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ON SPECTRA OF GLUED COMPLETE GRAPHS

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Abstract: In this article, we give spectra and characteristic polynomial of three partite complete graphs. We also give spectra of cartesian and tenor product of $K_{n,n,n}$ with itself. Finally, we give general closed forms of the characteristic polynomials of the graphs obtained by identifying two copies of K_n at a vertex and an edge.

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Key Words: characteristic polynomial, eigenvalue, spectrum

1. Introduction

Spectral graph theory is the study of algebraic properties of matrices associated with graphs. In particular, it is the study of spectra of adjacency matrices of graphs.

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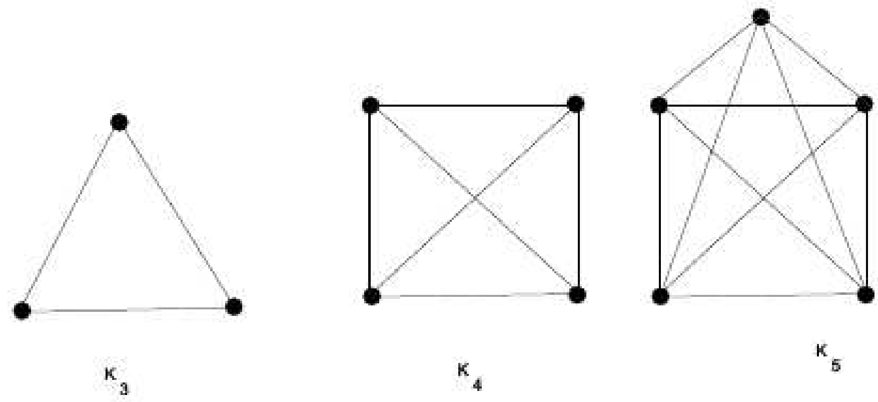

Figure 1. Complete graphs

Spectral graph theory has several important applications in computer science, differential geometry, Riemannian geometry, Markov chains, and astronomy. The founder of Google computed Perron-Frobenius eigenvector of the web graph, and thus applied it in computer sciences, see [9]. The second largest eigenvalue is used in statistics to gain information about expansion and randomness properties of graphs. The smallest eigenvalue is important for chromaticity. For details, see [13, 14, 17].

A *graph* G is a pair (V, G), where V is the set of vertices and E is the set of edges. The *adjacency matrix* associated to a finite connected graph G on vertices n is the $n \times n$ matrix $A = \{a_{ij}\}$, where a_{ij} is the number of edges between the vertices i and j. As an example, you can see the adjacency matrices of the complete graphs K_3, K_4 and K_5:

$$A_{K_3} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}; \ A_{K_4} = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}; \ A_{k_5} \ = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix}.$$

A real number λ is said to be an *eigenvalue* of an $n \times n$ matrix A if there exist a nonzero vector x of $\mathbb{R}^n$ such that $\lambda x = Ax$ and x is called the *eigenvector* corresponding to λ. Eigenvalues are usually computed as the roots of the characteristic polynomial $\Delta(A) = |\lambda I_n - A|$. The *spectrum* of a graph G is the

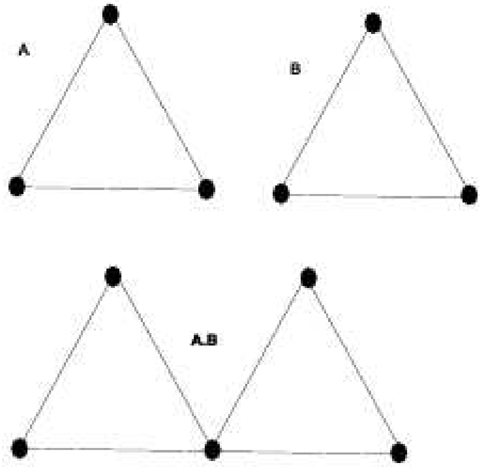

Figure 2. Vertex identification

collection of eigenvalues along with their multiplicities of the adjacency matrix of G.

Before going further, for some notations, by the graph $G_1.G_2$ we shall mean the identification of a vertex of G_1 with a vertex of G_2, and by the graph $G_1 \mathrm{I} G_2$ we shall mean the identification of an edge of G_1 with an edge of G_2.

2. Main results

This section contains the main results, including the characteristic polynomial of $K_{n,n,n}$, spectra of the cartesian and tenor products of $K_{n,n,n}$ with itself, and the closed forms of the characteristic polynomials of $K_n.K_n$ and $K_n \mathrm{I} K_n$.

Theorem 2.1. *The characteristic polynomial of $K_{n,n,n}$ is*

$$\lambda^{3n-3}(\lambda + n)^2(\lambda - 2n).$$

Proof. The adjacency matrix of $K_{n,n,n}$ is the $3n \times 3n$ matrix

$$A_{K_{\mathsf{n,n,n}}} = \begin{pmatrix} O & J & J \\ J & O & J \\ J & J & O \end{pmatrix},$$

where J is the $n \times n$ matrix of 1s and O_n is the null matrix of order n. In the lower upper factorization of $(A_{K_{\mathsf{n,n,n}}} - \lambda I_{3n})$, the determinant of the main

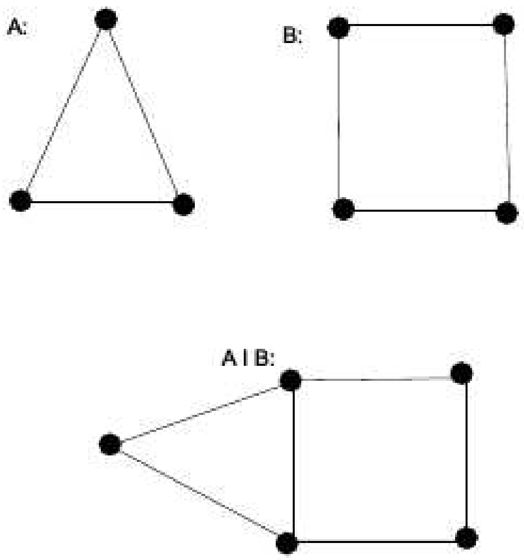

Figure 3. Edge identification

diagonal of lower triangular matrix L is 1 as $diag(L) = (1, 1, 1, \ldots, 1)$, and the determinant of the upper triangular matrix U is

$$\begin{aligned} det(U) = \lambda^n \left(\frac{\lambda^2 - n}{\lambda}\right)\left(\frac{\lambda^2 - n\lambda - 2n}{\lambda - n}\right) \\ \times \left(\prod_{m=1}^{n-1} \frac{A + (m-1)C}{B + (m-1)D},\ n \geqslant 2\right) \\ \times \left(\prod_{m=1}^{n-1} \frac{E + (m-1)G}{F + (m-1)H},\ n \geqslant 2\right), \end{aligned}$$

where

$$\begin{aligned} & A = (\lambda^2 - 2n)\lambda, \quad B = \lambda^2 - n, \\ & C = -n\lambda, \quad D = -n, \\ & E = (\lambda^2 - n\lambda - 4n)\lambda, \quad F = \lambda^2 - n\lambda - 2n, \\ & G = -2n\lambda, \quad H = -2n. \end{aligned}$$

Now, by simplifying, we get the desire result. □

Following Theorem 2.1 and [8] we get the following corollaries:

Corollary 2.2. *The spectrum of* $K^{*2}_{m,m,m}$ *is*

$$\begin{pmatrix} -2m & -m & 0 & m & 2m & 4m \\ 4 & 12(m-1) & (m-1) & 4 & 6(m-1) & 1 \end{pmatrix}.$$

Corollary 2.3. *The spectrum of* $K^{\otimes 2}_{m,m,m}$ *is*

$$\begin{pmatrix} -2m^2 & 0 & m^2 & (2m)^2 \\ 4 & 9(m^2-1) & 1 & 1 \end{pmatrix}.$$

Theorem 2.4. *The characteristic polynomial of* $K_n.K_n, n \geq 3$, *is*

$$(\lambda+1)^{2n-4}\big(\lambda-(n-2)\big)\big(\lambda^2-(n-2)\lambda-2(n-1)\big).$$

Proof. The adjacency matrix of $K_n.K_n$ is the $(2n-1)\times(2n-1)$ matrix

$$A_{K_\mathsf{n}.K_\mathsf{n}} = \begin{pmatrix} 0 & J & J \\ J^T & A_{K_{\mathsf{n}-1}} & O_{n-1} \\ J^T & O_{n-1} & A_{K_{\mathsf{n}-1}} \end{pmatrix},$$

where J is the $1\times(n-1)$ matrix of 1s, and $A_{K_{\mathsf{n}-1}}$ is the adjacency matrix of K_{n-1}. Here O is the null matrix. In the lower upper factorization of $(A_{K_\mathsf{n}.K_\mathsf{n}} - \lambda I_{2n-1})$, the determinant of the main diagonal of the lower triangular matrix L is 1 as $diag(L) = (1,1,1,\ldots,1)$, and the determinant of the upper triangular matrix U is

$$\det(U) = \lambda\left(\frac{\lambda^2-1}{\lambda}\right)\left(\prod_{m=3}^{n}\frac{\lambda^2-(m-2)\lambda-(m-1)}{\lambda-(m-2)}\right)$$
$$\times\left(\frac{\lambda^3-(n-2)\lambda^2-n\lambda+(n-2)}{\lambda^2-(n-2)\lambda-(n-1)}\right)\left(\prod_{m=1}^{n-2}\frac{C+(m-1)A}{D+(m-1)B}\right),$$

where

$$A = -\lambda^3+(n-4)\lambda^2+(3n-6)\lambda+(2n-3),$$
$$B = -\lambda^2+(n-3)\lambda+2n-3,$$
$$C = \lambda^4-(n-2)\lambda^3-(n+2)\lambda^2+(3n-8)\lambda+(3n-5),$$
$$D = \lambda^3-(n-2)\lambda^2-n\lambda+(n-2).$$

The result now follows from simple computations. □

Theorem 2.5. *The characteristic polynomial of* $K_n\mathrm{I}K_n, n \geq 4$, *is*

$$(\lambda+1)^{2n-5}(\lambda-(n-3))(\lambda^2-(n-2)\lambda+(-3n+5)).$$

Proof. The adjacency matrix of $K_n \mathrm{I} K_n$ is the $(2n-2)\times(2n-2)$ matrix

$$A_{K_n I K_n} = \begin{pmatrix} A_{K_n} & S^T_{(n-2,n)} \\ S_{(n-2,n)} & A_{K_{n-2}} \end{pmatrix},$$

where A_{K_n} is the adjacency matrix of K_n and $S_{(n-2,n)}$ is $(n-2)\times n$ matrix

$$S_{(n-2,n)} = \begin{pmatrix} J_{(n-2\times 2)} & O_{(n-2\times n-2)} \end{pmatrix}.$$

In the lower upper factorization of $(A_{K_n I K_n} - \lambda I_{2n-2})$, the determinant of the main diagonal of the lower triangular matrix L is 1 as $diag(L) = (1,1,1,\ldots,1)$, and the determinant of the upper triangular matrix U is

$$\begin{aligned}\det(U) = \lambda\left(\frac{\lambda^2-1}{\lambda}\right)&\left(\prod_{m=3}^{n}\frac{\lambda^2-(m-2)\lambda-(m-1)}{\lambda-(m-2)}\right)\\ &\times\left(\frac{\lambda^3-(n-2)\lambda^2-(n+1)\lambda+2n-6}{\lambda^2-(n-2)\lambda-(n-1)}\right)\\ &\times\left(\prod_{m=1}^{n-3}\frac{C+(m-1)A}{D+(m-1)B}\right),\end{aligned}$$

where

$$\begin{aligned} A &= \lambda^3+(n-5)\lambda^2+(4n-11)\lambda+3n-8,\\ B &= \lambda^2+\lambda-(3n+8)\\ C &= \lambda^4-(n-2)\lambda^3-(n+4)\lambda^2+(5n-18)\lambda+(5n-13),\\ D &= \lambda^3-(n-2)\lambda^2-(n+1)\lambda+(n-2).\end{aligned}$$

This completes the proof. □

3. Conclusions

In this article, we gave spectra and characteristic polynomials of three partite complete graphs using techniques of linear algebra and computer softwares. Technically speaking, we computed general closed forms of characteristic polynomials of the graphs obtained by gluing a vertex of K_n with a vertex of another copy of K_n, and gluing an edge of K_n with an edge of another copy of K_n.

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