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# AN UPPER ESTIMATE FOR THE MAXIMAL SINGULAR VALUE OF A SPECIAL MATRIX, I

Svetoslav I. Nenov
Department of Mathematics
University of Chemical Technology and Metallurgy
Sofia 1756, BULGARIA

**Abstract:** Our goal is to prove an upper bound for the maximal singular value of the following matrix

$$\boldsymbol{A}_0 = \boldsymbol{D}^{-1}\boldsymbol{E}\left(\boldsymbol{E}^t\boldsymbol{D}^{-1}\boldsymbol{E}\right)^{-1}\boldsymbol{E}^t.$$

Here $\cdot^t$ denotes the matrix transpose, $\boldsymbol{D}$ is a non-singular matrix, and $\boldsymbol{E}$ is thin matrix.

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**Key Words:** singular value, bounds on the singular values of a matrix, moving least-squares approximation

## 1. Statement

In many situation, we have to deal with a matrix in the form ($\cdot^t$ is matrix transpose)

$$\boldsymbol{A}_0 = \boldsymbol{D}^{-1}\boldsymbol{E}\left(\boldsymbol{E}^t\boldsymbol{D}^{-1}\boldsymbol{E}\right)^{-1}\boldsymbol{E}^t,$$

where:

1. $\boldsymbol{D}$ is a symmetric, positive definite $(m \times m)$-matrix.
2. $\boldsymbol{E}$ is a $(m \times n)$-matrix, $n \le m$ and rank $(\boldsymbol{E}) = n$.
3. It follows from (2) that det $\left(\boldsymbol{E}^t\boldsymbol{D}^{-1}\boldsymbol{E}\right) \neq 0$.

Our goal is to establish an upper bound for $\|\boldsymbol{A}_0\|$.

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In many applications $\boldsymbol{D}$ is a diagonal matrix, see for example the results in [1, 2, 4, 5, 6, 7, 10] for simple constructions of moving least squares approximations. The special case $l = 1$ is exactly the Shepard approach in moving least squares approximations (with many computer applications, for example SYMAP maps constructed in Harvard Laboratory for Computer Graphics and Spatial Analysis).

## 2. Main Result

First, we will prove that the matrix $\boldsymbol{E}^t\boldsymbol{D}^{-1}\boldsymbol{E}$ is non-singular.

**Lemma 2.1.** *Let:*

1. $\boldsymbol{D}$ *is a symmetric, positive definite* $(m \times m)$*-matrix.*

2. $\boldsymbol{E}$ *is a* $(m \times n)$*-matrix,* $n \leq m$ *and* $\operatorname{rank}(\boldsymbol{E}) = n$.

*Then* $\det\left(\boldsymbol{E}^t\boldsymbol{D}^{-1}\boldsymbol{E}\right) \neq 0$.

*Proof.* (1) $\boldsymbol{D}$ is a symmetric, positive definite $(m \times m)$-matrix. Let

$$\boldsymbol{D} = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^t$$

be the singular value decomposition of $\boldsymbol{D}$.

We will use the folowong notation: $\boldsymbol{D}^{\frac{1}{2}} = \boldsymbol{U}\boldsymbol{\Sigma}^{\frac{1}{2}}\boldsymbol{V}^t$. Then, obviously, $\boldsymbol{D}^{\frac{1}{2}}\boldsymbol{D}^{\frac{1}{2}} = \boldsymbol{D}$ and $\boldsymbol{D}^{-\frac{1}{2}} = \left(\boldsymbol{D}^{\frac{1}{2}}\right)^{-1}$, so $\boldsymbol{D}^{-\frac{1}{2}}\boldsymbol{D}^{-\frac{1}{2}} = \boldsymbol{D}^{-1}$.

(2) $\boldsymbol{D}^{-\frac{1}{2}}$ is a non-singular $(m \times m)$-matrix. So, $\operatorname{rank}\left(\boldsymbol{D}^{-\frac{1}{2}}\right) = m$. Moreover, $\operatorname{rank}(\boldsymbol{E}) = n$ and $n \leq m$. Thereofre $\operatorname{rank}\left(\boldsymbol{D}^{-\frac{1}{2}}\boldsymbol{E}\right) = n$.

Indeed, $\operatorname{rank}\left(\boldsymbol{D}^{-\frac{1}{2}}\boldsymbol{E}\right) \leq \min\left\{\operatorname{rank}\left(\boldsymbol{D}^{-\frac{1}{2}}\right), \operatorname{rank}(\boldsymbol{E})\right\} = \min\{m, n\} = n$. On the other hand, from Sylvester Rank Inequality $n = m + n - m = \operatorname{rank}\left(\boldsymbol{D}^{-\frac{1}{2}}\right) + \operatorname{rank}(\boldsymbol{E}) - m \leq \operatorname{rank}\left(\boldsymbol{D}^{-\frac{1}{2}}\boldsymbol{E}\right)$. Hence $\operatorname{rank}\left(\boldsymbol{D}^{-\frac{1}{2}}\boldsymbol{E}\right) = n$.

(3) Let

$$\boldsymbol{D}^{-\frac{1}{2}}\boldsymbol{E} = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^t$$

be the singular value decomposition of the full rank matrix $\boldsymbol{D}^{-\frac{1}{2}}\boldsymbol{E}$. Then

$$\boldsymbol{E}^t\boldsymbol{D}^{-1}\boldsymbol{E} = \boldsymbol{E}^t\boldsymbol{D}^{-\frac{1}{2}}\boldsymbol{D}^{-\frac{1}{2}}\boldsymbol{E}$$

$$
\begin{aligned}
&= \left(\boldsymbol{D}^{-\frac{1}{2}}\boldsymbol{E}\right)^t \left(\boldsymbol{D}^{-\frac{1}{2}}\boldsymbol{E}\right) \\
&= \left(\boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^t\right)^t \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^t \\
&= \boldsymbol{V}\boldsymbol{\Sigma}^t\boldsymbol{\Sigma}\boldsymbol{V}^t
\end{aligned}
$$

is the singular value decomposition of the symmetric $(n \times n)$-matrix $\boldsymbol{E}^t\boldsymbol{D}^{-1}\boldsymbol{E}$. Moreover, all singular values (i.e. all diagonal elements of $\boldsymbol{\Sigma}^t\boldsymbol{\Sigma}$) are positive. Hence $\boldsymbol{E}^t\boldsymbol{D}^{-1}\boldsymbol{E}$ is non-singular matrix. □

**Theorem 2.1.** *Let:*

1. $n = 1$;
2. $\boldsymbol{D} = \begin{pmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_m \end{pmatrix}$ *be a diagonal matrix with positive elements of the main diagonal:* $d_i > 0$, $i = 1, \ldots, m$;
3. $\boldsymbol{E} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$.

*Then*

$$\sigma_{\max}(\boldsymbol{A}_0) = \|\boldsymbol{A}_0\| \leq \sqrt{m}.$$

*Proof.* Direct calculations:

$$
\begin{aligned}
\boldsymbol{E}^t\boldsymbol{D}^{-1}\boldsymbol{E} &= \begin{pmatrix} 1 & 1 & \cdots & 1 \end{pmatrix} \frac{1}{2} \begin{pmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_m \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \\
&= \frac{1}{2} \begin{pmatrix} 1 & 1 & \cdots & 1 \end{pmatrix} \begin{pmatrix} d_1^{-1} \\ d_2^{-1} \\ \vdots \\ d_m^{-1} \end{pmatrix} \\
&= \frac{1}{2}\left(d_1^{-1} + d_2^{-1} + \cdots + d_m^{-1}\right), \\
\left(\boldsymbol{E}^t\boldsymbol{D}^{-1}\boldsymbol{E}\right)^{-1} &= \frac{2}{d_1^{-1} + d_2^{-1} + \cdots + d_m^{-1}}, \\
\boldsymbol{A}_0 &= \boldsymbol{D}^{-1}\boldsymbol{E}\left(\boldsymbol{E}^t\boldsymbol{D}^{-1}\boldsymbol{E}\right)^{-1}\boldsymbol{E}^t
\end{aligned}
$$

$$=\frac{2}{d_1^{-1}+d_2^{-1}+\cdots+d_m^{-1}}\boldsymbol{D}^{-1}\boldsymbol{E}\boldsymbol{E}^t$$

$$=\frac{2}{d_1^{-1}+d_2^{-1}+\cdots+d_m^{-1}}\boldsymbol{D}^{-1}\begin{pmatrix}1\\1\\\vdots\\1\end{pmatrix}\begin{pmatrix}1 & 1 & \cdots & 1\end{pmatrix}$$

$$=\frac{2}{d_1^{-1}+d_2^{-1}+\cdots+d_m^{-1}}\frac{1}{2}\begin{pmatrix}d_1^{-1} & 0 & \cdots & 0\\ 0 & d_2^{-1} & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & d_m^{-1}\end{pmatrix}$$

$$\times\begin{pmatrix}1 & 1 & \cdots & 1\\ 1 & 1 & \cdots & 1\\ \vdots & \vdots & \ddots & \vdots\\ 1 & 1 & \cdots & 1\end{pmatrix}$$

$$=\frac{1}{d_1^{-1}+d_2^{-1}+\cdots+d_m^{-1}}\begin{pmatrix}d_1^{-1} & d_1^{-1} & \cdots & d_1^{-1}\\ d_2^{-1} & d_2^{-1} & \cdots & d_2^{-1}\\ \vdots & \vdots & \ddots & \vdots\\ d_m^{-1} & d_m^{-1} & \cdots & d_m^{-1}\end{pmatrix}.$$

Hence

$$\begin{aligned}\|\boldsymbol{A}_0\|_{\max} &= \left|\frac{1}{d_1^{-1}+d_2^{-1}+\cdots+d_m^{-1}}\right|\max\left\{|d_i^{-1}| : i=1,\ldots,m\right\}\\ &=\frac{1}{d_1^{-1}+d_2^{-1}+\cdots+d_m^{-1}}\max\left\{d_i^{-1} : i=1,\ldots,m\right\}\\ &<1.\end{aligned}$$

or ($\|\boldsymbol{A}_0\|_2 \le \sqrt{mn}\,\|\boldsymbol{A}_0\|_{\max}$, see [8, Subsection 10.4.4])

$$\|\boldsymbol{A}_0\|_2 \le \sqrt{m}\,\|\boldsymbol{A}_0\|_{\max} < \sqrt{m}.$$ □

**Example 2.1.** Let $m=2$,

$$\boldsymbol{D}=\begin{pmatrix}1 & 0\\ 0 & d_2\end{pmatrix},\quad \boldsymbol{E}=\begin{pmatrix}1\\1\end{pmatrix}$$

Then

$$\boldsymbol{A}_0=\frac{1}{1+\frac{1}{d_2}}\begin{pmatrix}1 & 1\\ \frac{1}{d_2} & \frac{1}{d_2}\end{pmatrix}$$

and

$$\|\boldsymbol{A}_0\| = \frac{\sqrt{2}\sqrt{d_2^2+1}}{d_2+1}.$$

Obviously

$$\lim_{d_2\to\infty} \frac{\sqrt{2}\sqrt{d_2^2+1}}{d_2+1} = \sqrt{2} - 0 = \sqrt{m} - 0.$$

Therefore, we cannot expect a better upper bound in general case.

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