above *Theorem 3.1*, so the problem (5.1) – (5.3) has a unique fuzzy solution.

Next we prove the nonlocal controllability parts, let us take target set, $x_1 = \mathbf{2}$.

$$\begin{aligned}
[u(s)]^\alpha =& \Big[u_l^\alpha, u_r^\alpha\Big] \\
=& \Big[\widehat{\eta_l}^{-1}\Big((1-\alpha) - \sum_{k=1}^n c_k x_l^\alpha(t_k) - t(\alpha+1)(x_l^\alpha)^2 \\
& - \int_0^b U_l^\alpha(b,s)t(\alpha+1)(x_l^\alpha)^2(s)ds - \int_0^b U_l^\alpha(b,s)t(\alpha+2)(x_l^\alpha)^2(s)ds \\
& - \int_0^b U_l(b,s)t(\alpha+1)(x_l^\alpha)^2(s)ds \\
& - \sum_k \{1/(1+x_l^\alpha(t_k))\Big), \widehat{\eta_r}^{-1}\Big((1-\alpha) - \sum_{k=1}^n c_k x_l^\alpha(t_k)
\end{aligned}$$

$$
\begin{aligned}
&- t(\alpha+1)(x_l^\alpha)^2 - \int_0^b U_r^\alpha(b,s)t(\alpha+1)(x_r^\alpha)^2(s)ds \\
&- \int_0^b U_r^\alpha(b,s)t(\alpha+2)(x_r^\alpha)^2(s)ds \\
&- \int_0^b U_r^\alpha(b,s)t(\alpha+1)(x_r^\alpha)^2(s)ds \\
&- \sum_k U_r^\alpha(t,t_k)\{1/(1+x_r^\alpha(t_k))\Big)\Big].
\end{aligned}
$$

Then substituting this expression into the integral system with respect to (5.1)−(5.3) yields $\alpha-$ level set of $x(b)$ is $[x(b)]^\alpha = [\mathbf{2}]^\alpha = [x_1]$. Then all condition stated in theorem 4.1 are satisfied, so the system $(5.1) - (5.3)$ is controllable on $[0, b]$.

Remark. The controllability results obtained by using the representation Definition 4 in [14] is not correct. Hence a suitable controllability for fuzzy differential differential system is an interesting problem.

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