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FACIAL TOTAL-COLORING OF BIPARTITE PLANE GRAPHS

Július Czap[1 §], Peter Šugerek[2]

[1,2]Department of Applied Mathematics and Business Informatics
Faculty of Economics
Technical University of Košice
Němcovej 32, 040 01 Košice, SLOVAKIA

Abstract: Let G be a plane graph. Two edges are facially adjacent in G if they are consecutive edges on a boundary walk of a face of G. A facial edge-coloring of G is an edge-coloring such that any two facially adjacent edges receive distinct colors. A facial total-coloring of G is a coloring of vertices and edges such that no facially adjacent edges, no adjacent vertices, and no edge and its endvertices are assigned the same color. In this paper we prove that every plane graph admits a facial edge-coloring with at most four colors such that at most three colors appear at each vertex. Using this result we confirm a conjecture on facial total-coloring for bipartite plane graphs.

AMS Subject Classification: 05C10, 05C15
Key Words: plane graph, total-coloring

1. Introduction

All graphs considered in this paper are connected and simple. We use standard graph theory terminology according to the book [3]. However, the most frequent notions of the paper are defined through it. A *plane graph* is a particular drawing of a planar graph in the Euclidean plane. Let G be a connected plane graph with vertex set $V(G)$, edge set $E(G)$, and face set $F(G)$. The boundary of a face f is the boundary in the usual topological sense. It is a collection of all edges and vertices contained in the closure of f that can be organized into a

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§Correspondence author

closed walk in G traversing along a simple closed curve lying just inside the face f. This closed walk is unique up to the choice of initial vertex and direction, and is called the *boundary walk* of the face f (see [7], p. 101). Two edges are *facially adjacent* in G if they are consecutive edges on a boundary walk of a face of G. Colorings of graphs embedded in the plane with face-constrains have recently drawn a substantial amount of attention, see e.g. [4]. A *facial edge-coloring* of a plane graph G is a mapping $\varphi : E(G) \to \{1, \dots, k\}$ such that any two facially adjacent edges receive distinct colors. A *facial total-coloring* of a plane graph G is a mapping $\psi : E(G) \cup V(G) \to \{1, \dots, k\}$ such that no facially adjacent edges, no adjacent vertices, and no edge and its endvertices are assigned the same color. Fabrici, Jendroľ, and Voigt [5] proved that every plane graph admits a facial-edge coloring with at most four colors. Moreover, they showed that this bound is tight. In this paper we improve this result. We show that every plane graph admits a facial edge-coloring with at most four colors such that at most three colors appear at each vertex. By this result we confirm a conjecture of Fabrici, Jendroľ, and Vrbjarová [6] for bipartite plane graphs; we prove that every bipartite plane graph has a facial total-coloring with at most five colors. Moreover, we present a bipartite plane graph, which uses at least five colors in every facial total-coloring. Finally, we prove that every tree admits a facial total-coloring with at most four colors, moreover, if only the leaves have odd degree, then three colors suffice.

2. Results

The *facial chromatic index* $\overline{\chi}_e(G)$ of a plane graph G is the smallest integer k such that G has a facial edge-coloring with k colors.

Fabrici, Jendroľ, and Voigt proved that every plane graph admits a facial edge-coloring with at most four colors.

Theorem 1. [5] *Let G be a plane graph. Then*

$$\overline{\chi}_e(G) \leq 4.$$

Moreover, this bound is tight.

Now we improve this result, we show that every plane graph has a facial edge-coloring with at most four colors such that at most three colors appear at each vertex (we say that a color c appears at a vertex v in an edge-colored graph G if v is incident with an edge of color c). The *simplified medial* $M(G)$ of a plane graph G can be obtained as follows: Corresponding to each edge e of

G there is a vertex $m(e)$ of $M(G)$; two vertices $m(e_1)$ and $m(e_2)$ are joined by an edge in $M(G)$ if and only if their corresponding edges e_1 and e_2 are facially adjacent in G. It is easy to see that the simplified medial $M(G)$ of a plane graph G is itself a planar graph; moreover, there is a natural embedding of $M(G)$ in the plane, see Figure 1.

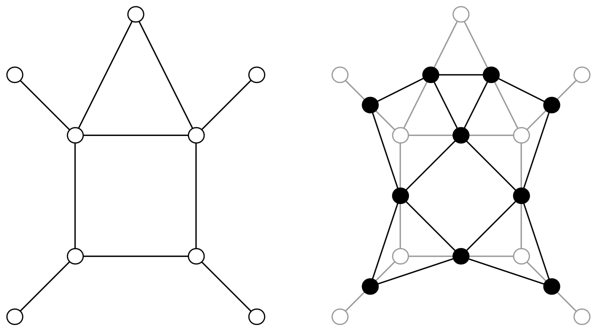

Figure 1: A plane graph and its simplified medial graph.

Observe that every vertex of G of degree d, $d \geq 3$, corresponds to a face of $M(G)$ of size d.

Theorem 2. *Every plane graph G admits a facial edge-coloring with at most four colors such that at most three colors appear at each vertex of G.*

Proof. Observe that every facial edge-coloring of G corresponds to a proper vertex-coloring (i.e. adjacent vertices have distinct colors) of $M(G)$ and vice versa. Since every vertex of G of degree d, $d \geq 3$, corresponds to a face of $M(G)$ of size d, it is sufficient to show that $M(G)$ has a proper vertex-coloring with at most four colors such that at most three colors appear on every face.

Insert a vertex v_f to every face f of $M(G)$ and join (without edge crossing) v_f with all vertices incident with f. In such a way we obtain a plane graph H. It is well known that every plane graph admits a proper vertex-coloring with at most four colors, see [1, 2, 8]. Every proper vertex-coloring of H with at most four colors induces a required vertex-coloring of $M(G)$. Since the vertex v_f is adjacent to every vertex incident with f, the color of v_f does not appear on the vertices incident with f, ergo, at most three colors appear on f. □

The *facial total chromatic number* $\overline{\chi}_{ve}(G)$ of a plane graph G is the smallest integer k such that G has a facial total-coloring with k colors. Fabrici, Jendroľ, and Vrbjarová [6] proved that every plane graph admits a facial total-coloring with at most six colors. They stated the following conjecture.

Conjecture 3. [6] *If G is plane graph, then $\overline{\chi}_{ve}(G) \leq 5$.*

Conjecture 3 was proved for plane triangulations and outerplane graphs [5]. In the next part of this paper we confirm the conjecture for bipartite plane graphs.

Theorem 4. *Every bipartite plane graph G admits a facial total-coloring with at most five colors, i.e. $\overline{\chi}_{ve}(G) \leq 5$. Moreover, this bound is tight.*

Proof. Let V_1 and V_2 be the two parts of G, i.e. $V = V_1 \cup V_2$, $V_1 \cap V_2 = \emptyset$, and every edge join a vertex in V_1 to one in V_2. Theorem 2 implies that G admits a facial edge-coloring with at most four colors, say $1, 2, 3, 4$, such that at most three colors appear at each vertex of G. Every such edge-coloring can be extended to a required facial total-coloring of G. It suffices to color each vertex of G with a color which does not appear on the incident edges, and then, recolor all the vertices from one part, say V_1, with color 5. Clearly, no edge and its endvertex have the same color. Adjacent vertices have different colors, because vertices in V_1 are colored with 5 and vertices in the other part are colored with $1, 2, 3, 4$. Facially adjacent edges have different colors, because we extended a facial edge-coloring. Therefore, the obtained coloring has the required properties.

Now we prove that the bipartite graph H depicted in Figure 2 has no facial total-coloring with four colors.

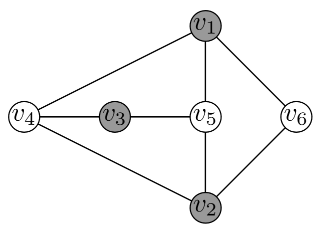

Figure 2: A bipartite plane graph H whose facial total chromatic number equals five.

Suppose to the contrary that H has a facial total-coloring c with colors $1, 2, 3, 4$. Since v_1 and v_5 are adjacent they have distinct colors, say $c(v_1) = 1$ and $c(v_5) = 2$. In the following we distinguish two cases according to the colors of v_1 and v_2.

Case 1. First assume that v_1 and v_2 have the same color, i.e. $c(v_1) = c(v_2) = 1$. Since c uses only four colors, each of the vertices v_1 and v_2 is incident with an edge of color 2. Therefore, either v_1v_4, v_2v_6 or v_1v_6, v_2v_4 are colored with 2. Without loss of generality assume that $c(v_1v_4) = c(v_2v_6) = 2$. Then $c(v_4) \in \{3, 4\}$. W.l.o.g. assume that $c(v_4) = 3$. Consequently, $c(v_2v_4) = 4$ and $c(v_2v_5) = 3$. Then $c(v_5v_1) = 4$ and $c(v_5v_3) = 1$. So there is no admissible color for the edge v_3v_4.

Case 2. Now assume that $c(v_1) \neq c(v_2)$. Since $c(v_1) = 1$ and $c(v_5) = 2$ we have $c(v_2) \in \{3, 4\}$. W.l.o.g. assume that $c(v_2) = 3$. Similarly as in the previous case we can assume that $c(v_1v_4) = c(v_2v_6) = 2$. Consequently, $c(v_4) = c(v_6) = 4$. Then $c(v_2v_4) = 1$ and $c(v_2v_5) = 4$. Finally, the edge v_1v_6 must have color 3, but in this case there is no admissible color for the edge v_1v_5, a contradiction. □

Note that it is easy to check whether a plane graph admits a facial total-coloring with three colors, because, in this case, every coloring of a single edge enforces the coloring of the whole graph. A challenging open problem in this direction is the following.

Problem 5. Characterize all bipartite plane graphs that admit a facial total-coloring with four colors.

Problem 5 cannot be solved using only structural properties of graphs. The particular embedding of the graph is very important. The bipartite graph depicted in Figure 3 has different facial total chromatic number depending on its embedding. With the embedding on the left its facial total chromatic number is three and with the embedding on the right, its facial total chromatic number is at least four.

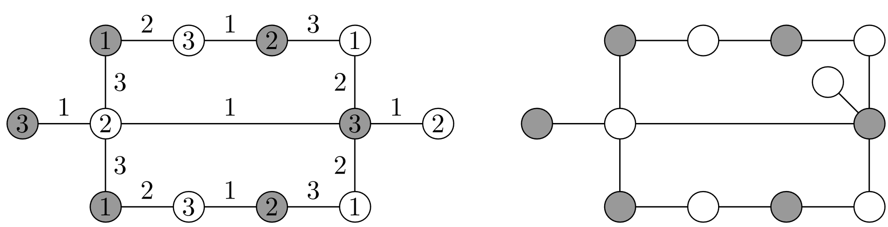

Figure 3: Two embeddings of the same graph with different facial total chromatic numbers.

In the rest of the paper we show that every tree has a facial total-coloring with at most four colors. Let v be a vertex of a tree T. If the degree of v equals one, then it is a *leaf*, otherwise we call it *internal vertex*.

Theorem 6. *If T is a nontrivial tree (i.e. $|V(T)| \geq 2$), then*

$$\overline{\chi}_{ve}(T) = \begin{cases} 3 & \textit{if each internal vertex has an even degree,} \\ 4 & \textit{otherwise.} \end{cases}$$

Proof. First assume that each internal vertex of T has an even degree.

If T is a star, then let v be its central vertex and let $e_1, \dots, e_k$ be the edges incident with v, listed in their clockwise order around v. Let v_i denote the second endvertex of e_i. In this case we color the vertex v with 1, the edges (with odd subscripts) $e_1, e_3, \dots, e_{k-1}$ and vertices (with even subscripts) $v_2, v_4, \dots, v_k$ with 2, and the other edges and vertices with 3.

If T is not a star, then pick any vertex of T to be the root. We color the edges and vertices of T starting from the root to the leaves. In each step it is sufficient to find a facial total-coloring of a star with (at most) one precolored edge. Let S_k be a star on k edges $e_1, \dots, e_k$. Assume that the edge e_1 of S_k has already been colored with color x and its endvertices with colors y and z, where y is the color of the central vertex of S_k. In this case we color the edges $e_3, e_5, \dots, e_{k-1}$ and vertices $v_2, v_4, \dots, v_k$ with x, and the the edges $e_2, e_4, \dots, e_k$ and vertices $v_3, v_5, \dots, v_{k-1}$ with z.

Now assume that at least one internal vertex, say u, has an odd degree. In this case we need at least four colors, because the edges incident with u cannot be colored with two colors (and the color of u cannot appear on the incident edges).

If T is a star, then we color the central vertex v with 1, the edge e_1 with 2, and the vertex v_1 with 3. Thereafter we color the edges $e_2, \dots, e_k$ alternately with 3 and 4, and color all the vertices $v_2, \dots, v_k$ with 2.

If T is not a star, then we pick any vertex of T to be the root and color the edges and vertices of T starting from the root to the leaves. Similarly as in the previous case, in each step we find a facial total-coloring of a star with (at most) one precolored edge. Let S_k be a star on k edges. Assume that the edge e of S_k has already been colored with color x and its endvertices with colors y and z, where y is the color of the central vertex of S_k. We color the uncolored edges of S_k alternately with z and w (w is the fourth color), and color all uncolored vertices with x. $\square$

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