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STRONGLY T-SEMISIMPLE MODULES AND STRONGLY T-SEMISIMPLE RINGS

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Abstract: In this paper, we introduce the notions of strongly t-semisimple modules and strongly t-semisimple rings as a generalization of semisimple modules, rings respectively. We investigate many characterizations and properties of each of these concepts. An R-module is called strongly t-semisimple if for each submodule N of M there exists a fully invariant direct summand K such that K t-essential in N. Also, the direct sum of strongly t-semisimple modules and homomorvarphic image of strongly t-semisimple is strongly t-semisimple.

A ring R is called right strongly t-semisimple if R_R is strongly t-semisimple. Various characterizations of right strongly t-semisimple rings are given.

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1. Introduction

Through this paper R be a ring with unity and M is a right R-module. Let

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Z_2 (M) be the second singular (or Goldi torsion) of M which is defined by $Z(M/(Z(M))) = (Z_2(M))/(Z(M))$ where $Z(M)$ is the singular submodule of M A module M is called Z_2-torsion if $Z_2(M) = M$ and a ring R is called right Z_2-torsion if $Z_2(R_R) = R_R$[8].,A submodule A of an R-module M is said to be essential in M (denoted by $A \leq_{ess} M$), if $A \bigcap W(0)$ for every non-zero submodule W of M. Equivalently $A \leq_{ess} M$ if whenever $A \bigcap W = 0$, then $W = 0$ [9],Asgari and Haghany [4] introduced the concept of t-essential submodules as generalization of essential submodules. A submodule N of M is said to be t-essential in M (denoted by ($N \leq_{tes} M$) if for every submodule B of M, $N \bigcap B \leq Z_2(M)$ implies that $B \leq Z_2(M)$. It is clear that every essential submodule is t-essential, but not conversely. However, the two concepts are equivalent under the class of nonsingular modules. A submodule N of M is called fully invariant if $f(N) \leq N$ for every R-endomorphism f of M.Clearly 0 and M are fully invariant submodules of M [15] . M is called duo module if every submodule of M is fully invariant. A submodule N of an R-module is called stable if for each homomorphism f:$N \to M$,$f(N) \leq N$. A module is called fully stable if every submodule of M is stable[1]. Asgari and Haghany [3] introduced the notion of t-semisimple modules as a generalization of semisimple modules. A module M is t-semisimple if for every submodule N of M, there exists a direct summand K such that $K \leq_{tes} N$. In this paper we introduce the notion of strongly t-semisimple modules as a generalization of t-semisimple modules. An R-module is called strongly t-semisimple if for each submodule N of M there exists a fully invariant direct summand K such that $K \leq_{tes} N$. It is clear that the class of strongly t-semisimple modules contains the class of t-semisimple. This paper consists of three sections. In Section 2 we introduce the concept of strongly t-semisimple and giving many characterizations and properties of this class of modules.

Section 3, concerns with strongly t-semisimple rings. Several, characterization of commutative strongly t-semisimple ring. Also we give some characterizations of nonsingular strongly t-semisimple ring. First, we list some known result, which will be needed in our work.

Proposition 1.1. *(see [2]) The following statements are equivalent for a submodule A of an R-module.*

(1) A is t-essential in M;

(2) $(A + Z_2(M))/Z_2(M)$ is essential in $M/Z_2(M)$;

(3) $A + Z_2(M)$ is essential in M;

(4) M/A is Z_2-torsion.

Lemma 1.2. *(see [4]) Let A_λ be submodule of $M\lambda$ for all λ in a set Λ.*

(1) If Λ is a finite and $A_\lambda \leq_{tes} M_\lambda$, then $\bigcap_\Lambda | A_\lambda \leq_{tes} \bigcap_\Lambda |M_\lambda$ for all $\lambda \in \Lambda$.
(2) $\oplus_\Lambda A_\lambda \leq_{tes} \oplus \Lambda M_\lambda$ If and only if $A_\lambda \leq_{tes} M_\lambda$ for all $\lambda \in \Lambda$.

Lemma 1.3. *(see [13]) Let R be a ring and let $L \leq K$ be submodules of an R-module M such that L is a fully invariant submodule of K and K is a fully invariant submodule of M. Then L is a fully invariant submodule of M.*

Theorem 1.4. *(see [3]) The following statements are equivalent for a module M:*
(1) M is t-semisimple;
(2) $M/Z_2(M))$ is semisimple;
(3) $M = Z_2(M \oplus M')$ where M' is a non-singular semisimple module;
(4) Every nonsingular submodule of M is a direct summand;
(5) Every submodule of M which contains $Z_2(M)$ is a direct summand.

2. Strongly t-Semisimple Modules

Definition 2.1. An R-module is called strongly t-semisimple if for each submodule N of M there exists a fully invariant direct summand K such that $K \leq_{tes} N$.

Remarks and Examples. (1) It is clear that every strongly t-semisimple module is t-semisimple, but the convers is not true as we shall see later.

(2) If M is Z_2-torsion, then M is strongly t-semisimple.
Proof. Since M is Z_2-torsion, $Z_2(M) = M$. So that for all $A \leq M$, $Z_2(A)$ $= Z_2(M) \bigcap A = M \bigcap A = A$, then $(0) + Z_2(A) = A \leq_{ess} A$, Thus $(0) \leq_{tes} A$ for all $A \leq M$.But (0) is a direct summand of M, and (0) is fully invariant. Hence M is strongly t-semisimple.

(3) Every singular module is strongly t-semisimple.
Proof. Let M be a singular R-module. Then $Z(M) = M$, it follows that $Z_2(M) = Z(M) = M$. Thus M is Z_2 torsion, hence M is strongly t-semisimple. Thus, in particular Z_n as Z-module is strongly t-semisimple for all $n \in Z_+$, $n > 1$.

(4) The converse of (3) is not true in general, for example Z_4 as Z_4 -module is not singular, but it is Z_2-torsion, so it is strongly t-semisimple.

(5) If M is t-semisimple module and weak duo (SS-module). Then M is strongly t-semisimple, where M is a weak duo(or SS-module) if every direct summand of M is fully invariant.

Proof. Let $N \leq M$, since M is t-semisimple, there exists $K \leq^{\oplus} M$ such that $K \leq_{tes} N$. But M is SS-module, so K is stable; hence K is fully invariant direct summand. Thus M is strongly t-semisimple,where M is a weak duo(or SS-module) if every direct summand of M is fully invariant.

(6) If M is t-semisimple and duo (or fully stable), then M is strongly t-semisimple.

Hence every t-semisimple multiplication R-module is strongly t-semisimple.

(7) If M is cyclic t-semisimple module over commutative ring R then M is a strongly t-semisimple.

Proof. Since M is cyclic module over commutative ring, then M is a multiplication module. Thus M is duo. Therefor the result follows by part (9).

(8) $M = Z_n \oplus Z$ as Z-module is not t-semisimple. For all $n \in Z_+$, $n > 1$.

Proof. Suppose M is t-semisimple. Then $M/Z_n \cong Z$ is t-semisimple [4, Corollary 2.4] which is a contradiction.

(9) t-semisimple module need not be strongly t-semisimple, for example:

Example 2.2. Let $T = M \oplus M$ where M is a non-singular semisimple R-module, $M \neq (0)$. Hence T is semisimple, and so T is a t-semisimple, let $N = M \oplus (0)$, so there exists $K \leq^{\oplus} M$ such that $K \leq_{tes} N$. Hence $K = K_1 \oplus (0)$ for some $K_1 \leq M$, if $K_1 = (0)$, then $K = < (0,0) >$ and $K \leq^{\oplus}_{tes} M \oplus (0)$. But $< (0,0) >+ Z_2\ (M + (0)) \leq_{ess} M \oplus (0)$ (by Proposition 1.1.(3)) Thus $Z_2(M) \leq_{ess} M$. But $Z_2(M) = (0)$, hence $(0) \leq_{ess} M$ and so $M = (0)$, which is a contradiction. It follows that $K_1 \neq (0)$, so $K \neq < (0,0) >$.But in this case K is not fully invariant submodule of T. To see this:

Let $f : T \to T$ defined by $T(x, y) = (y, x)$, for all $(x, y) \in T$, Then $T(K_1 \oplus (0)) = (0) \oplus K_1 \leq\neq K_1 \oplus (0)$. Thus $K = K_1 \oplus (0)$ is not fully invariant submodule of T, such that $K \leq_{tes}$N.Therefore T is not strongly t-semisimple.

In particular, R as R-module is simple non-singular R -module, so $R \oplus R$ as R -module is semisimple and so it is t-semisimple .But $R \oplus R$ is not strongly t-semisimple:

To see this, let $N = R \oplus (0)$. As $< (0,0) >$ is only direct summand fully invariant of $R \oplus R$, such that $< (0,0) > \leq N = R(0)$.But $< (0,0) > \leq\neq 2010_{tes}$ N because if we assume that $< (0,0) > \leq_{tes} N$ then $< (0,0) > + Z_2(N) \leq_{ess} N$, so that $< (0,0) > + < (0,0) > = < (0,0) > \leq_{ess} N$ which is a contradiction.

Now we shall give some characterizations of strongly t-semisimple.

Theorem 2.3. *The following statements are equivalent for an R -module M:*

(1) M is strongly t-semisimple,

(2) $\frac{M}{Z_2(M)}$ *is fully stable semisimple and isomorphic to a stable submodule of M,*

(3) $M = Z_2(M) \oplus M'$ *where* M' *is a nonsingular semisimple fully stable module and* M' *is a stable submodule in M,*

(4) Every nonsingular submodule of M is stable direct summand,

(5) Every submodule of M which contains $Z_2(M)$ *is a direct summand of M and* $\frac{M}{Z_2(M)}$ *is fully stable and isomorphic to a stable submodule of M.*

Proof. (1) $\Rightarrow$ (4) Let N be a nonsingular submodule of M. Since M is strongly t-semisimple, there exists a fully invariant direct summand K of M such that $K \leq_{tes} N$. Assume that $M = K \oplus K'$ for some $K' \leq M$.Hence $N = (K \oplus K') \bigcap N$ and so $N = K \oplus (K' \bigcap N)$ by modular law. Thus $K \leq^{\oplus} N$ and $\frac{N}{K} \cong (N \bigcap K')$. But $K \leq_{tes} N$ implies $\frac{N}{K}$ is Z_2-torsion that is $Z_2(\frac{N}{K}) = \frac{N}{K}$ by Proposition (1.1). On the other hand $(N \bigcap K') \leq N$ and N is nonsingular, so $(N \bigcap K')$ is nonsingular submodule, and hence $\frac{N}{K}$ is nonsingular, which implies that Z_2 $(\frac{N}{K}) = 0$.Thus $\frac{N}{K} = 0$ and hence $N = K$. Therefore N is a fully invariant direct summand, and hence N is a stable direct summand.

(4) $\Rightarrow$ (3) Let M' be a complement of $Z_2(M)$.Hence $M' \oplus Z_2(M) \leq_{ess} M$ And so $M' \leq_{tes} M$ by Proposition (1.1(3)).Thus $\frac{M}{M'}$ is Z_2-torsion, by proposition (1.1(4)).We claim that $M'$is nonsingular. To explain our assertion, suppose $x \in Z(M')$, so $x \in M' \leq M$ and $ann(x) \leq_{ess} R$. Hence $ann(x) \leq_{tes} R$ and this implies $x \in Z_2(M)$. Thus $x \in Z_2(M) \bigcap M' = (0)$, thus x=0 and M' is a nonsingular. So that by hypothesis, M' is a stable direct summand of M and so that $M = L \oplus M'$ for some $L \leq M$.Thus $L \cong \frac{M}{M'}$ which is Z_2-torsion, hence L is Z_2-torsion .On other hand, $Z_2(M) = Z_2(M') + Z_2(L) = 0 + L = L$. It follows that $M = Z_2(M) \oplus M'$, M' is a nonsingular. Now let $N \leq M'$, so N is a nonsingular and hence $N \leq^{\oplus} M$ by hypothesis. It follows that $M = N \oplus W$ for some $W \leq M$ and hence $M' = (N \oplus W) \bigcap M'$ and so M' $= N \oplus (W \bigcap M')$ by modular law. Thus $N \leq^{\oplus} M'$ and hence M' is semisimple .Next to prove M' is fully stable. It is sufficient to prove that every submodule of M' is fully invariant, so let $N \leq M' \leq M$ and let $f : M' \to M'$. Then $i \circ f \circ \rho \in End(M)$, where i inclusion map from M' to M and ρ is the projection of M onto M'. Then $(i \circ f \circ \rho)(N) \leq N$ since N is stable in M (by hypothesis). Now $(i \circ f \circ \rho)(N) = (i \circ f(\rho(N))$, but $N \leq M'$,so $\rho(N) = N$. Thus $i \circ f(\rho(N)) = i \circ f(N) = f(N) \leq N$. Thus N is fully invariant submodule of M', but $N \leq^{\oplus} M$, so that N is stable in M' and M' is fully stable.

(3) $\Rightarrow$ (1) Let $M = Z_2(M) \oplus M'$, $M'$is nonsingular semisimple fully stable

module, $M^{'}$ is stable in M. Let $N \leq M$, then $(N\bigcap M^{'}) \leq M^{'}$, so $(N\bigcap M^{'}) \leq^{\oplus} M^{'}$ (since $M^{'}$ is semisimple). It follows that $M^{'} = (N\bigcap M^{'}) \oplus W$ for some $W \leq M^{'}$ and hence $M = Z_2(M) \oplus (N\bigcap M^{'}) \oplus W$. Hence $(N\bigcap M^{'}) \leq^{\oplus} M$. On other hand, $\frac{N}{N\bigcap M^{'}} \cong \frac{N+M^{'}}{M^{'}} \leq \frac{M}{M^{'}} \cong Z_2(M)$. But $Z_2(M)$ is Z_2-torsion. Hence, $\frac{N}{N\bigcap M^{'}}$ is Z_2-torision and then by (Proposition 1.1(4) $(N\bigcap M^{'}) \leq_{tes} N$. But $(N\bigcap M^{'})$ is stable in $M^{'}$(since $M^{'}$ is fully stable) so $N\bigcap M^{'}$ is a fully invariant submodule in M. Thus by Lemma (1.3) $N\bigcap M^{'}$ is fully invariant in M. But $N\bigcap M^{'}$is direct summand of M. Thus $N\bigcap M^{'} \leq^{\oplus} M$,$N\bigcap M^{'} \leq N$, hence M is strongly t-semisimple.

(3) $\Rightarrow$ (5) Let $N \leq M$,$N \supseteq Z_2(M)$. Since $M = Z_2(M) \oplus M^{'}$, where $M^{'}$ is a nonsingular semisimple fully stable,$M^{'}$ is stable in M.Then $N = (Z_2(M) \oplus M^{'})\bigcap N = Z_2(M) \oplus (N\bigcap M^{'})$ by modular law. But $N\bigcap M^{'}) \leq M^{'}$ and $M^{'}$is semisimple implies$(N\bigcap M^{'}) \leq^{\oplus} M^{'}$. It follows that $(N\bigcap M^{'}) \oplus W = M^{'}$. Hence $M = Z_2(M)\oplus(N\bigcap M^{'})\oplus W = N\oplus W$. Thus $N \leq^{\oplus} M$. Also $\frac{M}{(Z_2(M))} \cong M^{'}$ and $M^{'}$ is a fully stable module and $M^{'}$ is stable in M, so that $\frac{M}{(Z_2(M)}$ is fully stable semisimple and isomorphic to stable submodule of M.

(2) $\Rightarrow$ (3) Since $Z_2(M)$ is t-closed,$\frac{M}{(Z_2(M))}$ is nonsingular. By condition (2), $\frac{M}{(Z_2(M))}$ is semisimple, hence $\frac{M}{(Z_2(M))}$ is projective (by [10, Coroallary 1.25,P.35] . Now let $\pi : M \to M/(Z_(2)(M))$ be the natural epiomorphism and as $\frac{M}{(Z_2(M))}$ is projective, we get $ker\pi = Z_2(M)$is a direct summand of M .Hence $M = Z_2(M) \oplus M^{'}$.Thus $M^{'} \cong \frac{M}{(Z_2(M))}$ which is nonsingular semisimple fully stable module. Then $M^{'}$ is nonsingular semisimple fully stable .Also $M^{'}$ is stable submodule of M by condition (2).

(3) $\Rightarrow$ (2) By condition (3),$M = Z_2(M) \oplus M^{'}$, where $M^{'}$, is a nonsingular semisimple fully stable module and $M^{'}$is stable in M. It follows that $\frac{M}{(Z_2(M))} \cong M^{'}$. Thus $\frac{M}{(Z_2(M))}$ is semisimple fully stable and isomorphic to stable submodule $M^{'}$ of M.

(2) $\Rightarrow$ (5) It follows directly (since (2) $\Leftrightarrow$ (3) $\Rightarrow$ (5) then(2) $\Rightarrow$ (5)).

(5) $\Rightarrow$ (2) Let $\frac{N}{Z_2(M)} \leq \frac{M}{Z_2(M)}$. Then $N \supseteq Z_2(M)$, so by condition (5), N is stable direct summand of M, so that $N \oplus W = M$ for some $W \leq M$.Thus $\frac{N}{(Z_2(M))} + \frac{(W+Z_2(M))}{(Z_2(M))} = \frac{M}{(Z_2(M))}$. But we can show that $\frac{N}{(Z_2(M))} \bigcap \frac{(N+Z_2(M))}{(Z_2(M))} = 0$, as follows:

Let $\overline{x} \in \frac{N}{(Z_2(M))} \bigcap \frac{(W+Z_2(M))}{(Z_2(M))}$. Then $\overline{x} = n + Z_2(M) = w + Z_2(M)$ for some $n \in N$,$w \in W$, and so $n - w \in Z_2(M) \subseteq N$. It follow that $n - w = n_1$ for some $n_1 \in N$ and hence $n - n_1 = w \in N \bigcap W = 0$. Thus $x = 0_{\frac{M}{Z_2(M)}}$ and so $\frac{N}{(Z_2(M))} \oplus \frac{(W+Z_2(M))}{Z_2(M)} = \frac{M}{(Z_2(M))}$. This implies $\frac{M}{Z_2(M)}$ is semisimple. By condition (5), $\frac{M}{(Z_2(M))}$ fully stable and isomorphic to stable submodule of M . But $\frac{M}{Z_2(M)}$ is nonsingular, so $\frac{M}{Z_2(M)}$ is projective and hence $M = Z_2(M) + M^{'}$. Thus $M^{'}$is nonsingular semisimple (since $M^{'} \cong \frac{M}{Z_2(M)}$). It follows that $M^{'}$ is fully stable module and $M^{'}$ is stable in M.

Now we shall give some other properties of strongly t-semisimple.

Recall that an R-module M is called quasi-Dedekind if $Hom(\frac{M}{N}, M) = 0$ for all nonzero submodule N of M.Equivantally, M is quasi-Dedkind if for each $f \in End(M)$,$f \neq 0$, then $kerf = 0$ [10]

Proposition 2.4. *If M is a quasi-Dedekind module, then M is t-semisimple if and only if M is strongly t-semisimple.*

Proof. $\Rightarrow$ since M is quasi-Dedekind, then for each $f \in EndM$ $f \neq 0$, $Kerf = 0$, and hence $kerf$ is stable and so that by [14], M is SS-module and so that M is strongly t-semisimple by Remarks and Examples 2.2(8).

$\Leftarrow$ It is clear.

To prove the next result, we state and prove the following Lemma.

Lemma 2.5. *Let N be a submodule of M and K is a direct summand of M such that $K \leq N$. If K is fully invariant submodule in M, then K is a fully invariant submodule in N.*

Proof. To prove K is a fully invariant submodule of N. Let $\varphi : N \to N$ be an R-homomorphism, to prove $\varphi(K) \leq K$.

Consider the sequence $M \overset{\rho}{\to} K \overset{inc}{\to} N \overset{\varphi}{\to} N \overset{j}{\to} M$. Where ρ is the natural projection and i,j are the inclusion mapping. Then $(j \circ \varphi \circ i \circ \rho) \in EndM$, and since K is a fully invariant in M, so $(j \circ \varphi \circ i \circ \rho)(K) \subseteq K$. But $j \circ \varphi(\rho(K))$=$j \circ \varphi(K)$=$\varphi(K)$, hence $\varphi(K) \leq K$. Thus K is a fully invariant submodule of N.

Proposition 2.6. *Every submodule of strongly t-semisimple module is strongly t-semisimple.*

Proof. Let $N \leq M$, let $W \leq N$, so $W \leq M$. Since M is strongly t-semisimple, there exists fully invariant direct summand K of M such that $K \leq_{tes} W \leq N$. As $K \leq^{\oplus} M$, $M = K \oplus K^{'}$ for some $K^{'} \leq M$ then, $N = N \bigcap (K \oplus K^{'})$=$K \oplus (K^{'} \bigcap \bigcap N)$. So that $K \leq^{\oplus} N$, and by Lemma (2.5) K is fully invariant submodule of N. Therefore, K is fully invariant direct summand of N such that $K \leq_{tes} W \leq N$.Thus N is a strongly t-semisimple module.

Now we consider the direct sum of strongly t-semisimple. First we notice that direct sum of strongly t-semisimple module need not be strongly t-semisimple for example:

Consider R as R -module R is strongly t-semisimple. But $M = R \oplus R$ is not strongly t-semisimple by Remarks and Examples 2.2(12). However, the direct sum of strongly t-semisimple is strongly t-semisimple under certain condition. Before giving our next result, we present the following lemma.

Lemma 2.7. *Let $M = M_1 \oplus M_2$ such that $annM_1 + annM_2 = R$.Then $Hom(M_1, M_2) = 0$ and $Hom(M_2, M_1) = 0$.*

Proof. since $R = annM_1 + annM_2$, then $M_1 = M_1(annM_1) + M_1(annM_2)$. Put $annM_1 = A_1$, $annM_2 = A_2$, therefore $M_1 = M_1A_1 + M_1A_2 = M_1A_2$, then for each $\varphi \in Hom(M_1, M_2)$,$\varphi(M_1) = \varphi(A_2M_1) = \varphi(M_1)A_2 \leq M_2A_2 = 0$,hence $\varphi = 0$.Thus $Hom(M_1, M_2) = 0$. Similarly, $Hom(M_2, M_1) = 0$.

Theorem 2.8. *Let $M = M_1 \oplus M_2$ such that $annM_1 + annM_2 = R$. Then M_1, M_2 are strongly t-semisimple if and only if $M = M_1 \oplus M_2$ is strongly t-semisimple.*

Proof. $\Leftarrow$ By Proposition(2.6).

$\Rightarrow$ Let $N \leq M$. Since $annM_1 + annM_2 = R$, $N = N_1 \oplus N_2$ for some N_1 and N_2 submodules of M_1 and M_2 respectively. As M_1 and M_2 are strongly t-semisimple, then there exist $K_1 \leq M_1$ and $K_2 \leq M_2$ such that K_1 is a direct summand of M_1, K_1 is fully invariant in M_1 and K_1 is t-essential in N_1, K_2 is a direct summand of M_2, K_2 is fully invariant in M_2 and K_2 is t-essential in N_2.But $K_1 \leq^{\square} M_1$ and $K_2 \leq^{\square} M_2$ imply $K_1 \oplus K_2 \leq^{\square} M_1 \oplus M_2$ and $K_1 \leq_{tes} N_1$, $K_2 \leq_{tes} N_2$ imply $K_1 \oplus K_2 \leq_{tes} N_1 \oplus N_2$ by Proposition (1.2).

Now, let

$$\varphi \in \operatorname{End}(M_1, M_2) \cong \begin{pmatrix} \operatorname{End} M_1 & \operatorname{Hom}(M_2, M_1) \\ \operatorname{Hom}(M_1, M_2) & \operatorname{End} M_2 \end{pmatrix} = \begin{pmatrix} \operatorname{End} M_1 & 0 \\ 0 & \operatorname{End} M_2 \end{pmatrix}$$

so

$$\varphi = \begin{pmatrix} \varphi_1 & 0 \\ 0 & \varphi_2 \end{pmatrix}$$

for some $\varphi_1 \in \operatorname{End} M_1$, $\varphi_2 \in \operatorname{End} M_2$. Then $\varphi(K_1 \oplus K_2) = \varphi_1(K_1) \oplus \varphi_2(K_2) \leq K_1 \oplus K_2$ since K_1 is fully invariant in M_1 and K_2 is fully invariant in M_2. Hence M is strongly t-semisimple.

Now we shall give other characterizations of strongly t-semisimple module.

Proposition 2.9. *The following statements are equivalent for a module M, such that any direct summand has a unique complement:*

(1) M is strongly t-semisimple,

(2) For each submodule N of M, there exists a decomposition $M = K \oplus L$ such that $K \leq N$ and L is stable in M and $N \bigcap L \leq Z_2\ L$,

(3) For each submodule N of M, $N = K \oplus K^{'}$ such that K is a direct summand stable in M and $K^{'}$ is Z_2-torsion.

Proof. (1) $\Rightarrow$ (2)
Let K be a complement of $Z_2(N)$ in N. Then $K + Z_2(N) \leq_{ess} N$ and let C be a complement of $K \oplus Z_2(M)$.So $K \oplus Z_2(M) \oplus C \leq_{ess} M$ and hence $K \oplus Z_2(M) \oplus C \leq_{tes} M$. But M is strongly t-semisimple implies M t-semisimple, hence $K \oplus Z_2(M) \oplus C$=M (by [4,Corollary 2.7].Put $Z_2(M) \oplus C$=L. Then $M = K \oplus L$ and hence $N = (K \oplus L) \bigcap N$= $K \oplus (N \bigcap L)$ (by modular law). But $K + Z_2(N) \leq_{ess} N$ implies $\frac{N}{K}$ is Z_2-torsion (by Proposition (1.1)).On other hand, $\frac{N}{K} \cong N \bigcap L$, so that $N \bigcap L$ is Z_2-torsion. Thus $N \bigcap L$=$Z_2(L \bigcap N) \leq Z_2(L)$. Now, C is a complement of $K \oplus Z_2(M)$ which is a direct summand of M, and by hypothesis, C is a unique complement and hence by [2, Theorem(1.4.8)] C is stable and hence $L = Z_2(M) \oplus C$ is stable submodule in M. Thus $M = K \oplus L$ is the desired decomposition.

(2) $\Rightarrow$ (3) By condition (2) $M = K \oplus L$ such that $K \leq N$, L is stable and $N \bigcap L \leq Z_2(L)$. Hence $N = (K \oplus L) \bigcap N = K \oplus (L \bigcap N)$, put $K^{'} = L \bigcap N$, so $N = K \oplus K^{'}$, $\frac{N}{K} \cong K^{'} = L \bigcap N$ is Z_2-torsion, K is stable in M (since K is complement of L which is direct summand of M).

(3) $\Rightarrow$ (1) By condition (3), $N = K \oplus K^{'}$, $K \leq^{\oplus} M$ and K is stable in M and $K^{'}$ is Z_2-torsion. Then $K \leq^{\oplus} M$ and $K \leq N$ and $\frac{N}{K} \cong K^{'}$ is Z_2-torsion. Hence $K \leq_{tes} N$ and so that M is strongly t-semisimple.

Definition 2.10. (see [7]) An R-module M is called comultiplication if $ann_M ann_R N = N$ for every submodule N of M.

Lemma 2.11. *Every comultiplication module is fully stable.*

Proof. Let M be a comultiplication R-module. Then $ann_M ann_R N = N$ for all $N \leq M$. Hence $ann_M ann_R(xR) = xR$ for all cyclic submodules xR in M.Thus M is fully stable, [2, Corollary(3.5)].

Corollary 2.12. *Let M be a comultiplication R-module. Then M is t-semisimple if and only if M is strongly t-semisimple.*

Proof. $\Leftarrow$ It is clear.

$\Rightarrow$ It follows directly by Lemma (2.11) and Remarks and Examples 2.2(6).

Recall that an R-module M is called a principally injective if for any $a \in R$, any homomorphism $f : Ra \to M$ extends to an R-homomorphism from R_R to M [12].

Corollary 2.13. *Let M be a principally injective. Then M is t-semisimple if and only if M strongly t-semisimple.*

Proof. $\Leftarrow$ It is clear.

$\Rightarrow$ M is principally injective implies that $ann_M ann_R(x) = (x)$ for each $x \in R$. Hence by [2, Corollary(3.5)] M is fully stable. Then by Remark and Examples 2.2(5), M is strongly t-semisimple.

Corollary 2.14. *(2.14):*
M is injective R- module. Then M is t-semisimple R- module if and only if M is strongly t-semisimple.

Definition 2.15. (2.15) [12]:
An R-module is called scalar if for all $\varphi \in \operatorname{End} M$, there exists $r \in R$ such that $\varphi(x) = xr$ for all $x \in M$, where R is a commutative ring.

Proposition 2.16. *(2.16):*
Let M be a scalar R-module. Then M is t-semisimple if and only if M is strongly t-semisimple, where R is commutative.

Proof: $\Leftarrow$It is clear.
$\Rightarrow$ Let N$\leq M, let \varphi \in \operatorname{End} M$. Since M is scalar, there exists $r \in R$ such that $\varphi(x) = xr$,for all $x \in M$. Hence $\varphi(N) = Nr \leq N$ and so that N is fully invariant submodule. Thus M is duo. But M is duo and t-semisimple implies M is strongly t-semisimple by Remarks and Examples 2.2(6).

Proposition 2.17. *(2.17):*
Let M be a duo R-module. Then the following statements are equivalent
(1) Every R-module is t-semisimple and $Z_2(M)$ is projective.
(2) Every R-module is strongly t-semisimple and $Z_2(M)$ is projective.
(3) R is semisimple.

Proof: (1) $\Rightarrow$ (3)
Let M be an R-module. Then M is t-semisimple by hypothesis. Hence $M = Z_2(M) \oplus M'$, where M' is a nonsingular semisimple. It follows that M' is projective, but by hypothesis $Z_2(M)$ is projective. Thus M is projective, that is every R-module is projective and so by [11, Corollary 8.2.2(e)]R is semisimple.
(3) $\Rightarrow$ (1)
Since R is semisimple, every R-module is semisimple by [11, Corollary 8.2.2(a)]

Hence every R-module is t-semisimple. Also R is semisimple, then every R-module is projective [11, Corollary 8.2.2(e)]. Thus $Z_2(M)$ is projective.
(1) ⇒ (2) It follows by Remark and Examples (2.2.(6))
(2) ⇒ (1) It is clear.

Proposition 2.18. *(2.18):*
Let M be a duo R-module if R is semisimple then every R-module is strongly t-semisimple, and conversely hold if R is nonsingular.

Proof: ⇒ R is semisimple implies every R-module M is semisimple and hence t-semisimple. But M is duo by hypothesis, so that M is strongly t-semisimple by Remark and Examples 2.2(6).
⇐ By hypothesis, R is t-semisimple. But R is nonsingular, so R is semisimple
Now we introduce the following:

Definition 2.19. (2.19):
An R-module M is called t-uniform if every submodule of M is t-essential.

Proposition 2.20. *(2.20):*
If M is t-uniform then M is strongly t-semisimple.

Proof: Since M is t-uniform, (0) $\leq_{tes}$ M . Hence $\frac{M}{(0)}$ is Z_2-torsion (by proposition. 1.1(4)); that is M is Z_2-torsion (so $M = Z_2(M)$). Now for all $N \leq M$, $Z_2(N) = Z_2(M) \bigcap N = N$.Hence (0) $\leq_{tes}$ N(since $(0) + Z_2(N) = 0 + N = N \leq_{ess} N$).But (0) is fully invariant direct summand of M .Thus M is strongly t-semisimple.

Remark 2.21. (2.21):
A uniform module need not be t-uniform.

Example 2.22. (2.22):
Consider Z- module Z_6, Z_6 is singular, hence Z_6 is Z_2-torsion; that is $Z_2(Z_6) = Z_6$. Hence for each $N \leq Z_6$, $N + Z_2(Z_6) = Z_6 \leq_{ess} Z_6$ and then by Proposition (1.1),$N \leq_{tes} Z_6$. Thus Z_6 is t-uniform. But Z_6 is not uniform.

Remark 2.23. (2.23):
It is clear that t-uniform module need not uniform, as the following example shows.

Example 2.24. (2.24):
Z$_6 as Z - module$,Z$_2(M) = Z_6 = M$,$(\overline{0}) \leq_{tes} M$ since $(\overline{0}) + Z_2(M) = M \leq_{ess} M$, Let $N_1 =< \overline{2} > \leq_{tes} M$ since $< \overline{2} > + Z_2(M) = M \leq_{ess} M$, similarly $N_2 =< \overline{3} > \leq_{tes} M$,$N_3 = M \leq_{tes} M$.Thus M is t-uniform, but M is not uniform.

Remark 2.25. (2.25):
M is t-uniform then $\frac{M}{N}$ is t-semisimple for all $N \leq M$.

Proof: For each $N \leq M$, $N \leq_{tes} M$. Then $\frac{M}{N}$ is Z_2-torsion (by proposition 1.1(4)). Hence $\frac{M}{N}$ is strongly t-semisimple by Remarks and Examples 2.2(2).

3. Strongly t-Semisimple Rings

Proposition 3.1. *(3.1):*
Every commutative t-semisimple ring R is strongly t-semisimple ring R.

Proof: Since R is commutative ring, then R is duo R-module and t-semisimple, implies R is strongly t-semisimple by Examples and Remarks 2.2(6).

Proposition 3.2. *(3.2):*
Let R be a commutative Artinian ring with $RadR \leq_{tes} R$. Then R is strongly t-semisimple. In particular every local Artinain ring is strongly t-semisimple.

Proof: By [4, Proposition 3.1], R is t-semisimple ring. Hence by Proposition (3.1), R is strongly t-semisimple.

Example 3.3. (3.3):
The ring Z_{P^∞} is Artnian and $RadZ_{P^\infty}$=$Z_{P^\infty} \leq_{ess} Z_{P^\infty}$. Hence by Proposition (3.2), Z_{P^∞} is strongly t-semisimple.

Proposition 3.4. *(3.4):*
The following statements are equivalent for a commutative ring
(1) R is strongly t-semisimple;
(2) R is t-semisimple;
(3) Every R-module is t-semisimple;
(4) Every nonsingular R-module is semisimple;
(5) Every nonsingular R-module is injective;
(6)Every R-module M there is an injective submodule M' such that $M = Z_2$ $(M) \oplus M'$;
(7) $\frac{R}{Z_2(R)}$ is a semisimple ring.
(8) Every maximal ideal which contains Z_2 (R) is a direct summand;
(9) R is a direct product of two ring, one is Z_2- torsion and other is semisimple ring.

Proof: (1) ⇒ (2) It is clear
(2) ⇒ (1) It is follows by (Proposition 3.1).

(2) ⇔ (3) ⇔ (4) ⇔ (5) ⇔ (6) ⇔ (7) [3, Theorem (3.2)].
(2) ⇔ (8) ⇔ (9) It follows by [3, Theorem 3.8]

Corollary 3.5. *(3.5) [4]:*
Let R be a t-semisimple ring.
(1) A maximal right ideal I of R is a direct summand if and only if it contains $Z_2(R)$.
(2) A minimal right ideal J of R is a direct summand if and only if it is nonsingular.

Corollary 3.6. *(3.6):*
Let R be a strongly t-semisimple. A maximal ideal I of R is a direct summand if and only if $I \supseteq Z_2(R)$. A minimum ideal I of R is a direct summand if and only if I is nonsingular.

Proof: It follows directly by (Corollary (3.5)).
Recall that a ring R is called quasi-Frobenius if R is self-injective and Noetherian.Equivalently "R is called quasi-Frobenius if R is self-injective and Artinian [9].

Corollary 3.7. *(3.7):*
Let R be a right nonsingular. Then R is quasi-Frobenius if and only if R is semisimple [3].

Proposition 3.8. *(3.8):*
Let R be a nonsingular ring. Then the following statements are equivalent:
(1) R is quasi-Frobenius;
(2) R is semisimple ;
(3) R is t-semisimple (R is strongly t-semisimple);
(4) Every R-module is t-semisimple;
(5) Every nonsingular R-module is semisimple;
(6) Every nonsingular R-module is injective;
(7) For every R-module M, there exists an injective submodule M' such that $M = Z_2\ (M) \oplus M'$;
(8) $\frac{R}{Z_2(R)}$ is a semisimple ring.

Proof: (3) ⇔ (4) ⇔ (5) ⇔ (6) ⇔ (7) ⇔ (8) by Proposition (3.4).
(1) ⇔ (2) It follows by Corollary (3.7)
(2) ⇔ (3) It follows by [3] and Proposition(3.4).

Proposition 3.9. *(3.9):*
The following statements are equivalent for a commutative ring R

(1)R is t-semisimple (R is strongly t-semisimple);
(2) Every weak duo module (SS-module) is strongly t-semisimple;
(3) Every R-module is t-semisimple.

Proof: (1) ⇔ (3) by Proposition (3.4)
(3) ⇔ (2) It follows by Remarks Examples 2.5(8).
(2) ⇒ (1) R is duo (because R is commutative ring with unity), so R is strongly t-semisimple.

Proposition 3.10. *(3.10):*
The following statements are equivalent for a commutative ring R:
(1) R is t-semisimple;
(2) Every nonsingualr R-module is strongly t-semisimple;
(3) For every R-module M, there exists a strongly t-semisimple R-module M' such that $M = Z_2(M) \oplus M'$.

Proof: (1) ⇒ (2) Let M be a nonsingular R-module. Hence M is t-semisimple by Proposition (3.4) (1 ⇒ 3), also M is injective by (Proposition (3.4) ((1) ⇒ (5)). It follows that M is strongly t-semisimple by (Corollary (2.15))
(2) ⇒ (1) By condition (2) every nonsingular module M is strongly t-semisimple, hence every nonsingular module M is t-semisimple. Thus every nonsingular is semisimple by (Remark and Examples 2.2(6)). It follows that R is t-semisimple by (Proposition (3.4) (4) ⇒ (1)).
(1) ⇒ (3) By (Proposition (3.4) (1) ⇒ (6)), $M = Z_2(M) \oplus M'$ for some injective R-module M' by But $M' \cong \frac{M}{Z_2(M)}$ which is nonsingular module. Hence M' is t-semisimple by (proposition (3.4) (1) ⇒ (4)). Thus M' is t-semisimple and injective, so M' is strongly t-semisimple by Corollary (2.15).
(3) ⇒ (1) $M = Z_2(M) \oplus M'$, where M' is strongly t-semisimple. Hence M' is t-semisimple. But $M' \cong \frac{M}{Z_2(M)}$ which is nonsingular, so $M'$is nonsingular t-semisimple. Thus $M'$is semisimple by Remarks and Examples 2.2(6). But $M'$is injective. Thus R is t-semisimple by (Proposition (3.4) (6) ⇒ (1)).

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